The distance between subdifferentials in the terms of functions

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Abstract. For convex continuous functions f, g defined respectively in neighborhoods of points x, y in a normed linear space, a formula for the distance between $\partial f(x)$ and $\partial g(y)$ in terms of f, g (i.e. without using the dual) is proved. Some corollaries, like a new characterization of the subdifferential of a continuous convex function at a point, are given. This, together with a theorem from [4], implies a sufficient condition for a family of continuous convex functions on a barrelled normed linear space to be locally uniformly Lipschitz.

Keywords: convex analysis, subdifferentials of convex functions, barrelled normed linear spaces

Classification: Primary 26B25, 52A41; Secondary 46A08

Let X be a real normed linear space, $x \in X$. Let f be a continuous convex function defined in a convex neighborhood U of x. Then *the subdifferential of f at x* is the set

$$\partial f(x) = \{x^* \in X^* \mid f(u) \geq f(x) + \langle u - x, x^* \rangle \text{ for all } u \in U\}.$$

The set $\partial f(x)$ is a nonempty *weak**-compact convex subset of the dual X^* of X (cf. [2, Proposition 1.11] or [1, p. 132]). It does not depend on U since

$$(1) \qquad \partial f(x) = \{x^* \in X^* \mid \langle v, x^* \rangle \leq f'_+(x, v) \text{ for all } v \in X\},$$

where

$$f'_+(x, v) = \lim_{t \to 0+} \frac{f(x + tv) - f(x)}{t}$$

is the one-sided derivative of f at x in the direction v (cf. [2, p. 43]). For connections between differentiability properties of f and properties of its subdifferential map $x \mapsto \partial f(x)$ we refer the reader to [1] and [2].

In particular, if $X = \mathbf{R}$ and φ is a convex function defined in an open interval that contains x, then

$$(2) \qquad \partial \varphi(x) = [\varphi'_-(x), \varphi'_+(x)]$$

where $\varphi'_+(x) = \varphi'(x, 1)$ and $\varphi'_-(x) = -\varphi'(x, -1)$ are the right and left derivative of φ at x (see also [3, p. 32]).

Let U, V be convex neighborhoods of respectively $x, y \in X$. Let the functions $f : U \to \mathbf{R}$ and $g : V \to \mathbf{R}$ be continuous and convex. The aim of the present paper is to express the distance

$$\operatorname{dist}\big(\partial f(x), \partial g(y)\big) = \inf_{\substack{x^* \in \partial f(x) \\ y^* \in \partial g(y)}} \|x^* - y^*\|$$

in terms of the functions f, g only. This is done in Theorem 2 followed by some corollaries. In concrete situations, these results make possible the calculation of distances between subdifferentials without knowing any representation of the dual, and without calculating explicitly the subdifferentials.

In the end of the present paper, these results are applied to a local uniform boundedness principle for monotone operators from [4] to obtain a principle of local uniform Lipschitz property for families of continuous convex functions: For each function f belonging to a family $\mathcal{F}$ of continuous convex functions on an open convex set $U \subset X$, and each point $x \in X$, we define a number $\lambda_f(x)$ (a lower estimate for a possible local Lipschitz constant of f at x). If X is barrelled and $\{\lambda_f(x) \mid f \in \mathcal{F}\}$ is bounded for each $x \in U$, then $\mathcal{F}$ is locally uniformly Lipschitz in U in the sense that each $x \in U$ has a neighborhood on which all functions from $\mathcal{F}$ are Lipschitz with the same constant (depending on x).

Let us begin with a one-dimensional auxiliary theorem.

Theorem 1. *Let φ be a convex function defined on an open interval $I \subset \mathbf{R}$, $x \in I$. Then*

$$\liminf_{\substack{s \to 0+ \\ t \to 0+}} \frac{|\varphi(x+t) - \varphi(x-s)|}{t+s} = \min_{k \in \partial\varphi(x)} |k|.$$

Proof: (a) First, suppose $\varphi'_-(x) < 0 < \varphi'_+(x)$. By (2) we have

$$\varphi(x+h) - \varphi(x) \geq \max\{\varphi'_-(x)h, \varphi'_+(x)h\} \quad \text{whenever } x+h \in I. \tag{3}$$

Take $h_0 > 0$ such that $[x+h_0, x-h_0] \subset I$ and put $\mu = \min\{\varphi(x+h_0), \varphi(x-h_0)\}$. Clearly, $\mu > \varphi(x)$ since (3) implies that x is a point of strict minimum for φ on I. Choose a sequence $\{\mu_n\} \subset (\varphi(x), \mu)$ such that $\mu_n \to \varphi(x)$. The properties of $\{\mu_n\}$ imply that there exist sequences $\{s_n\}$ and $\{t_n\}$ in $(0, h_0)$ such that $\varphi(x+t_n) = \varphi(x-s_n) = \mu_n$. By (3), both $\{s_n\}$ and $\{t_n\}$ converge to 0. Consequently

$$\begin{aligned} 0 &\leq \liminf_{\substack{s \to 0+ \\ t \to 0+}} \frac{|\varphi(x+t) - \varphi(x-s)|}{t+s} \\ &\leq \lim_{n \to \infty} \frac{|\varphi(x+t_n) - \varphi(x-s_n)|}{t_n+s_n} \\ &= 0 \\ &= \min_{k \in \partial\varphi(x)} |k|. \end{aligned}$$

(b) Now, suppose $0 \le \varphi'_-(x)$. By the convexity of φ we have

$$\frac{\varphi(x) - \varphi(x-s)}{s} \le \frac{\varphi(x+t) - \varphi(x-s)}{t+s}$$

for all $s, t > 0$. Consequently,

$$0 \le \varphi'_-(x) = \lim_{s \to 0+} \frac{\varphi(x) - \varphi(x-s)}{s} \le \liminf_{\substack{s \to 0+ \\ t \to 0+}} \frac{\varphi(x+t) - \varphi(x-s)}{t+s}. \tag{4}$$

At the same time,

$$\begin{aligned} \liminf_{\substack{s \to 0+ \\ t \to 0+}} \frac{\varphi(x+t) - \varphi(x-s)}{t+s} &\le \liminf_{s \to 0+} \left(\lim_{t \to 0+} \frac{\varphi(x+t) - \varphi(x-s)}{t+s} \right) \\ &= \liminf_{s \to 0+} \frac{\varphi(x) - \varphi(x-s)}{s} = \varphi'_-(x). \end{aligned} \tag{5}$$

From (4) and (5) we deduce

$$\liminf_{\substack{s \to 0+ \\ t \to 0+}} \frac{|\varphi(x+t) - \varphi(x-s)|}{t+s} = \liminf_{\substack{s \to 0+ \\ t \to 0+}} \frac{\varphi(x+t) - \varphi(x-s)}{t+s} = \varphi'_-(x) = \min_{k \in \partial \varphi(x)} |k|.$$

(c) The remaining case $\varphi'_+(x) \le 0$ is similar to the case (b). $\square$

The following lemma is a well-known consequence of the Hahn-Banach theorem.

Lemma 1. *Let f be a continuous convex function defined on an open convex neighborhood of a point x in a normed linear space X. Let $v \in X$. Then the function*

$$\varphi_v(t) = f(x + tv)$$

is a convex function defined on a neighborhood of $0 \in \mathbf{R}$ and the following subdifferential formula holds

$$\partial \varphi_v(0) = \{\langle v, x^* \rangle \mid x^* \in \partial f(x)\}.$$

Sketch of the proof: The inclusion "$\supset$" follows immediately from definitions. To prove "$\subset$", take any $k \in \partial \varphi_v(0)$. Then the linear functional $\xi(tv) = tk$, defined on $\mathbf{R}v$, satisfies $\xi(h) \le f'(x, h)$ for all $h \in \mathbf{R}v$. By the Hahn-Banach theorem, there exists an extension $x^* \in X^*$ of ξ such that $\langle h, x^* \rangle \le f'(x, h)$ for all $h \in X$. Thus $x^* \in \partial f(x)$ and $\langle v, x^* \rangle = \xi(v) = k$. $\square$

Lemma 2. *Let X be a normed linear space, K be a weak*-closed convex subset of X^*. Then*
$$\operatorname{dist}(0,K)=\sup_{\|v\|=1}\inf_{x^*\in K}|\langle v,x^*\rangle|.$$

Proof: (a) Clearly
$$\sup_{\|v\|=1}\inf_{x^*\in K}|\langle v,x^*\rangle|\le\inf_{x^*\in K}\|x^*\|=\operatorname{dist}(0,K).$$

If $0\in K$, the proof is complete.

(b) If $0\notin K$, take an arbitrary $0<r<\operatorname{dist}(0,K)$. Then $rB^*\cap K=\emptyset$ where B^* denotes the closed unit ball in X^*. By the Hahn-Banach separation theorem (cf. [1, p. 70]) there exists $v_r\in X$ such that $\|v_r\|=1$ and $r=\sup_{z^*\in rB^*}\langle v_r,z^*\rangle<\inf_{x^*\in K}\langle v_r,x^*\rangle$. Consequently, $r<\inf_{x^*\in K}\langle v_r,x^*\rangle\le\sup_{\|v\|=1}\inf_{x^*\in K}|\langle v,x^*\rangle|$. Since this holds for any $r\in\big(0,\operatorname{dist}(0,K)\big)$, we get $\operatorname{dist}(0,K)\le\sup_{\|v\|=1}\inf_{x^*\in K}|\langle v,x^*\rangle|$. □

Theorem 2. *Let U and V be open convex sets in a normed linear space X. Let $f:U\to\mathbf{R}$ and $g:V\to\mathbf{R}$ be continuous convex functions. Then for any $x\in U$ and any $y\in V$ the following formula holds:*
$$\operatorname{dist}\big(\partial f(x),\partial g(y)\big)=\sup_{\|v\|=1}\liminf_{\substack{s\to0+\\t\to0+}}\left|\frac{f(x+tv)-f(x-sv)}{t+s}-\frac{g(y+sv)-g(y-tv)}{s+t}\right|.$$

Proof: (a) Suppose first that $g\equiv0$. Then, for any $v\in X$, Theorem 1 and Lemma 1 imply

$$(6)\quad \liminf_{\substack{s\to0+\\t\to0+}}\left|\frac{f(x+tv)-f(x-sv)}{t+s}\right|=$$
$$=\liminf_{\substack{s\to0+\\t\to0+}}\frac{|\varphi_v(t)-\varphi_v(-s)|}{t+s}=\min_{k\in\partial\varphi_v(0)}|k|=\min_{x^*\in\partial f(x)}|\langle v,x^*\rangle|$$

where φ_v is as in Lemma 1. From (6) and Lemma 2, applied to $K=\partial f(x)$, we get
$$\operatorname{dist}\big(\partial f(x),0\big)=\sup_{\|v\|=1}\min_{x^*\in\partial f(x)}|\langle v,x^*\rangle|=\sup_{\|v\|=1}\liminf_{\substack{s\to0+\\t\to0+}}\left|\frac{f(x+tv)-f(x-sv)}{t+s}\right|.$$

(b) Let now g be an arbitrary continuous convex function on V. We can define a new function $\tilde g$ by the formula
$$\tilde g(x+h)=g(y-h)\quad\text{whenever}\quad y-h\in V.$$

Then $\tilde g$ is a continuous convex function defined on the open convex set $x+y-V$ that contains x. It follows easily from definitions that $\partial\tilde g(x)=-\partial g(y)$. Moreover,

$\partial(f+\tilde{g})(x) = \partial f(x) + \partial\tilde{g}(x)$ by the Moreau-Rockafellar theorem (cf. [2, Theorem 3.23], note that the proof works in incomplete spaces, too). Using the part (a) of the present proof, we can compute

$$\begin{aligned}
\operatorname{dist}\big(\partial f(x), \partial g(y)\big) &= \operatorname{dist}\big(\partial f(x) - \partial g(y)\,, 0\big) \\
&= \operatorname{dist}\big(\partial f(x) + \partial\tilde{g}(x)\,, 0\big) \\
&= \operatorname{dist}\big(\partial(f+\tilde{g})(x)\,, 0\big) \\
&= \sup_{\|v\|=1} \liminf_{\substack{s\to 0+\\ t\to 0+}} \left| \frac{f(x+tv)-f(x-sv)}{t+s} + \frac{\tilde{g}(x+tv)-\tilde{g}(x-sv)}{t+s} \right| \\
&= \sup_{\|v\|=1} \liminf_{\substack{s\to 0+\\ t\to 0+}} \left| \frac{f(x+tv)-f(x-sv)}{t+s} - \frac{g(y+sv)-g(y-tv)}{s+t} \right|.
\end{aligned}$$

□

Corollary 1. *Under the assumptions of Theorem 2, $\partial f(x) \cap \partial g(y) \neq \emptyset$ if and only if*

$$\liminf_{\substack{s\to 0+\\ t\to 0+}} \left| \frac{f(x+tv)-f(x-sv)}{t+s} - \frac{g(y+sv)-g(y-tv)}{s+t} \right| = 0 \quad \textit{for every} \quad v \in X. \tag{7}$$

Proof: The assertion follows immediately from the equivalence $\partial f(x) \cap \partial g(y) \neq \emptyset \iff \operatorname{dist}\big(\partial f(x), \partial g(y)\big) = 0$ (this because the two subdifferentials are *weak**-compact) and from the fact that the absolute value in (7) is positively homogeneous as a function of v. □

Corollary 2. *Let f be a continuous convex function defined in a neighborhood of a point x in a normed linear space X, $x^* \in X^*$. Then*

$$\operatorname{dist}\big(\partial f(x), x^*\big) = \liminf_{\substack{s\to 0+\\ t\to 0+}} \left| \frac{f(x+tv)-f(x-sv)}{t+s} - \langle v, x^*\rangle \right|.$$

Proof: Apply Theorem 2 for $g = x^*$ (note that $\partial g(0) = \{x^*\}$). □

Corollary 3. *Under the assumptions of Corollary 2, $x^* \in \partial f(x)$ if and only if*

$$\liminf_{\substack{s\to 0+\\ t\to 0+}} \left| \frac{f(x+tv)-f(x-sv)}{t+s} - \langle v, x^*\rangle \right| = 0 \quad \textit{for every} \quad v \in X.$$

Proof: The assertion follows directly from Corollary 2. □

As an application of the above results and of a Banach-Steinhaus theorem for monotone operators proved in [4], we state a local uniform Lipschitz property principle for families of convex functions. Note that the number $\lambda_f(x)$ from Theorem 3 is a lower estimate for a (possible) local Lipschitz constant of the function f at x, i.e. if f is locally Lipschitz with a constant L on a neighborhood of x then necessarily $L \geq \lambda_f(x)$.

Theorem 3. *Let U be an open convex subset of a barrelled normed linear space X. Let $\mathcal{F}$ be a family of continuous convex functions on U such that*

$$\sup_{f\in\mathcal{F}} \lambda_f(x) < +\infty \quad \text{for every} \quad x \in U,$$

where

$$\lambda_f(x) = \sup_{\|v\|=1} \liminf_{\substack{s\to 0+\\ t\to 0+}} \left| \frac{f(x+tv) - f(x-sv)}{t+s} \right|.$$

Then the family $\mathcal{F}$ is locally uniformly Lipschitz in U, i.e. for each $x \in U$ there exist its neighborhood $V_x \subset U$ and a number $L_x \geq 0$ such that $|f(y) - f(x)| \leq L_x\|y - z\|$ whenever $y, z \in V_x$ and $f \in \mathcal{F}$.

Proof: The family $\mathcal{T} = \{\partial f \mid f \in \mathcal{F}\}$ is a family of monotone operators defined on U, such that

$$\sup_{T\in\mathcal{T}} \operatorname{dist}\bigl(T(x), 0\bigr) < +\infty \quad \text{for each} \quad x \in U,$$

since $\operatorname{dist}\bigl(\partial f(x), 0\bigr) = \lambda_f(x)$ for $x \in X$, $f \in \mathcal{F}$ by Corollary 2. By [4, Corollary 2] the family $\mathcal{T}$ is locally uniformly bounded on U, i.e. for each $x \in U$ there is its neighborhood V_x and a constant $L_x \geq 0$ such that

$$\|y^*\| \leq L_x \quad \text{whenever} \quad y^* \in \partial f(y),\ y \in V_x,\ f \in \mathcal{F}.$$

For $y, z \in V_x$ and $f \in \mathcal{F}$, take $y^* \in \partial f(y)$ and $z^* \in \partial f(z)$ arbitrarily and compute

$$\begin{aligned} |f(y) - f(z)| &= \max\bigl\{f(y) - f(z), f(z) - f(y)\bigr\} \leq \max\bigl\{\langle y - z, y^*\rangle, \langle z - y, z^*\rangle\bigr\} \\ &\leq \max\bigl\{\|y^*\|, \|z^*\|\bigr\} \cdot \|y - z\| \leq L_x\|y - z\|. \end{aligned}$$

□

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(*Received* January 19, 1993)