On the existence of the price equilibrium by different methods

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Abstract. We have given several proofs on the existence of the price equilibrium — via variational inequality — via degree theory and via Brouwer's theorems.

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Smale [6] considered the price equilibrium of a pure exchange economy and unified the existence, algorithm and dynamic questions of the economy. In this paper we only give some new proofs of his existence theorems and show that the price equilibrium is a fixed point of continuous mapping of the price simplex into itself and indicate how the known algorithms can be used to compute the price equilibrium. The model of the exchange economy under consideration is described by: n commodities; a price system $p = (p_1, p_2, \dots, p_n)$, $p_i \geq 0$, where p_i represents the price of a unit of the i-th commodity; two functions $D : \mathbb{R}^n_+ \setminus \{0\} \to \mathbb{R}^n_+$ and $S : \mathbb{R}^n_+ \setminus \{0\} \to \mathbb{R}^n_+$ respectively called the demand and the supply function where $\mathbb{R}^n_+ = \{x \in \{x_1, x_2, \dots, x_n\} \in \mathbb{R}^n : x_i \geq 0,\ i = 1, 2, \dots, n\}$. The excess demand function $\xi : \mathbb{R}^n_+ \setminus \{0\} \to \mathbb{R}^n$ is defined by $\xi(p) = D(p) - S(p)$, $p \in \mathbb{R}^n_+ \setminus \{0\}$. The price p at which 'supply' equals 'demand', i.e. $D(p) = S(p)$, i.e. $\xi(p) = 0$, is called price (economic) equilibrium. We also use the notation $\xi(p) = (\xi_1(p), \xi_2(p), \dots, \xi_n(p))$ where $\xi_i(p)$ is the i-th coordinate function, i.e. the excess demand for the i-th commodity at the price p.

Let $\Delta_1 = \{p = \{p_1, p_2, \dots, p_n\} \in \mathbb{R}^n : p_i \geq 0 \text{ for } i = 1, 2, \dots, n \text{ and } \sum_{i=1}^n p_i = 1\}$ and $\Delta_0 = \{p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n : \sum_{i=1}^n p_i = 0\}$.

It is natural to impose the following conditions on the excess demand function ξ:

(i) $\xi(\lambda p) = \xi(p)$ for $\lambda \geq 0$ (homogeneity);

(ii) $p \cdot \xi(p) = 0$ for each $p \in \Delta_1$ (Walras law), where $p \cdot q$ denotes the inner product of p and q;

(iii) if $p \in \Delta_1$ and $p_i = 0$, then $\xi_i(p) \geq 0$ (weak boundary condition).

No explicit use of the condition (i) will occur in our works. We first prove that the following results due to Smale [6] are equivalent.

Theorem 1. *If the function $\Delta_1 \to \mathbb{R}^n$ is continuous and satisfies the Walras law and weak boundary condition, then there is a price vector $\bar{p} \in \Delta_1$ such that $\xi(\bar{p}) = 0$.*

Theorem 2. *If $\psi : \Delta_1 \to \Delta_0$ is continuous and satisfies the condition that $\psi_i(p) \geq 0$ if $p_i = 0$, then there is $\bar{p} \in \Delta_1$ such that $\psi(\bar{p}) = 0$. Here $\psi_i(p)$ is the i-th coordinate function of $\psi(\bar{p})$.*

We assume that Theorem 1 is true. Let $\psi : \Delta_1 \to \Delta_0$ be a continuous function satisfying $\psi_i(p) \geq 0$ if $p_i = 0$. We define $\xi : \Delta_1 \to \mathbb{R}^n$ by

$$\xi(p) = \psi(p) - \frac{p}{\|p\|^2}\Big(\sum_{i=1}^{n} \psi_i(p)p_i\Big), \qquad p \in \Delta_1.$$

Then ξ is continuous as $\|p\| > 0$ and $\xi_i(p) = \psi_i(p)$ if $p_i = 0$. Thus ξ satisfies the weak boundary condition. Also $p \cdot \xi(p) = \sum_{i=1}^n p_i\psi_i(p) - \sum_{i=1}^n p_i\psi_i(p) = 0$. Hence ξ satisfies the Walras law. Therefore by Theorem 1 there is a vector $\bar{p} \in \Delta_1$ such that $\xi(\bar{p}) = 0$. Now as $\sum_{i=1}^n \bar{p}_i = 1$ and $\sum_{i=1}^n \psi_i(\bar{p}) = 0$, it follows that

$$0 = \sum_{i=1}^{n} \xi_i(\bar{p}) = \sum_{i=1}^{n} \psi_i(\bar{p}) - \frac{\sum_{i=1}^n \psi_i(\bar{p})\bar{p}_i}{\|\bar{p}\|^2}\,.$$

Hence $\sum_{i=1}^n \psi_i(\bar{p})\bar{p}_i = 0$. Thus $0 = \xi(\bar{p}) = \psi(\bar{p})$ and therefore Theorem 2 is true. Next we assume that Theorem 2 is true and let $\xi : \Delta_1 \to \mathbb{R}^n$ be a continuous mapping satisfying the Walras law and weak boundary condition. We define the mapping $\psi : \Delta_1 \to \Delta_0$ by

$$\psi(p) = \xi(p) - \Big(\sum_{i=1}^{n} \xi_i(p)\Big)\, p, \qquad p \in \Delta_1.$$

Then ψ is continuous, $\sum_{i=1}^n \psi_i(p) = \sum_{i=1}^n \xi_i(p) - (\sum_{i=1}^n \xi_i(p)) \sum_{i=1}^n p_i = 0$ and if $p_i = 0$, $\psi_i(p) = \xi_i(p) \geq 0$. Thus by Theorem 2, there is a vector $\bar{p} \in \Delta_1$ such that $\psi(\bar{p}) = 0$. Hence $0 = \bar{p} \cdot \psi(\bar{p}) = \bar{p} \cdot \xi(\bar{p}) - (\sum_{i=1}^n \xi_i(\bar{p}))\|\bar{p}\|^2$ and hence by virtue of Walras law, $\sum_{i=1}^n \xi_i(\bar{p}) = 0$. Thus $\psi(\bar{p}) = \xi(\bar{p}) = 0$. We have, therefore, proved that Theorem 1 holds.

Proof of Theorem 1 via variational inequality.

We now first give an independent proof of Theorem 1 by using the variational inequality due to Hartman and Stampacchia [4, Lemma 3.1] which we give here as a lemma below. For more general results on variational inequality we refer to Browder [3].

We note that Aliprantis and Brown [1] used inequality to prove the existence of price equilibrium in the Riesz space under boundary condition different from that of Smale (see also Border [2]).

Lemma 1 (Hartman and Stampacchia)**.** *If K is a compact convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}^n$ is a continuous mapping, then there exists a vector $\bar{p} \in K$ such that $\bar{p} \cdot f(\bar{p}) \geq p \cdot f(\bar{p})$ for all $p \in K$. Such $\bar{p} \in K$ is called a solution of the variational inequality.*

Theorem 1′. *If $\xi : \mathbb{R}^n_+ \setminus \{0\} \to \mathbb{R}^n$ is a continuous mapping satisfying the Walras law and weak boundary condition, then there is a vector $\bar{p} \in \Delta_1$ such that $\xi(\bar{p}) = 0$.*

Proof of Theorem 1′: Let $\bar{\xi}$ denote the restriction of ξ to Δ_1. Then by Lemma 1 there is a vector $\bar{p} \in \Delta_1$ such that $\bar{p} \cdot \bar{\xi}(\bar{p}) \geq p \cdot \bar{\xi}(\bar{p})$ for every $p \in \Delta_1$. We will prove that each such $\bar{p}$ is a price equilibrium, i.e. $\xi(\bar{p}) = 0$. By Walras law it follows that $p \cdot \bar{\xi}(\bar{p}) \leq 0$ for every $p \in \Delta_1$. Now since for each $i = 1, 2, \ldots, n$, $e_i \in \Delta_1$ where $e_i = \{\delta_{i,j}\}_{j=1}^n$ and

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, \end{cases}$$

we have that for each $i = 1, 2, \ldots, n$, $e_i \cdot \bar{\xi}(\bar{p}) \leq 0$, i.e. $\bar{\xi}_i(\bar{p}) \leq 0$. Let $J = \{i \in \{1, 2, \ldots, n\} : \bar{p}_i = 0\}$ where $\bar{p}_i$ is the i-th coordinate of $\bar{p}$. Then by weak boundary condition $\bar{\xi}_i(\bar{p}) \geq 0$ for each $i \in J$. Hence for each $i \in J$, $\bar{\xi}_i(\bar{p}) = 0$. Let $P = \{i \notin J : \bar{\xi}_i(\bar{p}) \neq 0\}$, i.e. $P = \{i \notin J : \bar{\xi}_i(\bar{p}) < 0\}$. We complete the proof by showing that $P = \emptyset$. If possible, let $P \neq \emptyset$. Then $\sum_{i \in P} \bar{p}_i \cdot \bar{\xi}_i(\bar{p}) = l < 0$. Thus $\sum_{i=1}^n \bar{p}_i \bar{\xi}_i(\bar{p}) = \sum_{i \notin J} \bar{p}_i \cdot \bar{\xi}_i(\bar{p}) = l < 0$, which contradicts the Walras law. Hence $P = \emptyset$. □

Proof of Theorem 2 via degree theory.

Let $m = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \in \mathbb{R}^n$ and $\tilde{\Delta}_1 = \{q \in \Delta_0 : q = p - m \text{ for some } p \in \Delta_1\}$. We now define a mapping $\phi : \tilde{\Delta}_1 \to \Delta_0$ by $\phi(q) = \psi(p)$ where $q = p - m$. Then clearly ϕ is continuous. Let $H : [0,1] \times \tilde{\Delta}_1$ be the continuous mapping defined by $H(t,x) = -t\phi(x) + (1-t)x$, $t \in [0,1]$ and $x \in \tilde{\Delta}_1$. Now Δ_0 isomorphic (linear homeomorphic) to $\mathbb{R}^{n-1}$ (to see this it suffices to note that $\Delta_0 = f^{-1}\{0\}$ where $f : \mathbb{R}^n \to \mathbb{R}$ is the linear functional defined by $f(p) = \sum_{i=1}^n p_i$, $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$) and under this isomorphism $\partial\tilde{\Delta}_1 =$ boundary of $\tilde{\Delta}_1 = \{p = (p_1, p_2, \ldots, p_n) \in \tilde{\Delta}_1 : p_i = -\frac{1}{n}$ for at least one $i = 1, 2, \ldots, n\}$ since $\tilde{\Delta}_1 = \{p = (p_1, p_2, \ldots, p_n) \in \Delta_0 : p_i \geq -\frac{1}{n}$ for all $i = 1, 2, \ldots, n\}$.

Now if $x = (x_1, x_2, \ldots, x_n) \in \partial\tilde{\Delta}_1$, then $x_i = -\frac{1}{n}$ for some $i = 1, 2, \ldots, n$. This implies $-t\phi_i(x) \leq 0$, which in turn implies $H_i(t,x) \leq (1-t)(-\frac{1}{n})$. Thus $H(t,x) = 0$ implies $t = 1$ and $\phi(x) = 0$ and in this case we have the required solution $\psi(x+m) = 0$. If $H(t,x) \neq 0$ for all $(t,x) \in [0,1] \times \partial\tilde{\Delta}_1$, then by the homotopy invariance of the Brouwer's degree

$$d(-\phi, \tilde{\Delta}_1^0, 0) = d(I, \tilde{\Delta}_1^0, 0) = 1$$

and hence $-\phi(x) = 0$ has a solution in $\tilde{\Delta}_1^0$ where $\tilde{\Delta}_1^0$ is the interior of $\tilde{\Delta}_1$ and d denotes the Brouwer's degree. Thus in this case we have the required solution $\psi(x+m) = 0$. □

Proof of Theorem 2 by Brouwer's fixed point theorem.

For each $i = 1, 2, \ldots, n$ let us first define the function $K_i : \mathbb{R}^n \to \mathbb{R}$ by $K_i(p) = \max\{0, p_i\}$, $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$. Let $K : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $K(p) =$

$(K_1(p), K_2(p), \ldots, K_n(p))$. Then K is continuous. Now let us define $T : \Delta_1 \to \Delta_1$ by

$$T(p) = \frac{K(p+\psi(p))}{\sum_{i=1}^{n} K_i(p+\psi(p))}, \qquad p \in \Delta_1.$$

As $\sum_{i=1}^{n} K_i(p+\psi(p)) \geq \sum_{i=1}^{n}(p_i+\psi_i(p)) = 1$ and K and ψ are continuous, it follows that T is continuous and clearly $T(\Delta_1) \subset \Delta_1$. Hence by Brouwer's fixed point theorem there exists a point $\bar{p} \in \Delta_1$ such that $\bar{p} = T(\bar{p})$. Thus it suffices to show that $T(\bar{p}) = \bar{p}+\psi(\bar{p})$. If $0 < \bar{p}_i = T_i(\bar{p})$, then $K_i(\bar{p}+\psi(\bar{p})) > 0$ as $\sum_{i=1}^{n} K_i(\bar{p}+\psi(\bar{p})) \geq 1$. Thus $K_i(\bar{p}+\psi(\bar{p})) = \bar{p}_i+\psi(\bar{p})$. If $0 = \bar{p}_i = T_i(\bar{p})$, then by given condition $\psi_i(\bar{p}) \geq 0$, hence $\bar{p}_i + \psi(\bar{p}) \geq 0$. This implies that $K_i(\bar{p} + \psi(\bar{p})) = \bar{p}_i + \psi(\bar{p})$. Thus we always have $\sum_{i=1}^{n} K_i(\bar{p} + \psi(\bar{p})) = \sum_{i=1}^{n}(\bar{p}_i + \psi(\bar{p})) = 1$ and

$$\bar{p} = T(\bar{p}) = \frac{K(\bar{p}+\psi(\bar{p}))}{\sum_{i=1}^{n} K_i(\bar{p}+\psi(\bar{p}))} = \bar{p} + \psi(\bar{p}).$$

Hence $\psi(\bar{p}) = 0$. □

Remark on the computation of the price equilibrium.

Given a continuous demand function $\xi : \mathbb{R}^n_+ \setminus \{0\} \to \mathbb{R}^n$ we define the continuous mapping $\psi : \Delta_1 \to \Delta_0$ by

$$\psi(p) = \xi(p) - \Bigl(\sum_{i=1}^{n} \xi_i(p)\Bigr)\, p, \qquad p \in \Delta_1.$$

Finally we define the continuous mapping $T : \Delta_1 \to \Delta_1$ by

$$T(p) = \frac{K(p+\psi(p))}{\sum_{i=1}^{n} K_i(p+\psi(p))}, \qquad p \in \Delta_1.$$

Then by what has been done above, it follows that every fixed point of T is a price equilibrium of the economy for which ξ is an excess demand function. The Brouwer's fixed point theorem guarantees the existence of a fixed point theorem which is the price equilibrium. Beside the pioneering work of Scarf [5], several algorithms (see Todd [7]) by which fixed points of T can be computed are known.

Without contradicting the concluding remark in Smale [6] we would like to point out that out of the four existence proofs including the Smale's proof based on Sard implicit function theorem it appears to us that the proof by Brouwer's fixed point theorem is easier and more natural.

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