An example in the theory of approximate systems

N. Uglešić

Abstract. An approximate inverse sequence of plane continua is constructed which negatively answers a question of S. Mardešić related to approximate and usual inverse systems. The example also shows that an important result of M.G. Charalambous cannot be improved. As an application, it is shown that a procedure of making an approximate inverse sequence commutative ("taming") is discontinuous.

Keywords: inverse systems, (gauged) approximate systems, inverse limits, (gauged) approximate resolutions, compact metric spaces

Classification: 54B25

1. Introduction.

S. Mardešić and T. Watanabe [5] have developed a theory of (gauged) approximate resolutions of spaces and mappings. It allows a successful study of a very broad class of topologically complete spaces using the techniques of developing spaces into (gauged) inverse systems and resolutions. The key idea was to replace the rigid commutativity condition for inverse systems with a controlled nearness of bonding mappings. It was done (for metric compacta) a little earlier by S. Mardešić and L.R. Rubin [3]. That theory is based on three conditions, where two of them require prescribing (in advance) normal coverings — meshes. Recently it has been shown that only the "free" condition is essential ([1] partially for objects, [2] completely for objects, [6] completely for objects and mappings). However, it seems that the other two "gauging" conditions assure a useful and unavoidable technique required for that theory ([6]).

From the very beginning of the study of approximate systems, the question of possibility of their transformation (especially for sequences) into usual (commutative) ones arises. A partial answer for sequences of complete metric spaces is given by M.G. Charalambous [1].

It is shown here that such a transformation is generally impossible even for sequences of plane continua with surjective bonding mappings. Of course, it is an affirmative result for the mentioned theory, for it proves the approximate systems (even on the level of compact metric spaces) are something essentially new related to usual inverse systems.

For the sake of completeness, let us briefly recall the main definitions from [5].

1.1. A (gauged) approximate (inverse) system is a collection $\mathscr{X} = (X_a, \mathscr{U}_a, p_{aa'}, A)$ consisting of:

– a preordered set $A = (A, <)$ which is directed and unbounded;

– for each $a \in A$, a (topological) space X_a and a normal covering (mesh) $\mathscr{U}_a$ of X_a;
– for each related pair $a < a'$ in A, a (continuous) mapping $p_{aa'} : X_{a'} \to X_a$ ($p_{aa} = 1_{X_a}$ is the identity mapping on X_a).

These data are to satisfy the following three conditions:

(A1) $(p_{aa'}p_{a'a''}, p_{aa''}) < \mathscr{U}_a$ whenever $a < a' < a''$;
(A2) $(\forall\, a \in A)$ $(\forall\, \mathscr{U} \in \mathscr{C}ov\,(X_a))$ $(\exists\, a' > a)$
$(\forall\, a_2 > a_1 > a')$ $(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < \mathscr{U}$;
(A3) $(\forall\, a \in A)$ $(\forall\, \mathscr{U} \in \mathscr{C}ov\,(X_a))$ $(\exists\, a' > a)$ $(\forall\, a'' > a')$ $\mathscr{U}_{a''} < p_{aa''}^{-1}\mathscr{U}$.

Here, for any two mappings $f, g : X \to Y$ and any covering $\mathscr{V}$ of Y, $(f, g) < \mathscr{V}$ means that for every $x \in X$ there exists a $V \in \mathscr{V}$ such that $f(x)$ and $g(x)$ belong to V. Instead of $(f, g) < \mathscr{V}$ we will often write $f =_{\mathscr{V}} g$. For coverings $\mathscr{U}$, $\mathscr{U}'$ of X, $\mathscr{U}' < \mathscr{U}$ means that $\mathscr{U}'$ refines $\mathscr{U}$.

A normal covering of a space X is any open covering of X which admits a subordinate partition of unity. Therefore, normal coverings coincide with numerable open coverings. The set of all normal coverings of X is denoted by $\mathscr{C}ov\,(X)$. If an $X' \subseteq X$ and a covering $\mathscr{U}$ of X are given, then the star of X' with respect to $\mathscr{U}$ is the set

$$st(X', \mathscr{U}) = \bigcup \{U \in \mathscr{U} \mid X' \cap U \neq \emptyset\} \subseteq X.$$

Recall that every $\mathscr{U} \in \mathscr{C}ov\,(X)$ admits a $\mathscr{U}' \in \mathscr{C}ov\,(X)$ such that $st\mathscr{U}' = \{st(U', \mathscr{U}) \mid U' \in \mathscr{U}'\}$ belongs to $\mathscr{C}ov\,(X)$ and $st\mathscr{U}' < \mathscr{U}$. We inductively define $st^0\mathscr{U}' = \mathscr{U}', st^1\mathscr{U}' = st\mathscr{U}', \ldots, st^n\mathscr{U}' = \{st(U, \mathscr{U}') \mid U \in st^{n-1}\mathscr{U}'\}$, $n \in \mathbb{N}$, which all belong to $\mathscr{C}ov\,(X)$. The above definition may be written as $st^n\mathscr{U}' = st(st^{n-1}\mathscr{U}', \mathscr{U}')$. (Often $st^n\mathscr{U}'$ is incorrectly defined by $st^n\mathscr{U}' = st(st^{n-1}\mathscr{U}')$!)

1.2. A (gauged) approximate system $\mathscr{X}$ is called uniform provided the condition

(AU) $\quad \mathscr{U}_{a'} < p_{aa'}^{-1}\mathscr{U}_a, \quad a < a'$,

is satisfied.

1.3. A (gauged) approximate map q from a space Y into a (gauged) approximate system $\mathscr{X}$, $q : Y \to \mathscr{X}$, is any collection $q = \{q_a \mid a \in A\} = (q_a)$ of mappings $q_a : Y \to X_a$ such that

(AS) For every $a \in A$ and every $\mathscr{U} \in \mathscr{C}ov\,(X_a)$ there exists an $a' > a$ so that $(q_a, p_{aa''}q_{a''}) < \mathscr{U}$ whenever $a'' > a'$.

1.4. A (gauged) approximate map $p = (p_a) : X \to \mathscr{X}$ is called a limit of $\mathscr{X}$ provided it has the following universal property:

(UL) For any approximate map $q : Y \to \mathscr{X}$ there exists a unique mapping $g : Y \to X$ satisfying $p_a g = q_a$ for every $a \in A$.

Since a limit space X is determined up to a unique homeomorphism, we often speak of the limit X of $\mathscr{X}$ and we write $X = \lim \mathscr{X}$.

It is natural to take Theorem 2.8 from [5] as the definition of a (gauged) approximate resolution of a space.

1.5. An approximate resolution of a space X is an approximate map $p : X \to \mathscr{X}$ satisfying the following two conditions:

(B1) $(\forall\, \mathscr{U} \in \mathscr{C}ov\,(X))$ $(\exists\, a \in A)$ $(\forall\, a' > a)$ $p_{a'}^{-1}\mathscr{U}_{a'} < \mathscr{U}$;

(B2) $(\forall\, a \in A)$ $(\exists\, a' > a)$ $(\forall\, a'' > a')$ $p_{aa''}(X_{a''}) \subseteq st(p_a(X), \mathscr{U}_a)$.

A (gauged) approximate system $\mathscr{X}$ is said to be a (gauged) approximate resolution provided there exist a topologically complete space X and a (gauged) approximate resolution $p : X \to \mathscr{X}$ of X.

2. Example.

In the remaining part of this paper we consider only (gauged) approximate inverse sequences, especially, such resolutions. The following question was stated by S. Mardešić:

2.1 Question. Let $\mathrm{X} = (X_n, p_{nn'}, \mathbb{N})$ $(\mathscr{X} = (X_n, \mathscr{U}_n, p_{nn'}, \mathbb{N}))$ be an approximate (gauged) inverse sequence with limit $\lim \mathrm{X} = X$ $(\lim \mathscr{X} = X)$. Let $\underline{X} = (X_n, p_{n,n+1}, \mathbb{N})$ be the usual (commutative) inverse sequence obtained by replacing in X ($\mathscr{X}$) each $p_{nn'}$, $n' - n > 1$, with the composition $p_{n,n+1} \circ \cdots \circ p_{n'-1,n'}$. Is $\lim \underline{X} \approx X$?

The following result of M.G. Charalambous ([1, Proposition 8]) directly relates the above question:

2.2. Let $\mathrm{X} = (X_n, p_{nn'}, \mathbb{N})$ be an approximate sequence of complete metric spaces with the limit $\lim \mathrm{X} = X$. Then X is uniformly isomorphic to the limit $\lim \underline{X}'$, where $\underline{X}' = (X_m, p'_{mm'}, M)$ is the usual inverse sequence over a cofinal subset $M \subseteq \mathbb{N}$ and $p'_{mm'} = p_{mm'}$ whenever m' is an immediate successor of m in M.

Although that result encourages to answer the question affirmatively, we shall show that generally it is not the case. Namely, a cofinal $M \subseteq \mathbb{N}$ cannot be replaced by $\mathbb{N}$ even in the case of plane metric continua (and surjective bonding mappings; see 2.6 below). Our counterexample will be properly adapted from Example 1 of [4].

2.3 Example. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the standard unit circle based at $z = (1,0)$. Let $\bigvee_{k=1}^{\infty} S_k$ be the Hawaiian earring (i.e. the wedge of a sequence of copies S_k, $k \in \mathbb{N}$, of S^1) embedded into the product space $\prod_{k=1}^{\infty} S_k$ with the usual product metric. Then $\bigvee_{k=1}^{\infty} S_k$ is homeomorphic to $X \subseteq \mathbb{R}^2$, $X = \bigcup_{k=1}^{\infty} C_k$, where

$$C_k = \{x = (\xi, \eta) \in \mathbb{R}^2 \mid (\xi - (2k-1)/2k)^2 + \eta^2 = 1/4k^2\}.$$

For every $k \in \mathbb{N}$ choose the homeomorphism $h_k : C_k \to S_1$ defined by the radial projection from the center point $((2k-1)/2k, 0)$ of the circle C_k. Let us define the sequence of mappings $\varphi_n : X \to X$, $n \in \mathbb{N}$, by putting

$$\varphi_n(x) = \begin{cases} x, \ x \in C_k \ \text{ and } \ k \neq n+1 \\ h_n^{-1}(h_{n+1}(x)^2), \ x \in C_{n+1}\,. \end{cases}$$

Observe that, for every $n \in \mathbb{N}$ and every $x \in C_{n+1}$, $x, \varphi_n(x) \in C_n \cup C_{n+1}$. Take now $X_n = X$ for all n and define $p_{nn'} : X_{n'} \to X_n$, $n' \geq n$, by

$$p_{nn'} = \begin{cases} 1_X, & n' - n \neq 1 \\ \varphi_n, & n' - n = 1. \end{cases}$$

Further, let $\mathbb{N}_n = \{k \in \mathbb{N} \mid k \geq n\}$ and $H_n = \bigcup_{k\in\mathbb{N}_n} C_k \subseteq X_n$. Note that $x, \varphi_n(x) \in H_n$ for every $x \in H_{n+1}$. If $n = 1$, choose $\mathscr{U}_1 = \{H_1 = X_1\}$. If $n > 1$, consider for each $k \in \{1,\dots,n-1\}$ all counterclockwise ordered n-tuples $(x_1^k(n),\dots,x_n^k(n))$ of points $x_j^k(n) \in C_k$, $x_j^k(n) \neq x_0 = (1,0)$, $j = 1,\dots,n$, such that each n-tuple determines n arcs on C_k of the same length. Obviously, for each $(x_1^k(n),\dots,x_n^k(n))$, only one arc, say $\langle x_n^k(n), x_1^k(n)\rangle$, contains the point x_0. Denote by $\mathscr{L}_n^0$ the set of all such open arcs $L_n^k(x_0)$ on all C_k containing x_0, and by $\mathscr{L}_n$ the set of all remaining such open arcs L_n^k on all C_k, $k = 1,\dots,n-1$. Note that for every $L_{n+1}^k(x_0)$ (L_{n+1}^k) there exists an $L_n^k(x_0)$ (L_n^k) such that $L_{n+1}^k(x_0) \subseteq L_n^k(x_0)$ ($L_{n+1}^k \subseteq L_n^k$). Take now

$$\mathscr{U}_n = \Big\{H_n \cup \Big(\bigcup_{k=1}^{n-1} L_n^k(x_0)\Big) \mid L_n^k(x_0) \in \mathscr{L}_n^0\Big\} \cup \mathscr{L}_n, \quad n > 1.$$

Thus, $\mathscr{U}_n \in \mathscr{C}ov\,(X_n)$ for every n. In this way we have constructed the collection $\mathscr{X} = (X_n, \mathscr{U}_n, p_{nn'}, \mathbb{N})$. Note that $(\mathscr{U}_n)$ is a decreasing sequence in $(\mathscr{C}ov\,(X), <)$. Furthermore, if $\delta_n = \sup\{\operatorname{diam} U \mid U \in \mathscr{U}_n\}$, then $\delta_1 = 1$ and $\delta_n = \operatorname{diam}(H_n \cup L_n^1(x_0)) = \operatorname{diam} L_n^1 = \sin(\pi/n)$, $n > 1$. Therefore, (δ_n) is a decreasing sequence of positive numbers converging to zero.

Lemma. *$\mathscr{X} = (X_n, \mathscr{U}_n, p_{nn'}, \mathbb{N})$ is a uniform gauged approximate system.*

PROOF: We are verifying the condition (A1), (A2), (A3) and (AU).

(A1) Let any $n \leq n' \leq n''$ in $\mathbb{N}$ be given. Among potentially nine essentially various cases, only the following three are non-trivial:

$$(p_{nn''}, p_{nn'}p_{n'n''}) = \begin{cases} (1, \varphi_n\varphi_{n+1}), & n'' = n'+1 > n' = n+1 \\ (1, \varphi_n), & n'' > n'+1 > n' = n+1 \\ (1, \varphi_{n'}), & n'' = n'+1 > n' > n+1. \end{cases}$$

In the first case, only the points of $(C_{n+1} \cup C_{n+2}) \setminus \{x_0\} \subseteq H_{n+2}$ are moving. Because of $\varphi_n\varphi_{n+1}(C_{n+1} \cup C_{n+2}) = C_n$, $1(C_{n+1} \cup C_{n+2}) = C_{n+1} \cup C_{n+2}$ and $C_n \cup C_{n+1} \cup C_{n+2} \subseteq H_n \subseteq U \in \mathscr{U}_n$, the condition (A1) for $\mathscr{X}$ is satisfied. In the second (third) case, only the points of $C_{n+1} \setminus \{x_0\}$ ($C_{n'+1} \setminus \{x_0\}$) are moving. Thus the same argument holds.

(A2) Let any $n \in \mathbb{N}$ and $\mathscr{U} \in \mathscr{C}ov\,(X_n)$ be given. Let $\delta > 0$ be a Lebesgue number of $\mathscr{U}$. Choose an $n_0 \in \mathbb{N}$ such that $\sin(\pi/n_0) \leq \delta$. Then $\mathscr{U}_{n'} < \mathscr{U}$ whenever $n' \geq n_0$. Take now $n' = \max\{n_0, n+2\}$, and let any $n_2 \geq n_1 \geq n'$ be given. Observe that $p_{nn_1} = p_{nn_2} = 1$, hence

$$(p_{nn_2}, p_{nn_1}p_{n_1n_2}) = \begin{cases} (1,1), & n_2 - n_1 \neq 1 \\ (1, \varphi_{n_1}), & n_2 - n_1 = 1. \end{cases}$$

Only the second case restricted to $C_{n_1+1} \setminus \{x_0\} \subseteq H_{n_1+1}$ is non-trivial. Because of $\varphi_{n_1}(C_{n_1+1}) = C_{n_1}$, $1(C_{n_1+1}) = C_{n_1+1}$ and $C_{n_1} \cup C_{n_1+1} \subseteq H_{n_1} \subseteq U \in \mathscr{U}_{n_1}$,

$$(1, \varphi_{n_1}) < \mathscr{U}_{n_1} < \mathscr{U}_{n'} < \mathscr{U}$$

holds true. This verifies the condition (A2) for $\mathscr{X}$.

(A3) Let any $n \in \mathbb{N}$ and $\mathscr{U} \in \mathscr{C}ov\,(X_n)$ be given. Take a Lebesgue number $\delta > 0$ of $\mathscr{U}$. Choose an $n_0 \in \mathbb{N}$ satisfying $\sin(\pi/n_0) \leq \delta$. Then $\mathscr{U}_{n'} < \mathscr{U}$ whenever $n' \geq n_0$. Take now $n' = \max\{n_0, n+2\}$, and let any $n'' \geq n'$ be given. Then $p_{nn''} = 1$, hence

$$\mathscr{U}_{n''} < \mathscr{U}_{n'} < \mathscr{U} = p_{nn''}^{-1}\mathscr{U}$$

which establishes the condition (A3) for $\mathscr{X}$.

(AU) Let any $n \leq n'$ in $\mathbb{N}$ be given. If $n' = n$ or $n' > n+1$, then $p_{nn'} = 1$. Since $(\mathscr{U}_n)$ decreases, the condition (AU) in those cases is trivially fulfilled. Let $n' = n+1$ and let any $U \in \mathscr{U}_{n+1}$ be given. If $U \cap C_{n+1} = \emptyset$ then $U = L^k_{n+1} \in \mathscr{L}_{n+1}$ for some $k \in \{1, \dots, n\}$. Therefore,

$$p_{n,n+1}(U) = \varphi_n(L^k_{n+1}) = L^k_{n+1} \subseteq \begin{cases} H_1 = X_1 \in \mathscr{U}_1, & n = 1 \\ L^k_n \in \mathscr{L}_n \subseteq \mathscr{U}_n, & n > 1 \end{cases}.$$

If $U \cap C_{n+1} \neq \emptyset$ then $U = H_{n+1} \cup (\bigcup_{k=1}^{n} L^k_{n+1}(x_0))$ for some $L^k_{n+1}(x_0) \in \mathscr{L}^0_{n+1}$, $k = 1, \dots, n$. Consequently,

$$p_{n,n+1}(U) = \varphi_n(U) \subseteq \begin{cases} H_1 = X_1 \in \mathscr{U}_1, & n = 1 \\ H_n \cup \left(\bigcup_{k=1}^{n-1} L^k_{n+1}(x_0)\right) \subseteq H_n \cup \left(\bigcup_{k=1}^{n-1} L^k_n(x_0)\right) \in \mathscr{U}_n, & n > 1. \end{cases}$$

This verifies the condition (AU) for $\mathscr{X}$ and finally proves the lemma. □

2.4. Take $p_n = 1 : X \to X = X_n$, $n \in \mathbb{N}$. Then $p = (p_n) : X \to \mathscr{X}$ is an approximate resolution of X. Indeed, one trivially checks the conditions (AS), (B1)* and (B2)* for p (see [5, § 2]). Since each $X_n = X$ is a metric compactum, p is a limit, i.e. $X = \lim \mathscr{X}$ (see [5, 4.2] and [2, Remark 3]).

2.5. Let $\underline{X} = (X_n, p_{n,n+1}, \mathbb{N})$ be the usual inverse sequence associated with $\mathscr{X}$, i.e. $(p_{n,n+1} = \varphi_n)$

$$p_{nn'} = \begin{cases} 1, & n' = n \\ \varphi_n \circ \cdots \circ \varphi_{n'-1}, & n' > n \end{cases}.$$

As in Example 1 from [4], the limit space $\lim \underline{X}$ contains a diadic solenoid. Therefore, $\lim \underline{X}$ cannot be isopmorphic to $X = \lim \mathscr{X}$. This negatively answers the above stated question.

2.6 Remark. Note that the bonding mappings $p_{n,n+1} = \varphi_n$ of $\mathscr{X}$ are not surjective. Slightly modifying φ_n into $\varphi'_n : X \to X$, $n \in \mathbb{N}$,

$$\varphi'_n(x) = \begin{cases} x, & x \in C_k \text{ and } k < n+1 \\ h_n^{-1}(h_{n+1}(x)^2), & x \in C_{n+1} \\ h_{k-1}^{-1} h_k(x), & x \in C_k \text{ and } k > n+1 \end{cases}$$

and defining $p'_{nn'} : X_{n'} \to X_n$ by means of φ'_n and 1 (quite analogously to $p_{nn'}$ by φ_n and 1), one obtains the uniform gauged approximate inverse sequence $\mathscr{X}' = (X_n, \mathscr{U}_n, p'_{nn'}, \mathbb{N})$ with all bonding mappings surjective, which also negatively answers the question.

3. An application — “taming” of approximate sequences is discontinuous.

3.1. We will construct a procedure of transforming of a (gauged) approximate sequence into the corresponding commutative inverse sequence keeping at each step the same limit space. Even more, for (gauged) approximate resolutions, the isomorphism class of a starting resolution will be preserved throughout the whole procedure. However, the commutative “limit” resolution will be, in general, out of that class. (This result should be again treated affirmatively as well as the previous one. An analogy with that one can find in the homotopy theory, where an expansion of homotopy equivalent spaces may yield the homotopy non-equivalent “limit” space. That was the origin for the shape theory!)

3.2. Let $\mathscr{X} = (X_n, \mathscr{U}_n, p_{nn'}, \mathbb{N})$ be a gauged approximate sequence. For each $k \in \mathbb{N}_0 = (\mathbb{N} \cup \{0\}, \leq)$, define the collection $\mathscr{X}^k = (X_n, \mathscr{U}_n^k, p_{nn'}^k, \mathbb{N})$ by putting

$$\mathscr{U}_n^k = \begin{cases} \mathscr{U}_n, & k = 0 \ \text{ or } \ k > 0 \ \text{ and } \ n > k \\ st^{k-n+1}\mathscr{U}_n, & k > 0 \ \text{ and } \ 1 \leq n \leq k \end{cases}$$

and

$$p_{nn'}^k = \begin{cases} p_{nn'}, & k = 0 \ \text{ or } \ n' - n \leq 1 \ \text{ or } \ n' > k+2 \\ p_{n,n+1} \circ \cdots \circ p_{k+1,n'}, & k > 0, 1 \leq n \leq k \ \text{ and } \ n+1 < n' \leq k+2 \end{cases}.$$

Observe the following: $\mathscr{X}^0 = \mathscr{X}$;
$\mathscr{X}^1$ is obtained by replacing of (in $\mathscr{X}^0$) $p_{13}^0 = p_{13}$ with $p_{13}^1 = p_{12}^0 p_{23}^0 = p_{12}p_{23}$, and $\mathscr{U}_1^0 = \mathscr{U}_1$ with $\mathscr{U}_1^1 = st\mathscr{U}_1$;
$\mathscr{X}^2$ is obtained by replacing of (in $\mathscr{X}^1$) p_{14}^1 with $p_{14}^2 = p_{13}^1 p_{34}^1 = p_{12}p_{23}p_{34}$, p_{24}^2 with $p_{24}^2 = p_{23}^1 p_{34}^1 = p_{23}p_{34}$, $\mathscr{U}_1^1$ with $\mathscr{U}_1^2 = st^2\mathscr{U}_1$ and $\mathscr{U}_2^1 = \mathscr{U}_2$ with $\mathscr{U}_2^2 = st\mathscr{U}_2$;
$\mathscr{X}^3$ is obtained by replacing of (in $\mathscr{X}^2$) p_{15}^2 with $p_{15}^3 = p_{14}^2 p_{45}^2 = p_{12}p_{23}p_{34}p_{45}$, p_{25}^2 with $p_{25}^3 = p_{24}^2 p_{45}^2 = p_{23}p_{34}p_{45}$, p_{35}^2 with $p_{35}^3 = p_{34}^2 p_{45}^2 = p_{34}p_{45}$, $\mathscr{U}_1^2$ with $\mathscr{U}_1^3 = st^3\mathscr{U}_1$, $\mathscr{U}_2^2$ with $\mathscr{U}_2^3 = st^2\mathscr{U}_2$ and $\mathscr{U}_3^2 = \mathscr{U}_3$ with $\mathscr{U}_3^3 = st\mathscr{U}_3$; etc.
Consequently, the “limit” object of the procedure from above is the collection $\mathscr{X}^\infty = (X_n, \mathscr{U}_n^\infty, p_{nn'}^\infty, \mathbb{N})$, which is the corresponding commutative inverse sequence $\underline{X} = (X_n, p_{n,n+1}, \mathbb{N})$ of $\mathscr{X}$. Indeed,

$$p_{nn'}^\infty = \begin{cases} p_{nn} = 1, & n' = n \\ p_{n,n+1} \circ \cdots \circ p_{n'-1,n'}, & n' > n \end{cases}$$

holds by the construction. Note that, for each $k \in \mathbb{N}_0$, $\mathscr{X}^k$ is a gauged approximate system and that $\mathscr{X}$ and $\mathscr{X}^k$ share the common cofinal subsequence $\mathscr{Y}_k = (X_m, \mathscr{U}_m, p_{mm'}, M_k)$, where $M_k = (\mathbb{N}_{k+1}, \leq)$. Therefore, $\lim \mathscr{X}^k \approx \lim \mathscr{Y}_k \approx \lim \mathscr{X}$ ([5, 1.18]). But, as our previous example shows, in general $\lim \underline{X} \not\approx \lim \mathscr{X}$.

3.3. In the case of an approximate sequence that is an approximate resolution, the following holds true:

Claim. *If a starting approximate sequence* $\mathscr{X}$ *belongs to the category* APRES, *then* $\mathscr{X} = \mathscr{X}^0 \cong \mathscr{X}^1 \cong \cdots \cong \mathscr{X}^k \cong \ldots$ *However, generally,* $\mathscr{X} \ncong \underline{X}$ *in* APRES.

In order to prove it, observe that $\mathscr{Y}_k \in Ob\,(\text{APRES})$, $\mathscr{Y}_k \cong \mathscr{X}$ and $\mathscr{Y}_k \cong \mathscr{X}^k$ in APRES (see [5, 8.12] and [7, 1.5]). Hence, $\mathscr{X}^k \cong \mathscr{X}$ for all $k \in \mathbb{N}_0$. Now, if we provide $\underline{X}$ with gauges (see [2, Theorem 1] or [6, 2.3]) such that it becomes an approximate resolution $\mathscr{X}^*$, it may occur $\mathscr{X}^* \ncong \mathscr{X}$ in APRES. Namely, $\mathscr{X} \cong \mathscr{X}^*$ implies $\lim \mathscr{X} \approx \lim \mathscr{X}^* = \lim \underline{X}$ which contradicts our example.

References

[1] Charalambous M.G., *Approximate inverse systems of uniform spaces and an application of inverse systems*, Comment. Math. Univ. Carolinae **32** (1991), 551–565.

[2] Mardešić S., *On approximate inverse systems and resolutions*, to appear in Fund. Math..

[3] Mardešić S., Rubin L.R., *Approximate inverse systems of compacta and covering dimension*, Pacific J. Math. **138** (1989), 129–144.

[4] Mardešić S., Segal J., *Stability of almost commutative inverse systems of compacta*, Topology of Appl. **31** (1989), 285–299.

[5] Mardešić S., Watanabe T., *Approximate resolutions of spaces and mappings*, Glasnik Mat. **24** (44) (1989), 587–637.

[6] Matijević V., Uglešić N., *A new approach to the theory of approximate resolutions*, to appear.

[7] Uglešić N., *Isomorphisms in the category of approximate systems*, to appear in Glasnik Mat.

Department of Mathematics, Teslina 12/III, 58000 Split, Croatia

(*Received* October 26, 1992)