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Some adaptive estimators for slope parameter

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Abstract. An adaptive estimator (of a slope parameter) based on rank statistics is constructed and its asymptotic optimality is studied. A complete orthonormal system is incorporated in the adaptive determination of the score generating function. The proposed sequential procedure is based on a suitable stopping rule. Various properties of the sequential adaptive procedure and the stopping rule are studied. Asymptotic linearity results of linear rank statistics are also studied and some rates of the convergence are established.

Keywords: asymptotically optimal score generating function, Fisher information, orthonormal system, rank ($R-$)-estimator, stopping rule, asymptotically optimal estimators

Classification: 62G05, 62G20

1. Introduction.

For $n = 1, 2, \dots,$ let $X_1, \dots, X_n$ be independent observations such that

$$X_i = \theta_0 + \theta_1 c_i + e_i, \quad i = 1, 2, \dots, n, \tag{1.1}$$

where θ_0, θ_1 are unknown parameters, $c_1, \dots, c_n$ are known regression constants, $e_1, \dots, e_n$ are independent random errors fulfilling certain regularity assumptions.

The problem is to estimate θ_1 (slope parameter). The estimator is based on ranks and a score-generating function φ defined on $(0, 1)$. If the distribution function F of e_i's possesses an absolutely continuous density function (p.f.d) f with a finite Fisher information $I(f) = \int (\frac{f'}{f})^2\, dF$ $(< \infty)$, where f' stands for the first derivative of f, then for the estimation problem, the score-generating function $\varphi_f = -\frac{f'(F^{-1})}{f(F^{-1})}$ (F^{-1} stands for the quantile function corresponding to F) is asymptotically optimal in the sense that the asymptotic variance of the estimator of θ_1 attains the Cramer-Rao lower bound.

In practice, however, F and hence φ_f are rarely known, so the estimation of φ_f is of a considerable interest. Several types of estimations of φ_f have been developed. Here we shall concentrate on the Fourier expansion type via the estimation of the Fourier coefficients of φ_f. In this approach, Beran [1] used the trigonometric system to study this type, he described the construction of uniformly asymptotically efficient rank estimates in the two-sample location model. For this two-sample location model, Hušková [4] used a general type of Fourier series to estimate the Fourier coefficients and these estimators were based on the linearity of rank statistics. And in [6], Hušková and Sen considered the Legendre polynomials to estimate φ_f. Using similar ideas, Rödel [7] developed an adaptive rank statistics for testing independence. Towards this problem, he used the system of Legendre polynomials and the Fourier expansion to estimate the bivariate density.

In the present paper, asymptotically efficient rank estimators are constructed, and differ from those mentioned above. In fact, we use the general type of Fourier series to estimate the score function and construct an asymptotically optimal estimator for the slope parameter in the simple regression model (1.1). Our proposed procedure is a sequential one based on a well defined stopping rule. This is presented in Section 2. The main results on the asymptotic optimality of the proposed procedure are considered in Section 3. The proofs of the main results are mentioned in Section 4.

2. Assumptions and notation.

We shall adopt the following assumptions in the sequel:

Assumption A. The regression constants $c_1, \ldots, c_n$ fulfil:

(i) $n^{-1}C_n \to C > 0$,
(ii) $n^{-1}\sum_{i=1}^n c_i^4 = O(1)$ as $n \to \infty$, with

$$C_n = \begin{bmatrix} n & \sum_{i=1}^n c_i \\ \sum_{i=1}^n c_i & \sum_{i=1}^n c_i^2 \end{bmatrix}$$

and $C > 0$ is a positively definite matrix.

The condition (ii) is slightly stronger than one usually considered for rank estimates, however, still reasonable when φ_f is known.

Assumption B. $e_1, \ldots, e_n$ are iid random variables with distribution function F satisfying:

(i) $f(x) = \frac{dF(x)}{dx}$ exists and is absolutely continuous on $(-\infty, \infty)$,
(ii) the Fisher information in nonzero and finite, i.e.

$$0 < I(f) = \int_{-\infty}^{\infty} \Big(\frac{f'(x)}{f(x)}\Big)^2 \, dx < \infty\,;$$

(i) and (ii) are the usual regularity assumptions.

Further, throughout the paper we shall work with a complete orthonormal system $\{P_k(u),\ 0 \le u \le 1,\ k \ge 0\}$ in $L^2([0,1])$. Suppose that the system satisfies the following properties:

Assumption C. $\{P_k(u),\ 0 \le u \le 1,\ k \ge 0\}$ is a complete orthonormal system in $L^2([0,1])$ fulfilling: The first three derivatives $P_k^{(i)}(u)$, $i = 0,1,2,3$ exist and

$$D_{ki} = \sup_{0 \le u \le 1} |P_k^{(i)}(u)| < \infty, \quad i = 0,1,2,3,$$

where

$$P_k^0 = P_k\,.$$

The Legendre polynomial system and trigonometric system satisfy this assumption.

If Assumption B is satisfied, one can easily realize that $I(f)=\|\varphi_f\|^2$ $(=\int(\frac{f'}{f})^2\,dF)$ and $\varphi_f \in L^2([0,1])$. Hence φ_f can be written:

$$\varphi_f(u) \sim \sum_{k\geq 1} \gamma_k P_k(u), \quad 0 \leq u \leq 1, \tag{2.1}$$

where

$$\gamma_k = \langle \varphi_f, P_k\rangle = \int_0^1 \varphi_f(u) P_k(u)\,du = \int_{-\infty}^{\infty} P_k'(u)(F(x)) f^2(x)\,du. \tag{2.2}$$

Following the idea of Hušková and Sen [6], we introduce the stopping rules as follows:

$$L_n = \min\{k \geq K : \sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 \leq \varepsilon_n\}, \tag{2.3}$$

where K is a predetermined positive integer and $\hat{\gamma}_{n,j}$ is the estimator of γ_j defined by: (for $t \neq 0$)

$$\hat{\gamma}_{n,j} = \tag{2.4}$$

$$= -\frac{1}{t}\sum_{i=1}^{n} c_{in}\Big[P_k\Big((n+1)^{-1}\overset{\circ}{R}_i\Big(\overline{\theta}_{1n} - \frac{t}{[\sum_{i=1}^{n}(c_i-\overline{c}_n)^2]^{1/2}}\Big)\Big) - P_k\Big(\frac{\overset{\circ}{R}_i(\overline{\theta}_{1n})}{n+1}\Big)\Big]$$

with $\overset{\circ}{R}_i(u)$ being the rank of $X_i - c_i u$ among $X_1 - c_1 u, \dots, X_n - c_n u$, and

$$c_{in} = \frac{c_i - \overline{c}_n}{[\sum_{i=1}^{n}(c_i-\overline{c}_n)^2]^{1/2}} \quad \Big(\overline{c}_n = n^{-1}\sum_{i=1}^{n} c_i\Big), \tag{2.5}$$

$\overline{\theta}_{1n}$ is a preliminary estimator of θ_1 satisfying:

$$\Big[\sum_{i=1}^{n}(c_i-\overline{c}_n)^2\Big]^{1/2}(\overline{\theta}_{1n}-\theta_1) = 0_p(1) \quad \text{as} \quad n\to\infty \tag{2.6}$$

and $\{r_n\}$ and $\{\varepsilon_n\}$ are sequences of positive integers and positive real numbers such that

Assumption D.

(i) $\{r_n\}$ is increasing but $r_n n^{-s} \to 0$ as $n\to\infty$, for some $s>0$,

(ii) $\{\varepsilon_n\}$ is nonincreasing with $\lim_n \varepsilon_n = 0$;

if $r_n = O(\log n)$ and $\varepsilon_n = O(n^{-\alpha}(\log n)^{\beta})$ with $\alpha, \beta > 0$, then (i) and (ii) are satisfied.

Along with L_n we need

$$L_n^*(\lambda) = \min\{k \geq K : \sum_{j=k+1}^{k+r_n} \gamma_j^2 \leq \lambda \varepsilon_n\}, \quad \lambda > 0. \tag{2.7}$$

Together with the stopping rules we consider the following adaptive estimators $\hat{\varphi}_n(u)$ of $\varphi_f(u)$ (following the idea of [1] and [6])

$$\hat{\varphi}_n(u) = \sum_{k \leq L_n + r_n} \hat{\gamma}_{n,k} P_k(u), \quad u \in [0,1], \tag{2.8}$$

where L_n, $\hat{\gamma}_{n,k}$ are given by (2.3), (2.4), respectively.

As an estimator of the Fisher information, we use

$$\hat{I}_n = \sum_{k \leq L_n + r_n} \hat{\gamma}_{n,k}^2. \tag{2.9}$$

Finally, we are ready to introduce the adaptive estimator of θ_1 as follows:

$$\hat{\theta}_{1n} = \overline{\theta}_{1n} + \frac{1}{\hat{I}_n [\sum_{i=1}^n (c_i - \overline{c}_n)^2]^{1/2}} \sum_i^n c_{in} \hat{\varphi}_n(\overset{\circ}{R}_i(\overline{\theta}_{1n})/(n+1)), \tag{2.10}$$

where $\hat{\varphi}_n$, $\hat{I}_n$ are given by (2.8) and (2.9).

We shall investigate the asymptotic properties of the stopping rules, of $\hat{\varphi}_n$, $\hat{I}_n$ and, of course, of the resulting adaptive estimators.

The following assumptions will be needed:

Assumption E. For some $\delta > 0$, some $0 < \lambda_2 < 1 < \lambda_1$ and some $s > 0$, the first three derivatives of $\{P_k\}_{k=1}^{\infty}$ satisfy:

(i) $\sum_{k=L_n^*(\lambda_2)+1}^{L^*(\lambda_2)+r_n} [D_{k1}^2 n^{-1+\delta} + (D_{k2}^2 + D_{k3}^2) n^{-2+\delta}]\, \varepsilon_n^{-1} \to 0$ as $n \to \infty$,

(ii) $\max_{K \leq k < L_n^*(\lambda_1)} \sum_{j=k+1}^{k+r_n} [D_{k1}^2 n^{-1+\delta} + (D_{k2}^2 + D_{k3}^2) n^{-2+\delta}]\, \varepsilon_n^{-1} \to 0$ as $n \to \infty$,

(iii) $(L_n^*(\lambda_1) + r_n) n^{-s} \to 0$ as $n \to \infty$,

where K is a predetermined positive integer.

Assumption F. For some $\delta > 0$, some $0 < \lambda_2 < 1 < \lambda_1$ and some $s > 0$

(i) $\sum_{k=1}^{L_n^*(\lambda_2)+r_n} [D_{k1}^2 n^{-1+\delta} + (D_{k2}^2 + D_{k3}^2) n^{-2+\delta}] \to 0$ as $n \to \infty$,

(ii) $\sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k^2 \to 0$ as $n \to \infty$,

(iii) $(L_n^*(\lambda_2) + r_n) n^{-s} \to 0$ as $n \to \infty$,

where K is a predetermined positive integer.

Assumption G. For some $\delta > 0$ and some $0 < \lambda_2 < 1$

$\sum_{k=1}^{L_n^*(\lambda_2)+r_n} [D_{k1} n^{-1/2+\delta} + (D_{k2} + D_{k3}) n^{-1+\delta}] \to 0$ as $n \to \infty$.

In practice one can consider either the trigonometric system or Legendre polynomial system (see [6], [5]) and Assumptions E, F, G can be replaced by:

Assumption H (trigonometric system)**.** For some $0 < \lambda_2 < 1 < \lambda_1$ and some $0 < \delta < \frac{1}{2}$

(i) $L_n^*(\lambda_1) + r_n \to \infty$ as $n \to \infty$,
(ii) $(L_n^*(\lambda_2) + r_n)^2 n^{-1/2+\delta} \to 0$ as $n \to \infty$,
(iii) $\limsup\limits_{n\to\infty} \frac{r_n n^{-1/2}}{\varepsilon_n} < \infty$.

Assumption I (Legendre polynomials)**.** For some $0 < \lambda_2 < 1 < \lambda_1$ and some $0 < \delta < \frac{1}{2}$

(i) $L_n^*(\lambda_1) + r_n \to \infty$ as $n \to \infty$,
(ii) $(L_n^*(\lambda_2) + r_n)^{7/2} n^{-1/2+\delta} \to 0$ as $n \to \infty$,
(iii) $\limsup\limits_{n\to\infty} \frac{r_n (L_n^*(\lambda_2)+r_n)^{3/2} n^{-1/2}}{\varepsilon_n} < \infty$,
(iv) $(L_n^*(\lambda_2) + r_n)^8 n^{-1} \to 0$ as $n \to \infty$.

In these examples some assumptions are stronger, some are weaker than the above ones.

F (ii) is fulfilled e.g. if $L_n^*(\lambda_1) + r_n \to \infty$ as $n \to \infty$ or $\gamma_k = 0$, for all $k \geq M$ and $L_n^*(\lambda_1) + r_n > M$.

If $D_{ki} \geq D_{k-1,i}$, $D_{ki} \leq D_{k,i+1}$, $k = 1, 2, 3, \ldots$, $i = 0, 1, 2, 3$, we can formulate the above assumptions in a simple way.

3. Main theorems.

In this section the results concerning properties of the stopping rules, $\hat{\varphi}_n$ as well as the asymptotic distribution of the adaptive estimator $\hat{\theta}_{1n}$ of θ_1 will be formulated.

Theorem 3.1. *Let Assumptions* A–E *and* (2.6) *be satisfied, then*

$$L_n^*(\lambda_1) \leq L_n \leq L_n^*(\lambda_2) \ \text{ in probability as } \ n \to \infty\,,$$

for some $0 < \lambda_2 < 1 < \lambda_1$ *in the sense that for every* $\varepsilon > 0$ *there exists a positive integer* n_0 *such that:*

$$P(L_n^*(\lambda_1) \leq L_n \leq L_n^*(\lambda_2)) \geq 1 - \varepsilon, \quad \text{for } \ n \geq n_0,$$

where L_n *and* $L_n^*(\lambda)$ *are defined by* (2.3) *and* (2.7)*, respectively, i.e. the stopping rule* L_n *is bounded in probability by a nonrandom lower and upper bound in the above sense.*

Next, we state a result concerning $\hat{\varphi}_n$ and $\hat{I}_n$.

Theorem 3.2. *Let Assumptions* A–F *and* (2.6) *be satisfied, then*

$$\|\hat{\varphi}_n - \varphi_f\| \to 0, \quad \text{in probability as } n \to \infty, \tag{3.2}$$

$$\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 \le \hat{I}_n \le \sum_{k=1}^{L_n^*(\lambda_2)+r_n} \gamma_k^2 \quad \text{in probability as } n \to \infty \tag{3.3}$$

in the sense as in Theorem 3.1 *and hence*

$$\lim_{n\to\infty} \cdot \hat{I}_n = I(f) \quad \text{in probability as } n \to \infty,$$

where $\hat{\varphi}_n$ *and* $\hat{I}_n$ *are given by* (2.8) *and* (2.9).

Thus $\hat{\varphi}_n$ and $\hat{I}_n$ are consistent estimators of φ_f and $I(f)$, respectively.

Now, we shall present a result on the asymptotic distribution of $[\sum_1^n (c_i - \overline{c}_n)^2]^{1/2}$ $(\hat{\theta}_{1n} - \theta_1)$ as n tending to infinity with $\hat{\theta}_{1n}$ given by (2.10).

Theorem 3.3. *Let Assumptions* A–G *and* (2.6) *be satisfied, then* $[\sum_1^n (c_i - \overline{c}_n)^2]^{1/2}$ $(\hat{\theta}_{1n} - \theta_1)$ *has asymptotically normal distribution of* $(0, I(f)^{-1})$, *i.e. is an asymptotically optimal estimator of* θ_1.

Theorems 3.2 and 3.3 imply that $[\sum_1^n (c_i - \overline{c}_n)^2]^{1/2}$ $(\hat{\theta}_{1n} - \theta_1)$ has asymptotically normal distribution $N(0,1)$ and hence, for some given $\alpha \in (0,1)$, we can find a $(1-\alpha)$ confidence interval for θ_1.

4. Proof of Theorems 3.1, 3.2, 3.3.

At first, we derive a certain extension of the asymptotic linearity result of [6] and then we use it as a main tool in the proof of Theorems 3.1, 3.2 and 3.3.

Let $Z_1, Z_2, \ldots, Z_n$ be iid random variables. Let $R_i(t)$ denote the rank of $Z_i - c_{in}t$ among $Z_1 - c_{1n}t, \ldots, Z_n - c_{nn}t$ with c_{in} defined by (2.5). Let P_k be defined on $[0,1]$. Define

$$S_n(t, P_k) = \sum_{i=1}^{n} c_{in} \cdot P_k\Big(\frac{R_i(t)}{n+1}\Big). \tag{4.1}$$

Theorem 4.1. *Let* $Z_1, Z_2, \ldots, Z_n$ *be iid random variables with absolute continuous density* f *satisfying Assumption* B. *Let Assumptions* A, C *be satisfied. Then for every* $s > 0$, $\delta > 0$ *and* $A > 0$ *there exist* $d > 0$ *and* n_0 *such that for all* $n \ge n_0$:

$$P\Big(\sup_{|t|\le a} \{|S_n(t, P_k) - S_n(0, P_k) + t\gamma_k|\} \ge d u_{nk}\Big) < n^{-s}, \tag{4.2}$$

where

$$u_{nk} = n^{-1/2+\delta} \cdot D_k + n^{-1+\delta}(D_{k2} + D_{k3}), \tag{4.3}$$

$\delta > 0$ *arbitrary,* S_n *is defined by* (4.1), γ_k *is defined by* (2.2).

PROOF: The proof is similar to that of Theorem 4 in [6]. Therefore, we shall provide only the necessary modifications.

In order to prove (4.2) we shall use exponential inequalities and replace the "sup" in (4.2) by a "max" over a set of gridpoints, noticing

$$\begin{aligned} &\sup_{|t|\le a}\{|S_n(t,P_k)-S_n(0,P_k)-t\gamma_k|\}\le \\ (4.4)\qquad &\le \max_{q=1,\dots,N}\{|S_n(t_q,P_k)-S_n(0,P_k)-t_q\gamma_k|+ \\ &+\sup_{t_q\le t\le t_{q+1}}\{|S_n(t,P_k)-S_n(t_q,P_k)-(t-t_q)\gamma_k|\}\}, \end{aligned}$$

where

$$t_0=-a,\quad t_1=-a+1a/n,\dots,t_n=a,\quad N=2n.$$

Now, we have

$$\begin{aligned} &\sup_{t_q\le t\le t_{q+1}}\{|S_n(t,P_k)-S_n(t_q,P_k)|\}\le \\ (4.5)\qquad &\le \sup_{t_q\le t\le t_{q+1}}\Big\{\sum_{i=1}^n|c_{in}|\cdot\Big|\frac{R_i(t)-R_i(t_q)}{n+1}\Big|\cdot D_{k1}\Big\}= \\ &=\sum_{i=1}^n|c_{in}|(n+1)^{-1}D_{k1}\cdot\sum_{j\ne i}W_{ij}(Z_i), \end{aligned}$$

where

$$\begin{aligned} W_{ij}=I\{&\min((c_{jn}-c_{in})t_q,(c_{jn}-c_{in})t_{q+1})\le Z_j-z\le \\ &\le\max((c_{jn}-c_{in})t_q,(c_{jn}-c_{in})t_{q+1})\},\quad z\in R_1,\quad 1\le i,\quad j\le n. \end{aligned}$$

Then one observes that by the exponential inequality and the independence of $Z_1,\dots,Z_n$ for $z\in R_1$, $\lambda>0$,

$$\begin{aligned} &P\Big\{\sum_{j\ne i}(W_{ij}(z)\ge\lambda)\Big\}\le e^{-\lambda}\cdot E\Big[\exp\Big\{\sum_{j\ne i}W_{ij}(z)\Big\}\Big]\le \\ (4.6)\qquad &\le e^{-\lambda}\prod_{j=1}^n\Big(1+\sum_{\nu=1}^\infty\frac{1}{\nu!}\cdot EW_{ij}(z)\Big)\le e^{-\lambda}\exp\Big\{e\cdot\sum_{j\ne i}EW_{ij}(z)\Big\}\le \\ &\le\exp\{-\lambda+e\cdot K_1(|c_{in}|+\frac{1}{\sqrt{n}})\}\ \text{ for some }\ K_1>0, \end{aligned}$$

where we used the fact that for $z\in R_1$

$$E\{W_{ij}^p(z)\}\le EW_{ij}(z)\le K_1|c_{in}-c_{jn}|/n,\quad p=1,2,\dots\qquad\text{for some }\ K_1>0.$$

Then putting $\lambda = d_1 \log n$ one can realize for $z \in R_1$, $d_1 > 0$ and n large

$$(4.7)\qquad P\Big\{\sum_{j\neq i}(W_{ij}(z) \geq d_1 \log n)\Big\} \leq K_2^{-d_1} \text{ for some } K_2 > 0.$$

From (4.5)–(4.7) one can conclude that for every $d_1 > 0$ there exist $d_2 > 0$ and n_0 such that for $n \geq n_0$

$$P\big(\sup_{t_q \leq t \leq t_{q+1}} \{|S_n(t, P_k) - S_n(t_q, P_k)|\} \geq d_2 \cdot D_{k1} n^{-1/2} \log n\big) \leq K_2 \cdot n^{-d_1},$$

which further yields

$$P\big(\max_{q=1,\dots,N} \sup_{t_q \leq t \leq t^{q+1}} \{|S_n(t, P_k) - S_n(t_q, P_k) - (t - t_q) \cdot \gamma_k|\} > \\ > d_2 D_{k1} n^{-1/2} \log n\big) < n^{-d_1+2} < n^{-s}$$

if d_1 is chosen such that $d_1 - 2 > s$ and $n \geq n_0$.

Consequently, it remains to show that for every $d_1 > 0$ there exist $d_2 > 0$ and $n_0 > 0$ such that for $n \geq n_0$

$$P[|S_n(t, P_k) - S_n(0, P_k) - t\gamma_k| \geq d_2(D_{k1} n^{-1/2+\delta} + (D_{k2} + D_{k3}) n^{-1+\delta})] < \\ < n^{-d_1} \text{ for } |t| \leq a$$

the proof of which is similar to that of Theorem 4 of [6] with t fixed, and hence it can be omitted. □

Proof of Theorem 3.1: At first, we prove the first inequality.

For every $\lambda > 0$ and positive integer $p \leq L_n^*(\lambda) + r_n$, put

$$A_n(\lambda) = \bigcap_{k=L_n^*(\lambda)+1}^{L_n^*(\lambda)+r_n} A_{nk}$$

and

$$\mathring{A}_n(\lambda) = \bigcap_{k=p}^{L_n^*(\lambda)+r_n} A_{nk},$$

where

$$(4.8)\qquad A_{nk} = \{|\hat{\gamma}_{n,k} - \gamma_k| \leq d u_{nk}\}$$

with u_{nk} and $\hat{\gamma}_{n,k}$ being defined by (4.3) and (2.4), respectively.

Then for every $\varepsilon > 0$ from Theorem 4.1 and (2.6) one can easily prove that there exists a positive integer n_0 such that for $n \geq n_0$

$$(4.9)\qquad P(\mathring{A}_n^c) \leq (L_n^*(\lambda) + r_n - p) n^{-s} + \varepsilon,$$

where $\overset{\circ}{A}{}_n^c$ is the complement of the event $\overset{\circ}{A}_n$.

It follows that

$$P(A_n^c(\lambda)) \to 0 \quad \text{as} \quad n \to \infty\,, \tag{4.10}$$

where $A_n^c(\lambda)$ is the complement of the event $A_n(\lambda)$.

Further, from the inequality

$$\Big|\Big(\sum_1^m a_i^2\Big)^{1/2} - \Big(\sum_1^m b_i^2\Big)^{1/2}\Big| \le \Big(\sum_1^m (a_i - b_i)^2\Big)^{1/2} \tag{4.11}$$

one receives

$$\Big[\sum_{k=L_n^*(\lambda)+1}^{L_n^*(\lambda)+r_n} \hat\gamma_{n,k}^2\Big]^{1/2} \le \Big[\sum_{k=L_n^*(\lambda)+1}^{L_n^*(\lambda)+r_n} (\hat\gamma_{n,k} - \gamma_k)^2\Big]^{1/2} + (\lambda\varepsilon_n)^{1/2},$$

which together with the definition of L_n implies that (for $\lambda \in (0,1)$)

$$\begin{aligned} P(L_n^*(\lambda) < L_n) = P\Big(\Big[\sum_{k=L_n^*(\lambda)+1}^{L_n^*(\lambda)+r_n} \hat\gamma_{n,k}^2\Big]^{1/2} > \varepsilon_n^{1/2}\Big) \le \\ \le P\Big(\Big[\sum_{k=L_n^*(\lambda)+1}^{L_n^*(\lambda)+r_n} (\hat\gamma_{n,k} - \gamma_k)^2\Big]^{1/2} > \varepsilon_n^{1/2}(1 - \lambda^{1/2})\Big). \end{aligned}$$

Hence

$$\begin{aligned} & P(L_n^*(\lambda) < L_n) \le \\ & \le P\Big(\sum_{k=L_n^*(\lambda)+1}^{L_n^*(\lambda)+r_n} (\hat\gamma_{n,k} - \gamma_k)^2 > \varepsilon_n(1 - \lambda^{1/2})^2, A_n(\lambda)\Big) + P(A_n^c(\lambda)). \end{aligned} \tag{4.12}$$

On the other hand, on the set $A_n(\lambda)$ and for $0 < \varepsilon < (1 - \lambda^{1/2})^2$ with $0 < \lambda < 1$ and under Assumptions E (i) and D one can easily see that there exists a positive integer n_0 such that the first summand term on the r.h.s. of (4.12) is equal to 0 for $n \ge n_0$ and by (4.10) one obtains

$$P(L_n^*(\lambda_2) < L_n) \to 0 \quad \text{as} \quad n \to \infty\,, \quad \text{for any} \quad 0 < \lambda_2 < 1.$$

Next, we prove the second inequality. We first note that:

$$\begin{aligned} P(L_n^*(\lambda) > L_n) = P\Big(\bigcup_{k=K}^{L_n^*(\lambda)-1} \{L_n = k\}\Big) \le \\ \le P\Big(\bigcup_{k=K}^{L_n^*(\lambda)-1} \Big\{\sum_{j=k+1}^{k+r_n} \hat\gamma_{n,j}^2 \le \varepsilon_n\Big\}\Big) = P\Big(\min_{K \le k < L_n^*(\lambda)} \sum_{j=k+1}^{k+r_n} \hat\gamma_{n,j}^2 \le \varepsilon_n\Big). \end{aligned}$$

Hence

$$P(L_n^*(\lambda) > L_n) \le P\Big(\min_{K\le k<L_n^*(\lambda)} \sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 \le \varepsilon_n, B_n(\lambda)\Big) + P(B_n^c(\lambda)), \tag{4.13}$$

where

$$B_n(\lambda) = \bigcap_{K\le k<L_n^*(\lambda)} A_{nk} \quad \text{for every} \quad \lambda > 1, \tag{4.14}$$

with A_{nk} being defined by (4.8).

Using the inequality (4.11) with $a = \gamma_j$, $b = \gamma_j - \hat{\gamma}_{n,j}$, and for every $K \le k < L_n^*(\lambda)$, then one has

$$(\lambda\varepsilon_n)^{1/2} \le \Big[\sum_{j=k+1}^{k+r_n} \gamma_j^2\Big]^{1/2} \le \Big[\sum_{j=k+1}^{k+r_n} (\hat{\gamma}_{n,j} - \gamma_j)^2\Big]^{1/2} + \Big[\sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2\Big]^{1/2}. \tag{4.15}$$

which implies that on the set B_n

$$\min_{K\le k<L_n^*(\lambda)} \Big[\sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2\Big]^{1/2} > (\lambda\varepsilon_n)^{1/2} - \max_{K\le k<L_n^*(\lambda)} d\Big[\sum_{j=k+1}^{k+r_n} u_{n,j}^2\Big]^{1/2}. \tag{4.16}$$

By Assumption E (ii), there exists a positive integer n_0 such that for all $n \ge n_0$ the second member on the r.h.s. of (4.16) is larger than $-\varepsilon\varepsilon_n$ $(0 < \varepsilon < \lambda^{1/2} - 1)$. It follows that

$$\min_{K\le k<L_n^*(\lambda)} \sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 > (\lambda^{1/2} - \varepsilon)\varepsilon_n > \varepsilon_n, \quad \text{for} \quad n \ge n_0,$$

which implies that for $n \ge n_0$, the first summand term on the r.h.s. of (4.12) is equal to 0. Hence there exists $\lambda_1 > 1$ for every $\varepsilon > 0$ and n large such that

$$P(L_n^*(\lambda) > L_n) \le P(B_n^c) \le (L_n^*(\lambda_1) + r_n)n^{-s} + \varepsilon \quad \text{(by (4.10) and (4.14))},$$

which tends to 0 by Assumption E (iii). This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2: At first, we prove (3.2). Putting

$$C_n(\lambda_2) = \bigcap_{k=1}^{L_n^*(\lambda_2)+r_n} A_{nk}, \tag{4.17}$$

where A_{nk} is defined by (4.8), then by (4.9) and F (iii) one has

$$P(C_n^c(\lambda_2)) \to 0 \quad \text{as} \quad n \to \infty\,,$$

where $C_n^c(\lambda_2)$ is the complement of $C_n(\lambda_2)$ for $\lambda_2 < 1$. Note that

$$\|\hat{\varphi}_n - \varphi\|^2 = \int_0^1 (\hat{\varphi}_n - \varphi)^2(u)\,du = \sum_{k=1}^{L_n+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 + \sum_{k=L_n(\lambda)+r_n+1}^{\infty} \gamma_k^2$$

and

$$\begin{aligned} (4.18) \quad P(\|\hat{\varphi}_n - \varphi\| \geq \varepsilon) \leq P(\|\hat{\varphi}_n - \varphi\| \geq \varepsilon, L_n \leq L_n^*(\lambda_2), C_n(\lambda_2)) + \\ + P(L_n > L_n^*(\lambda_2)) + P(C_n^c(\lambda_2)). \end{aligned}$$

By Assumption F (i) and (ii), for every $\varepsilon > 0$ there exists n_1 such that for $n \geq n_1$ the first probability on the r.h.s. of (4.18) is equal to 0 and one gets

$$P(\|\hat{\varphi}_n - \varphi\| \geq \varepsilon) \leq P(L_n > L_n^*(\lambda_2)) + P(C_n^c(\lambda_2)), \text{ for } n \geq n_1,$$

which together with (4.17) and Theorem 3.1 implies (3.2).

Now, we shall prove (3.3) in two steps:

(i) $\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 \leq \hat{I}_n$ in probability as $n \to \infty$.

For every $\varepsilon > 0$, we have

$$\begin{aligned} &P\Big(\hat{I}_n - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 < -\varepsilon\Big) \leq \\ &\qquad \leq P\Big(\hat{I}_n - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 < -\varepsilon, L_n \geq L_n^*(\lambda_1)\Big) + P(L_n < L_n^*(\lambda_1)). \end{aligned}$$

On the other hand,

$$\begin{aligned} -\sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k}^2 - \gamma_k^2) &= -2\sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)\gamma_k - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 \leq \\ &\leq 2\Big[I(f) \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2\Big]^{1/2} + \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2. \end{aligned}$$

Hence

$$\begin{aligned} (4.19) \quad &P\Big(\hat{I}_n - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 < -\varepsilon\Big) \leq P(C_n^c(\lambda_1)) + \\ &\quad + P\Big(\Big[I(f) \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2\Big]^{1/2} > \varepsilon, C_n(\lambda_1)\Big) + P(L_n < L_n^*(\lambda_1)), \end{aligned}$$

where $C_n(\lambda_1)$ is defined similarly as $C_n(\lambda_2)$ with $0 < \lambda_2 < 1 < \lambda_1$. From (4.9), F (iii) and Theorem 3.1 one obtains $P(C_n^c(\lambda_1)) \to 0$ as $n \to \infty$. By the same way as in the proof of (3.2), it follows from F (i) that the second probability on the r.h.s. (4.19) is equal to 0 as $n \geq n_2$ (for some positive integer n_2). And hence

$$P\Big(\hat{I}_n - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 < -\varepsilon\Big) \to 0 \quad \text{as } n \to \infty, \quad \text{for every} \quad \varepsilon > 0,$$

which implies the first inequality of (3.3).

(ii) $\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 \geq \hat{I}_n$ in probability as $n \to \infty$.

Choosing $a_k = \hat{\gamma}_{n,k}$, $b_k = \gamma_k$ in the inequality (4.11) then, for every $\varepsilon > 0$, we have

$$P\Big(\hat{I}_n > \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 + \varepsilon\Big) \leq$$
$$\leq P\Big(\sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 + 2\Big[I(f) \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2\Big]^{1/2} > \varepsilon\Big) + $$
$$+ P(L_n > L_n^*(\lambda_2)),$$

which ensures

$$P\Big(\hat{I}_n > \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 + \varepsilon\Big) \leq \tag{4.20}$$
$$\leq P\Big(\sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 + 2\Big[I(f) \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2\Big]^{1/2} > \varepsilon, C_n(\lambda_2)\Big) + $$
$$+ P(L_n > L_n^*(\lambda_2)) + P(C_n^c(\lambda_2)),$$

where $C_n(\lambda_2)$ is defined by (4.17) and $C_n^c(\lambda_2)$ is its complement.

It follows from Assumption F (i) that the first probability on the r.h.s. of (4.20) is equal to 0 for $n \geq n_3$ (n_3 is some positive integer); and hence, the second inequality of (3.3) is proved. □

PROOF OF THEOREM 3.3: Recall that

$$\hat{\theta}_{1n} = \overline{\theta}_{1n} + \Big[\hat{I}_n\Big(\sum_1^n (c_i - \overline{c}_n)^2\Big)^{1/2}\Big]^{-1} \sum_1^n c_{in}\hat{\varphi}_n(\overset{\circ}{R}_i(\overline{\theta}_{1n})/(n+1)),$$

where $\hat{\varphi}_n$, $\hat{I}_n$ are defined by (2.8) and by (2.9), respectively. $\overset{\circ}{R}_i(\overline{\theta}_{1n})$ is the rank of $X_i - c_i\overline{\theta}_{1n}$ among $X_1 - c_1\overline{\theta}_{1n}, \ldots, X_n - c_n\overline{\theta}_{1n}$.

In order to be able to apply Theorem 4.1 we rewrite $\hat{\theta}_{1n}$ as follows:

$$\begin{aligned}\hat{\theta}_{1n} = \overline{\theta}_{1n} + \Big[\hat{I}_n\Big(\sum_1^n (c_i-\overline{c}_n)^2\Big)^{1/2}\Big]^{-1}\cdot \\ \sum_1^n c_{in}\hat{\varphi}_n\Big(R_i\Big(\Big[\sum_1^n (c_i-\overline{c}_n)^2\Big]^{1/2}(\overline{\theta}_{1n}-\theta_1)\Big)/(n+1)\Big),\end{aligned} \tag{4.21}$$

where $R_i(t)$ is the rank of $e_i - c_{in}t$ among $e_1 - c_{1n}t, \ldots, e_n - c_{nn}t$.

Next,

$$\begin{aligned}&\sup_{|t|\le a}\Big|\sum_1^n c_{in}[\hat{\varphi}_n(R_i(t)/(n+1)) - \hat{\varphi}_n(R_i(0)/(n+1))] + \\ &\quad + t\int_0^1 \hat{\varphi}_n(u)\varphi_f(u)\,du\Big| = \\ &= \sup_{|t|\le a}\Big|\sum_{k=1}^{L_n+r_n}\hat{\gamma}_{n,k}\Big(\sum_1^n c_{in}[P_k(R_i(t)/(n+1)) - P_k(R_i(0)/(n+1))] + \\ &\quad + t\int_0^1 P_k(u)\varphi_f(u)\,du\Big)\Big| \le \\ &\le \sup_{|t|\le a}\Big|\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\hat{\gamma}_{n,k}^2\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\Big[\sum_1^n c_{in}\Big[P_k\Big(\frac{R_i(t)}{n+1}\Big) - P_k\Big(\frac{R_i(0)}{n+1}\Big)\Big] + \\ &\quad + t\int_0^1 P_k(u)\varphi_f(u)\,du\Big]^2\Big|^{1/2} \text{ in probability as } n\to\infty\,.\end{aligned} \tag{4.22}$$

Since $\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\hat{\gamma}_{n,k}^2 \le 2\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\gamma_k^2 + 2\sum_{k=1}^{L_n^*(\lambda_2)+r_n}(\hat{\gamma}_{n,k}-\gamma_k)^2$ and $\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\gamma_k^2 \le I(f)$, one gets from Assumption F, Theorem 4.1, (2.6) and (4.21)

$$\begin{aligned}&\sup_{|t|\le a}\Big|\sum_1^n c_{in}[\hat{\varphi}_n(R_i(t)/(n+1)) - \hat{\varphi}_n(R_i(0)/(n+1))] + \\ &\qquad + t\int_0^1 \hat{\varphi}_n(u)\varphi_f(u)\,du\Big| \xrightarrow{P} 0 \text{ as } n\to\infty\,.\end{aligned} \tag{4.23}$$

Further, we have

$$\int_0^1 \hat{\varphi}_n(u)\varphi_f(u)\,du = \sum_{k=1}^{L_n+r_n}\hat{\gamma}_{n,k}\gamma_k = \sum_{k=1}^{L_n+r_n}\hat{\gamma}_{n,k}(\gamma_k - \hat{\gamma}_{n,k}) + \hat{I}_n.$$

Hence, by the Cauchy-Schwarz inequality one has

$$\Big|1 - (\hat{I}_n)^{-1}\int_0^1 \hat{\varphi}_n(u)\varphi_f(u)\,du\Big| \xrightarrow{P} 0 \text{ as } n\to\infty\,. \tag{4.24}$$

From (4.21), (4.23) and (4.24) we can write

$$\text{(4.25)}\quad \Big[\sum_1^n (c_i-\overline{c}_n)^2\Big]^{1/2}(\hat\theta_{1n}-\theta_1)=(\hat I_n)^{-1}\sum_1^n c_{in}\hat\varphi_n(R_i(0)/(n+1))+ \\ +o_P(1)\quad \text{as}\quad n\to\infty\,.$$

Now, in order to prove the assertion of our theorem, it is sufficient to show that

$$\text{(4.26)}\qquad (\hat I_n)^{-1}\sum_1^n c_{in}\hat\varphi_n(R_i(0)/(n+1))\xrightarrow{D} N(0,(I(f))^{-1})$$

where $I(f)$ is the Fisher information.

Note that (4.26) will be implied by the following two assertions:

$$\text{(4.27)}\qquad H_n=\sum_{k=1}^{L_n+r_n}(\hat\gamma_{n,k}-\gamma_k)\sum_1^n c_{in}P_k(R_i(0)/(n+1))\xrightarrow{P}0\quad\text{as } n\to\infty$$

and

$$\text{(4.28)}\qquad \sum_1^n c_{in}\sum_{k=1}^{L_n+r_n}\gamma_kP_k(R_i(0)/(n+1))\xrightarrow{D} N(0,I(f))\quad\text{as } n\to\infty\,.$$

Putting

$$H_{1n}=\sum_{k=1}^{L_n+r_n}(\hat\gamma_{n,k}-\gamma_k)I_{A_{nk}}\sum_1^n c_{in}P_k(R_i(0)/(n+1)),$$

where A_{nk} is defined by (4.8), I_A is the indicator function of the set A and

$$H_{2n}=\sum_{k=1}^{L_n+r_n}(\hat\gamma_{n,k}-\gamma_k)I_{A^c_{nk}}\sum_1^n c_{in}P_k(R_i(0)/(n+1)).$$

We shall prove that $H_{in}\xrightarrow{P}0$ as $n\to\infty$, $i=1,2$. Clearly

$$\text{(4.29)}\qquad E|H_{1n}|\le d\sum_{k=1}^{L^*_n(\lambda_2)+r_n}u_{nk}\Big[E\Big(\sum_1^n c_{in}P_k(R_i(0)/(n+1))^2\Big)\Big]^{1/2},$$

where

$$E\Big(\sum_1^n c_{in}P_k(R_i(0)/(n+1))^2\Big)=\operatorname{var}\Big(\sum_1^n c_{in}P_k(R_i(0)/(n+1))^2\Big)=$$
$$=(n-1)^{-1}\sum_{i=1}^n\Big[P_k(i/(n+1))-n^{-1}\sum_{j=1}^nP_k(j/(n+1))\Big]^2\le$$
$$\le(n-1)^{-1}\sum_{i=1}^nP_k^2(i/(n+1))\le$$
$$\le 1+\frac{n}{n-1}\Big|\sum_{i=1}^n\int_{(i-1)/n}^{i/n}(P_k^2(i/(n+1))-P_k^2(u))\,du\Big|\le 1+4(D_{k1}+D_{k1}^2)n^{-1}$$

(by the mean value theorem for the function $P_k^2(u)$).

Hence, by Assumptions F, G and (4.29), one receives

$$EH_{1n} \to 0 \text{ and then } H_{1n} \xrightarrow{P} 0 \text{ as } n \to \infty .$$

As for H_{2n}, by (4.8) and (4.9) we have for every $\varepsilon > 0$ and n large

$$\begin{aligned} P(|H_{2n}| > \eta) &\le P(H_{2n} \ne 0) \le P(H_{2n} \ne 0, L_n \le L_n^*(\lambda_2)) + P(L_n > L_n^*(\lambda_2)) \le \\ &\le P(L_n > L_n^*(\lambda_2)) + P\Big(\bigcup_{k=1}^{L_n^*(\lambda_2)+r_n} \{|\hat{\gamma}_{n,k} - \gamma_k| > du_{nk}\}\Big) \le \\ &\le P(L_n > L_n^*(\lambda_2)) + (L_n^*(\lambda_2) + r_n)n^{-s} + \varepsilon, \end{aligned}$$

which tends to 0 as n goes to infinity. Hence (4.27) is proved.

Next, we turn to (4.28). Obviously, it is sufficient to show that

$$\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k \sum_{1}^{n} c_{in} P_k(R_i(0).(n+1)) \xrightarrow{D} N(0, I(f)), \tag{4.30}$$

and

$$\sum_{k=L_n^*(\lambda_1)+r_n}^{L_n+r_n} \gamma_k \sum_{1}^{n} c_{in} P_k(R_i(0)/(n+1)) \xrightarrow{P} 0 \text{ as } n \to \infty . \tag{4.31}$$

We start with (4.30). Note that Assumption A implies that

$$\max_{1 \le i \le n} c_{in}^2 \to 0 \text{ as } n \to \infty . \tag{4.32}$$

Putting

$$a_n(i) = \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k P_k(i/(n+1)),$$

one can write

$$\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k \sum_{1}^{n} c_{in} P_k(R_i(0)/(n+1)) = \sum_{i=1}^{n} c_{in} a_n(R_i(0))$$

and

$$\int_0^1 (a_n(1+[un]) - \varphi_f(u))^2\,du = \int_0^1 \Big\{ \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k \Big(P_k\Big(\frac{1+[un]}{n+1}\Big) - P_k(u)\Big) +$$

$$+ \sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k P_k(u) \Big\}^2\,du \leq$$

$$\leq 2\int_0^1 \Big\{ \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k((P_k(1+[un])/(n+1)) - P_k(u)) \Big\}^2\,du +$$

$$+ 2\int_0^1 \Big\{ \sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k P_k(u) \Big\}^2\,du \leq$$

$$\leq 2\Big\{ \sum_{k=1}^{L_n^*(\lambda_1)+r_n} |\gamma_k| D_{k1} n^{-1} \Big\}^2 + 2 \sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k^2,$$

which implies that

$$\int_0^1 (a_n(1+[un]) - \varphi_f(u))^2\,du \leq 2 \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 \sum_{k=1}^{L_n^*(\lambda_1)+r_n} D_{k1}^2 n^{-2} + \\ + 2 \sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k^2 \to 0 \quad \text{as} \quad n \to \infty\,.$$

Hence, by Theorem 1.6 (a) in [3, p. 163] and (4.32) we can conclude that

$$\sum_1^n c_{in} \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k P_k(R_i(0)/(n+1)) = \sum_1^n c_{in} a_n(R_i(0)) \xrightarrow{D} N(\mu, \sigma_f^2) \quad \text{as} \quad n \to \infty\,,$$

where

$$\mu = E \sum_1^n c_{in} a_n(R_i(0)) = 0$$

and

$$\sigma_f^2 = \lim_{n\to\infty} \mathrm{var}\Big(\sum_1^n c_{in} a_n(R_i(0)) \Big) = \\ = \lim_{n\to\infty} \sum_1^n c_{in}^2 \int_0^1 (\varphi(u) - \overline{\varphi})^2\,du = \int_0^1 (\varphi(u) - \overline{\varphi})^2\,du = I(f)$$

with $\overline{\varphi} = \int_0^1 \varphi(u)\,du$.

Now, we prove (4.31). Putting

$$S_j = \sum_{k=L_n^*(\lambda_1)+r_n}^{L_n^*(\lambda_1)+r_n+j} \gamma_k \sum_1^n c_{in} P_k(R_i(0)/(n+1)), \quad 1 \le j \le N,$$

where $N = L_n^*(\lambda_2) - L_n^*(\lambda_1)$,

$S_0 = 0$,

$M_N = \max_{0 \le j \le N} |S_j|$.

$M'_N = \max_{0 \le j \le N} \min\{|S_j|, |S_n - S_j|\}$,

then

$M'_N \le M_N$

and by [2, 12.4] we have

$M_N \le M'_N + S_N$.

In order to prove (4.31), it is sufficient to check that M'_N and S_N converge in probability to 0. We start with the first assertion. To do this we will compute

$$E\Big[\sum_1^n c_{in} \sum_{k=p+1}^{q} \gamma_k P_k(R_i(0)/(n+1))\Big]^4 \text{ for every } p, q:$$

$$L_n^*(\lambda_1) + r_n \le p \le q \le L_n^*(\lambda_2) + r_n.$$

After straightforward but long computations (cf. [8]) one receives

$$E\Big[\sum_1^n c_{in} \sum_{k=p+1}^{q} \gamma_k P_k(R_i(0)/(n+1))\Big]^4 \le K_3\Big(\sum_{p+1}^{q} \gamma_k^2\Big)^2, \tag{4.33}$$

for some $K_3 > 0$ and n large. It follows that for every $i, j, k : 0 \le i \le j \le k \le N$ and for every $\lambda > 0$

$$P(|S_j - S_i| > \lambda, |S_k - S_j| > \lambda) \le P(|S_j - S_i| > \lambda) \le \lambda^{-4} E(S_j - S_i)^4 \le$$

$$\le K_4\Big(\sum_{m=i+1}^{k} \gamma_m^2\Big)^2, \text{ for some } K_4 > 0.$$

So, the assumption (12.11) of Theorem 12.1, [2], is satisfied, and now this theorem can be applied, and one gets

$$P(M'_N > \lambda) \le \lambda^{-4} K_5 \Big[\sum_{k=L_n^*(\lambda_1)+r_n+1}^{L_n^*(\lambda_2)+r_n} \gamma_k^2\Big]^2 \quad (\gamma = \alpha = 2),$$

for some $K_5 > 0$, i.e. $M'_N \xrightarrow{P} 0$, and if we take $p = L_n^*(\lambda_1) + r_n$ and $q = L_n^*(\lambda_2) + r_n$ in (4.32) then $S_N \xrightarrow{P} 0$. These ensure that $M_N = \max_{0 \le j \le N} |\sum_{k=L_n^*(\lambda_1)+r_n}^{L_n^*(\lambda_1)+r_n+j} \gamma_k \sum_1^n c_{in} P_k(R_i(0)/(n+1))| \xrightarrow{P} 0$ as $n \to \infty$, hence (4.31) is proved. The proof of the theorem is complete. □

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