Biframe compactifications

A. Schauerte

Abstract. Compactifications of biframes are defined, and characterized internally by means of strong inclusions. The existing description of the compact, zero-dimensional coreflection of a biframe is used to characterize *all* zero-dimensional compactifications, and a criterion identifying them by their strong inclusions is given. In contrast to the above, two sufficient conditions and several examples show that the existence of smallest biframe compactifications differs significantly from the corresponding frame question.

Keywords: biframe compactification, strong inclusion

Classification: 06D20, 54E55

1. Preliminaries.

Background information on frames may be found in [8], and on biframes in [6], [2].

A **frame** is a complete lattice (with bottom element 0 and top element e) that satisfies the infinite distributivity law $a \wedge \bigvee X = \bigvee a \wedge x \ (x \in X)$.

A **frame map** (or homomorphism) is a function between frames that preserves $\wedge$ and $\bigvee$ (and 0 and e).

A **biframe** is a triple $L = (L_0, L_1, L_2)$ where L_1 and L_2 are subframes of the frame L_0 such that L_0 is generated by $L_1 \cup L_2$. A **biframe map** (or homomorphism) $h : L \to M$ is a frame map $h : L_0 \to M_0$ for which $h(L_i) \subseteq M_i$ $(i = 1, 2)$. In the sequel, we use L_i, L_k to denote L_1 or L_2, always assuming that $i, k = 1, 2$, $i \neq k$.

The following kinds of biframes have been discussed in the literature (see [6], [2]):

- **Compact biframes.** L is compact iff L_0 is a compact frame, that is, $e = \bigvee X$ for some $X \subseteq L_0$ implies that $e = \bigvee F$ for some finite subset $F \subseteq X$.
- **Regular biframes.** We write $x \prec_i y$ (read "x is rather below y in L_i") iff $x, y \in L_i$ and there exists $c \in L_k$ $(i \neq k)$ such that $x \wedge c = 0$ and $c \vee y = e$. Then the biframe L is regular iff $x = \bigvee y \ (y \prec_i x)$ for all $x \in L_i$.
- **Zero-dimensional biframes.** For any $x \in L_i$, denote by $x^\bullet$ the element of L_k given by $x^\bullet = \bigvee z (z \wedge x = 0, \ z \in L_k)$. The biframe L is zero-dimensional if each L_i is generated by those $x \in L_i$ for which $x \vee x^\bullet = e$ (in particular, these elements are complemented).

A biframe map $h : L \to M$ is called

- **dense** iff $h(a) = 0$ implies that $a = 0$, for all $a \in L_0$, and
- **onto** iff $h \mid L_1$ and $h \mid L_2$ are both onto (and hence $h \mid L_0$ is also onto).

The bulk of this work has been taken from my Ph.D. Thesis, written at McMaster University under the supervision of Professor B. Banaschewski, whose assistance is here gratefully acknowledged

2. Compactifications vs strong inclusions.

In frame theory, the compactifiable frames are exactly those which admit strong inclusions (see [3]). We present definitions of compactifications and strong inclusions for biframes which allow us to obtain a similar relationship here: we acknowledge our debt to [4].

Definition 1. *A* **compactification** *of a biframe L is a dense, onto biframe homomorphism $h : M \to L$ from a compact, regular biframe M to L.*

Definition 2. *A* **strong inclusion** *on a biframe L is a pair $\lhd = (\lhd_1, \lhd_2)$ of relations on L_1 and L_2 respectively such that (for $i, k = 1, 2$, $i \neq k$)*

(SI1) *$y \leq x \lhd_i a \leq b$ implies that $y \lhd_i b$;*
(SI2) *$\lhd_i$ is a sublattice of $L_i \times L_i$;*
(SI3) *$x \lhd_i a$ implies that $x \prec_i a$;*
(SI4) *$x \lhd_i a$ implies that there exists $y \in L_i$ with $x \lhd_i y \lhd_i a$;*
(SI5) *if $x \lhd_i a$ then there exist $u, v \in L_k$ such that $u \lhd_k v$, $x \wedge v = 0$ and $a \vee u = e$;*
(SI6) *$a = \bigvee x$ $(x \lhd_i a)$ for all $a \in L_i$.*

Remark. The condition (SI5) above (in the presence of the other properties of strong inclusions) is the same as: $x \lhd_i a$ implies that $a^\bullet \lhd_k x^\bullet$.

($\Longrightarrow$) If $u \lhd_k v$, $x \wedge v = 0$, $a \vee u = e$ then $a^\bullet \leq u \lhd_k v \leq x^\bullet$.

($\Longleftarrow$) Take $x \lhd_i y \lhd_i a$. Then $a^\bullet \lhd_k y^\bullet \lhd_k x^\bullet$, so that $y^\bullet \lhd_k x^\bullet$, $x \wedge x^\bullet = 0$, $a \vee y^\bullet = e$ (since $y \lhd_i a$ implies that $y \prec_i a$, which is the same as $y^\bullet \vee a = e$).

Lemma 1. *On a compact, regular biframe, $(\prec_1, \prec_2)$ is a strong inclusion.*

PROOF: We check only (SI5):
If $x \prec_i a$ then $x \wedge v = 0$, $a \vee v = e$ for some $v \in L_k$. Compactness and regularity give $u \prec_k v$ with $a \vee u = e$. ☐

Lemma 2. *For any onto $h : N \to L$, if $\lhd = (\lhd_1, \lhd_2)$ is a strong inclusion on N, then $\widehat{\lhd} = (h \times h[\lhd_1], h \times h[\lhd_2])$ is a strong inclusion on L.*

PROOF:

(SI1) Take $x \leq h(a) \widehat{\lhd}_i h(b) \leq y$ with $a \lhd_i b$ in N_i and $x, y \in L_i$. Now $x = h(s)$, $y = h(t)$ so $h(s \wedge a) \leq h(a) \widehat{\lhd}_i h(b) \leq h(b \vee t)$ and $s \wedge a \lhd_i b \vee t$.
(SI2) $h \times h$ preserves sublattices.
(SI3) If $x \lhd_i a$ in N_i (that is, $h(x) \widehat{\lhd}_i h(a)$), then $x \prec_i a$, so $h(x) \prec_i h(a)$.
(SI4) If $h(x) \widehat{\lhd}_i h(a)$ then $x \lhd_i y \lhd_i a$ so that $h(x) \widehat{\lhd}_i h(y) \widehat{\lhd}_i h(a)$.
(SI5) If $x \lhd_i a$ in N_i, there exist $u \lhd_k v$ in N_k with $x \wedge v = 0$, $a \vee u = e$. Then $h(x) \widehat{\lhd}_k h(v)$, $h(x) \wedge h(v) = 0$, $h(a) \vee h(u) = e$.
(SI6) For $a \in N_i$, $a = \bigvee x$ $(x \lhd_i a)$ so $h(a) = \bigvee h(x)$ $(x \lhd_i a)$ hence also $h(a) = \bigvee z$ $(z \widehat{\lhd}_i h(a))$.

☐

Corollary 1. *If L has a compactification, it has a strong inclusion.*

We now consider the converse of this corollary, that is, we construct a compactification of L from a given strong inclusion.

Let $(\lhd_1, \lhd_2)$ be a strong inclusion on L. An ideal J of L_0 is a downset (that is, $x \le y \in J$ implies that $x \in J$) that is closed under binary joins. Now, an ideal J that is generated by $J \cap L_i$ will be called **strongly regular** iff $x \in J \cap L_i$ implies that there exists a $y \in J \cap L_i$ with $x \lhd_i y$. Let $\mathcal{R}_i$ consists of these J and $\mathcal{R}_0$ be the subframe of the frame of all ideals of L_0 that is generated by $\mathcal{R}_1 \cup \mathcal{R}_2$. Now $\mathcal{R}_i$ is a subframe of this total frame: it is closed under binary meets and binary joins by (SI2); 0 and $\downarrow e$ are in it, and it is trivially closed under updirected joins (unions). Thus $\mathcal{R}$ is a compact biframe; we verify that it is regular.

Define $r_i : L_i \to \mathcal{R}_i$ by $r_i(a) = [x \mid x \lhd_i a]$, where $[\dots]$ denotes the ideal generated in L_0. Then $r_i(a) \in \mathcal{R}_i$ by (SI2), (SI4).

Claim. $a \lhd_i b$ implies that $r_i(a) \prec_i r_i(b)$.

Proof: Take $a \lhd_i c \lhd_i b$ and $w \lhd_k u$ with $a \wedge u = 0$, $c \vee w = e$. Then $r_i(a) \cap r_k(u) = 0$ and $c \vee w \in r_i(b) \vee r_k(u)$. Hence $r_i(b) \vee r_k(u) = \downarrow e$ and so $r_i(a) \prec_i r_i(b)$.

Finally, for any $J \in \mathcal{R}_i$, $J = \bigvee r_i(a)$ $(a \in J \cap L_i)$ and $a \lhd_i b \in J \cap L_i$ implies that $r_i(a) \prec_i r_i(b) \subseteq J$, which gives the regularity of $\mathcal{R}$.

The join map $\tau_L : \mathcal{R} \to L$ provides the required compactification of L. It maps $\mathcal{R}_i$ onto L_i since $\bigvee r_i(a) = a$, $a \in L_i$, by (SI6). That it is dense is clear.

We have just shown the first part of the next proposition. For its second part, we need the following terminology:

The compactifications (up to isomorphism) of a biframe L form a partially ordered set under the partial order given by $h : M \to L \le \tilde{h} : \widetilde{M} \to L$ iff there exists a biframe map $f : M \to \widetilde{M}$ satisfying $\tilde{h} \cdot f = h$.

The strong inclusions of L form a partially ordered set under set inclusion.

We denote these two sets by $\mathbf{K}L$ and $\mathbf{S}L$, respectively.

Proposition 1. *A biframe L has a compactification if and only if it has a strong inclusion. Moreover, the above constructions provide isomorphisms between $\mathbf{K}L$ and $\mathbf{S}L$ inverse to each other.*

Proof: We first check that both constructions are order-preserving:

Given $\lhd_i \subseteq \widehat{\lhd}_i$, one gets $\mathcal{R}_i \subseteq \widehat{\mathcal{R}}_i$, hence $\mathcal{R} \subseteq \widehat{\mathcal{R}}$ so that $\mathcal{R} \to L \le \widehat{\mathcal{R}} \to L$ by the inclusion map $\mathcal{R} \to \widehat{\mathcal{R}}$.

Given $h : M \to L \le \hat{h} : \widehat{M} \to L$ with $f : M \to \widehat{M}$ satisfying $\hat{h} \cdot f = h$, we obtain $h \times h[\prec] = \hat{h} \cdot f \times \hat{h} \cdot f[\prec] = (\hat{h} \times \hat{h})(f \times f)[\prec] \subseteq \hat{h} \times \hat{h}[\widehat{\prec}]$, where $\widehat{\prec}$ denotes the relation $\prec$ on $\widehat{M}$.

Next we verify that they are inverse to each other.

Let $\lhd$ be a strong inclusion on L, $\tau_L : \mathcal{R} \to L$ the compactification associated with it, and $\widehat{\lhd}$ the strong inclusion associated with that. Then $a \lhd_i b$ implies that $r_i(a) \prec_i r_i(b)$, which, by the definition of $\widehat{\lhd}$, gives $\bigvee r_i(a) \widehat{\lhd}_i \bigvee r_i(b)$ and hence $a \widehat{\lhd}_i b$. Conversely, $a \widehat{\lhd}_i b$ means that $a = \bigvee J$, $b = \bigvee I$ where $I, J \in \mathcal{R}_i$ and $J \prec_i I$. Then there exists $H \in \mathcal{R}_k$ with $J \cap H = 0$ and $I \vee H = \downarrow e$. Thus $x \vee y = e$ for some

$x \in I$, $y \in H$. Also, $x \lhd_i b$ since I is strongly regular. Now $J \cap H = 0$ implies that $a \wedge y = 0$, so $a = a \wedge x \lhd_i b$. Hence $\lhd_i = \widehat{\lhd}_i$.

Beginning with a compactification $h : M \to L$ of L, let $\lhd$ be its associated strong inclusion, and $\tau_L : \mathcal{R} \to L$ the compactification associated with that. Consider the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\tau_L} & L \\ {\scriptstyle \hat{h}}\uparrow & & \uparrow{\scriptstyle h} \\ \widehat{\mathcal{R}} & \xrightarrow[\tau_M]{} & M \end{array}$$

where $\widehat{\mathcal{R}}$ is given by the strongly regular ideals with respect to $\prec_i$ on M_i, and $\hat{h}$ is the restriction of $\mathcal{J}h$. Since $\prec_i$ is a strong inclusion on the compact, regular biframe M, $\tau_M : \widehat{\mathcal{R}} \to M$ is a compactification. It is codense (that is, its total part maps only the top to the top) by compactness of M, hence an isomorphism. We show that $\hat{h}$ is also an isomorphism. Now $\hat{h}$ is dense because h is, and so one-one. (See [1].) It remains to show $\hat{h}$ onto, and for this it suffices to prove $r_i(a) \in Image\,(\hat{h})$, $a \in L_i$. Let $J = [z \in M_i \mid h(z) \in r_i(a)]$. Then $(\mathcal{J}h)(J) = r_i(a)$ because h is onto. To check that $J \in \widehat{\mathcal{R}}$, take $z \in J \cap M_i$, so $h(z) \lhd_i a$. Then $h(z) = h(u)$, $a = h(v)$ for some $u \prec_i v$. Take $u \prec_i w \prec_i v$, hence $u \wedge s = 0$, $w \vee s = e$ for some $s \in L_k$. Now $h(z \wedge s) = h(u \wedge s) = 0$, so $z \wedge s = 0$, by the density of h. So $z \prec_i w \in J \cap M_i$, as required. $\square$

Remark. If (L_0, L_1, L_2) has a compactification, so does L_0, because the total part of a compact, regular biframe is a compact, regular frame (see [6]). The converse is not true, as may be seen by considering $(L, L, \mathbf{2})$ for any non-trivial, compactifiable frame L.

3. Zero-dimensional compactifications.

A natural variation on the theme of compactifications is given by the notion of a zero-dimensional compactification:

Definition 3. *$h : M \to L$ is a* **zero-dimensional compactification** *of L iff M is compact, zero-dimensional and h is dense, onto.*

Since a zero-dimensional biframe is automatically regular, a zero-dimensional compactification *is* a compactification.

In [2], Banaschewski finds the compact, zero-dimensional coreflection of a biframe. In the process he uses the following definitions:

- a **Boolean bilattice** $B = (B_0, B_1, B_2)$ is a triple in which B_0 is a Boolean algebra, B_1 and B_2 are sublattices of B_0 such that B_0 is generated by $B_1 \cup B_2$ and an element of B_0 is in B_i, if and only if its complement is in B_k $(i \neq k)$;
- the **Boolean part** BL of a biframe L is the bilattice whose i-th part is $(BL)_i = \{x \in L_i \mid x \vee x^{\bullet} = e\}$ and whose total part $(BL)_0$ is the sublattice of L_0 generated by $(BL)_1 \cup (BL)_2$;

– the **ideal biframe** $\mathcal{J}A$ of a Boolean lattice A has total part $(\mathcal{J}A)_0$ the ideal frame of A_0 and i-th part
$(\mathcal{J}A)_i = \{J \in (\mathcal{J}A)_0 \mid J \text{ is generated by } J \cap L_i\}$.

In fact, Banaschewski shows that the categories of compact, zero-dimensional biframes and Boolean bilattices are equivalent, via the functors B and $\mathcal{J}$. The natural transformations are $\alpha_A : A \to B\mathcal{J}A$ by $\alpha_A(a) = \downarrow a$ (for bilattices A) and $\sigma_L : \mathcal{J}BL \to L$ by the map taking joins of ideals (for biframes L). Further, the coreflection maps to the compact, zero-dimensional biframes are given by the join maps $\sigma_L : \mathcal{J}BL \to L$.

As a consequence, we see that L has a zero-dimensional compactification iff the join map $\mathcal{J}BL \to L$ is onto iff L is zero-dimensional.

Hence we consider only zero-dimensional biframes in this section. Let $\mathbf{K}_0 L$ be the partially ordered set of zero-dimensional compactifications of L, modulo isomorphism.

Definition 4. *A* **basic Boolean bilattice** *of a (zero-dimensional) biframe L is any Boolean sublattice A of BL which generates L (that is, A_i generates L_i). Ordering these basic Boolean lattices of L by inclusion gives a partially ordered set, which we denote by* $\mathbf{BBB}(L)$.

We now describe two correspondences between zero-dimensional compactifications and basic Boolean bilattices of a biframe.

- For any such bilattice A, the join map $\sigma_L : \mathcal{J}A \to L$ is a zero-dimensional compactification of L.
- For any zero-dimensional compactification $h : M \to L$, $h[BM] \subseteq BL$ is a basic Boolean bilattice of L.

Proposition 2. *The correspondences above are mutually inverse isomorphisms between* $\mathbf{BBB}(L)$ *and* $\mathbf{K}_0(L)$.

Proof: Let A be a basic Boolean bilattice of L, $\sigma : \mathcal{J}A \to L$ the compactification associated with it, and $\sigma[B\mathcal{J}A] \subseteq BL$ the bilattice associated with that. Since $(B\mathcal{J}A)_i = \{\downarrow a \mid a \in A_i\}$, we obtain $\sigma[B\mathcal{J}A] = A$.

Let $h : M \to L$ be a zero-dimensional compactification of L, $A = h[BM]$ and $\sigma : \mathcal{J}A \to L$ the compactification associated with A. Consider

$$\begin{array}{ccc} \mathcal{J}A & \xrightarrow{\sigma} & L \\ {\scriptstyle \hat{h}}\uparrow & & \uparrow{\scriptstyle h} \\ \mathcal{J}BM & \xrightarrow[\bigvee]{} & M \end{array}$$

where $\hat{h}$ is given by $\hat{h}(J) = \bigcup\{\downarrow h(a) \mid a \in J\}$ for $J \in (\mathcal{J}BM)_0$. The bottom join map is an isomorphism because M is compact, regular. $\hat{h}$ is onto since $h : (BM)_i \to A_i$ is onto, and $\hat{h}$ is dense because h is (and dense maps between compact, regular biframes are one-one). Thus $\hat{h}$ is an isomorphism too, and σ is isomorphic to h.

$\square$

The next result shows how zero-dimensional compactifications may be identified by looking at their strong inclusions.

Proposition 3. *The compactification associated with $\lhd$ is zero-dimensional if and only if, for any $a \lhd_i b$, there exists $c \in L_i$ with $a \leq c \lhd_i c \leq b$.*

PROOF: ($\Longrightarrow$) Suppose we are given a zero-dimensional compactification $h : M \to L$ with associated $\lhd_i = h \times h[\prec_i]$. Let $a \lhd_i b$ and $u \prec_i v$ with $a = h(u)$, $b = h(v)$. By applying compactness of M we obtain from $u \prec_i v$ and $v = \bigvee z$ $(z \in M_i \mid z \vee z^\bullet = e)$, that there exists a complemented $w \in M_i$ with $u \leq w \leq v$. Then $w \prec_i w$ so $c = h(w)$ satisfies $c \lhd_i c$ and $a \leq c \leq b$.

($\Longleftarrow$) Claim. If $A_i = \{c \in L_i \mid c \lhd_i c\}$ and A_0 is generated by $A_1 \cup A_2$, then (A_0, A_1, A_2) is a basic Boolean bilattice of L.

Proof: A_i is a lattice by (SI2). A is Boolean since $c \lhd_i c$ implies $c \prec_i c$, that is, c is complemented with complement in L_k. By (SI6) and our assumption, A_i generates L_i, hence is basic.
Further, any strongly regular ideal $J \in \mathcal{R}_i$ is generated by $J \cap A_i$: $x \in J$ implies $x \lhd_i y \in J$, which gives $x \leq c \lhd_i c \leq y \in J$, and thus $x \leq c \in J \cap A_i$. Conversely, any ideal J generated by $J \cap A_i$ is in $\mathcal{R}_i$: $x \in J$ implies $x \leq y \in J \cap A_i$, so that $x \leq y \lhd_i y \in J$.
Hence $\mathcal{J}A \cong \mathcal{R}$ and $\mathcal{J}A$ is a zero-dimensional compactification. $\square$

4. Least biframe compactifications.

We now turn to the question of determining which biframes have least compactifications. It is well known ([7]) that a topological space has a least (that is, one-point) compactification iff it is locally compact and Hausdorff. To state the corresponding frame and biframe results, we require these definitions ([5]):

For a *frame* N, we say:

- $a \ll b$ iff $b \leq \bigvee X$ for some $X \subseteq N$ implies that $a \leq \bigvee F$ for some finite $F \subseteq X$,
- N is **continuous** iff $a = \bigvee b$ $(b \ll a)$ for all $a \in N$, and
- N is **stably continuous** iff N is continuous and $\ll$ is closed under finite meets (including the top) of N.

In [3] it is shown that a frame has a least compactification iff it is regular and continuous. Further, a zero-dimensional frame has a smallest zero-dimensional compactification iff it is continuous, and that compactification is then the smallest.

We do not have analogous characterizations in the case of biframes, but we do present two partial results, with examples.

Lemma 3. *Let L be a regular biframe such that each L_i is stably continuous and the condition* (SI5) *holds for $\ll_i$ (the relation $\ll$ with respect to the frame L_i). This is a necessary and sufficient condition for $\ll_i$ to be a strong inclusion on L, and it is then necessarily the least.*

PROOF: (SI1) and (SI2) follow from the properties of $\ll$, with stable continuity.

(SI3) comes from the regularity of L.

(SI4) holds since $\ll$ interpolates in a continuous lattice.

(SI5) is postulated explicitly.

(SI6) holds because each L_i is continuous.

Furthermore, if $\lhd_i$ is any strong inclusion on L, then $x \ll_i y = \bigvee z \ (z \lhd_i y)$, so that $x \lhd_i y$. $\square$

Example 1. Let $\mathcal{L}_0$ = all open subsets of the **rational** unit interval E,

$\mathcal{L}_1$ = all open downsets,

$\mathcal{L}_2$ = all open upsets.

In $\mathcal{L}_i$, $U \ll_i V$ iff $U \subset V$ or $U = V = E$ or $U = V = \emptyset$. Hence $\mathcal{L}_i$ is stably continuous (note that $\mathcal{L}_i$ is compact). $\mathcal{L}$ is regular because $U \subset V$ implies $U \prec_i V$; and (SI5) is equally easy to check. So $\mathcal{L}$ has a smallest compactification, by the lemma above. It is given by taking $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$ analogous to $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ for the **real** unit interval. $\mathcal{M}$ is compact, regular, and the restriction map $\mathcal{M} \to \mathcal{L}$ is dense, onto. In $\mathcal{M}_i$, $U \prec_i V$ iff $U \subset V$ iff $U \cap E \subset V \cap E$. Hence the strong inclusion induced by this compactification is the smallest, and $\mathcal{M}$ is the least compactification of L.

Remarks.

- The smallest compactification of $\mathcal{L}$ has many new points.
- $\mathcal{L}$ is zero-dimensional, but its smallest compactification is not.
- $\mathcal{L}_0$ is not continuous, and thus does not have a smallest frame compactification.

Lemma 4. *Let L be a regular biframe in which each L_i is continuous, and $a \prec_i b$ implies $a \ll_i b$ whenever $a < e$. Then L has a unique compactification.*

PROOF: We verify that $\prec_i$ is a strong inclusion on L.

(SI1)–(SI3) always hold.

(SI4) for $a < e$, $a \prec_i b$ iff $a \ll_i b$ (by regularity) and $\ll$ interpolates on a continuous frame.

(SI5) If $a \prec_i b$ there exists $c \in L_i$ with $a \prec_i c \prec_i b$, witnessed by $s, t \in L_k$, $a \wedge s = 0$, $c \vee s = e$, $c \wedge t = 0$, $b \vee t = e$. Then $t \prec_k s$, as required.

(SI6) holds since L_i is continuous.

Now $\prec_i$ is certainly the largest strong inclusion on L. It is also the smallest: if $\lhd_i$ is another, $a < e$, $a \prec_i b$ implies $a \ll_i b$, so that $a \lhd_i b$. $\square$

Example 2. $\mathcal{L}_0$ = all open subsets of the **open** unit interval E,

$\mathcal{L}_1$ = all open downsets,

$\mathcal{L}_2$ = all open upsets.

$\mathcal{L}$ is certainly regular. Also, $U \prec_i V$, $U \neq E$ holds iff $U \subset V$, which implies that $U \ll_i V$, thus $\mathcal{L}_i$ is continuous. So $\mathcal{L}$ has a unique compactification.

Remarks.

- The unique compactification is again the biframe given by the closed unit interval (mentioned in the previous example), with the relevant restriction map.
- The least compactification of $\mathcal{L}$ is not obtained from the least compactification of $\mathcal{L}_0$, since the open unit interval is locally compact, Hausdorff and has a one-point compactification.

Example 3. $\mathcal{L}_0 =$ all subsets of the natural numbers, $\mathbf{N} = \{0, 1, 2, 3, \dots\}$,

$\mathcal{L}_1 =$ all downsets,

$\mathcal{L}_2 =$ all upsets.

$\mathcal{L}$ is a biframe, using $\wedge = \cap$ and $\bigvee = \bigcup$. It is clearly Boolean, and hence regular. $\mathcal{L}_1$ and $\mathcal{L}_2$ are continuous. Also if $U \neq N$, $U \prec_i V$ implies $U \subseteq V$, which implies $U \ll_i V$. Thus $\mathcal{L}_i$ has a unique compactification by the lemma above.
This compactification is given as follows. Let the set $M = \mathbf{N} \cup \{*\}$, where $* \notin \mathbf{N}$ and $n \leq *$ for all $n \in \mathbf{N}$. Let

$\mathcal{M}_1 =$ all downsets of M,

$\mathcal{M}_2 = \{U \cup \{*\} \mid U \text{ is an upset of } \mathbf{N}\}$,

$\mathcal{M}_0$ be generated by $\mathcal{M}_1 \cup \mathcal{M}_2$.

Then $\mathcal{M}$ is compact, zero-dimensional and the restriction map $\mathcal{M} \to \mathcal{L}$ is clearly dense, onto.

The analogous example with $\mathbf{N}$ replaced by $\mathbf{Z}$, the integers, also has a compactification, using two points at infinity instead of one, as above.

Remark. Every compactifiable biframe has a largest compactification, since there exists a compact regular coreflection (see [6]).

References

[1] Banaschewski B., *Compact regular frames and the Sikorski theorem*, Kyungpook Math. J. **28** (1988), 1–14.

[2] ———, *Universal zero-dimensional compactifications*, Categorical Topology and its Relations to Modern Analysis, Algebra and Combinatorics (Prague, 1988), 257–269, World Sci. Publishing, Teaneck, NJ, 1989.

[3] ———, *Compactification of frames*, Math. Nachr. **149** (1990), 105–116.

[4] ———, *Biframe compactifications*, manuscript, 1989.

[5] Banaschewski B., Brümmer G.C.L., *Stably continuous frames*, Math. Proc. Cambr. Phil. Soc. **104** (1988), 7–19.

[6] Banaschewski B., Brümmer G.C.L., Hardie K.A., *Biframes and bispaces*, Quaestiones Math. **6** (1983), 13–25.

[7] Bourbaki N., *General Topology*, Addison Wesley Publishing Company, Reading, Massachussetts, 1966.

[8] Johnstone P.T., *Stone Spaces*, Cambridge Univ. Press, Cambridge, 1982.

[9] Schauerte A., *Biframes*, Ph.D. Thesis, McMaster University, 1992.

Department of Mathematics, University of Cape town, Rondebosch 7700, South Africa

(*Received* December 17, 1992)