A note on the existence of solution for semilinear heat equations with polynomial growth nonlinearity*

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Abstract. The existence of weak solution for periodic-Dirichlet problem to semilinear heat equations with superlinear growth non-linear term is treated.

Keywords: periodic-Dirichlet problem, semilinear heat equation, superlinear growth

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1. Introduction.

Let Z^+, Z and R be the set of all positive integers, integers and real numbers, respectively, and let $\Omega = [0, 2\pi] \times [0, \pi]$ and $I = [0, \pi]$. Let $p \in [1, \infty)$; by $L^p(\Omega)$ we denote the space of all measurable real valued functions $u(t, x)$ for which $|u(t, x)|^p$ is Lebesgue integrable.

The norm is given by

$$\|U\|_{L^p} = \Big[\int\int_\Omega |u(t,x)|^p \, dt \, dx\Big]^{1/p}.$$

In particular, $L^2(\Omega)$ is a Hilbert space having the usual inner product $\langle \, , \rangle$ and the usual norm $\|\cdot\|_{L^2}$. Let $C^k(\Omega)$ be the space of all continuous functions $u(t,x)$ such that the partial derivatives up to order k with respect to both variables exist and are continuous on Ω, while $C(\Omega)$ is used for $C^0(\Omega)$ with the usual norm $\|\cdot\|$ and we write $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$.

Consider the periodic-Dirichlet problem

$$u_t(t,x) - u_{xx}(t,x) + g(x, u(t,x)) = h(t,x) \quad \text{in} \quad \Omega \tag{1.1}$$

$$\begin{aligned} &u(0,x) - u(2\pi, x) = u_t(0,x) - u_t(2\pi,x) = 0 \quad \text{for all} \quad x \in I \\ &u \mid \partial I = 0, \end{aligned} \tag{1.2}$$

where $g : I \times R \to R$ is a Carathéodory function, that is, $g(\cdot, u)$ is measurable on I for each $u \in R$ and $g(x, \cdot)$ is continuous on R with the continuity uniform with respect to a.e. $x \in I$. This holds, for example, if g is continuous on $I \times R$, but

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it holds in many other cases; e.g. $g(x,u) = p(x)F(u)$, where $p \in L^\infty(I)$ and F is continuous. Moreover, we assume that for each $d > 0$ there exists a constant $M_d > 0$ such that $|g(x,u)| \le M_d$ for all $(x,u) \in I \times R$ with $|u| \le d$ and $h \in L^2(\Omega)$. Let $H^1(\Omega)$ be the completion of the space $C^\infty(\Omega)$ with respect to the norm given by

$$\|u\|_1^2 = \int\int_\Omega \left[|u^2(t,x)|^2 + |u_t(t,x)|^2 + |u_x(t,x)|^2\right] dt\, dx \tag{1.3}$$

and $H^{1.2}(\Omega)$ be the completion of the space $C^\infty(\Omega)$ with respect to the norm given by

$$\|u\|_{1.2}^2 = \int\int_\Omega \left[|u(t,x)|^2 + |u_t(t,x)|^2 + |u_x(t,x)|^2 + |u_{xx}(t,x)|^2\right] dt\, dx. \tag{1.4}$$

Note that $H^1(\Omega)$ $(H^{1.2}(\Omega))$ has distributional derivatives $u_t, u_x \in L^2(\Omega)$ $(u_t, u_x, u_{xx} \in L^2(\Omega))$ and these derivatives can be obtained as limit in $L^2(\Omega)$ of the corresponding derivatives of a sequence of $C^\infty(\Omega)$ functions which tend to u in $H^1(\Omega)$ $(H^{1.2}(\Omega))$. Moreover if derivatives are interpreted in distributional sense, the norms in $H^1(\Omega)$ and $H^{1.2}(\Omega)$ are given by (1.3) and (1.4), respectively, and $H^{1.2}(\Omega) \subseteq C(\Omega)$, and the imbedding of $H^{1.2}(\Omega)$ in $C(\Omega)$ is continuous. Let $H^{0.1}(\Omega)$ be the closure in $H^1(\Omega)$ of all functions $u(t,x)$ in $C^\infty(\Omega)$.

A weak solution to the periodic-Dirichlet problem (1.1), (1.2) on Ω will be $u \in H^{1.2}(\Omega) \cap H^{0.1}(\Omega)$ which satisfies the equation (1.1) a.e. on Ω and the boundary condition (1.2).

Several authors deal with the periodic-Dirichlet problem for semilinear heat equations. For example, Brezis and Nirenberg [1], Fučík [2], Nkashama and Willem [3], Šťastnová and Fučík [5] and the references in [6]. In particular, Sanchez [4] shows the existence of solutions for some power-like behavior of g by using Leray-Schauder principle.

In this note, we will investigate the existence of weak solution of the problem (1.1) (1.2) when g has polynomial growth and satisfies a sign condition. The rate of growth allowed in g is any polynomial growth; i.e. there exist $a(x), b(x)$ in $L^\infty(\Omega)$ such that

$$|g(x,u)| \le a(x)|u|^p + b(x) \quad \text{for} \quad x \in I, |u| \ge d_0 \quad \text{and} \quad p > 0. \tag{H_1}$$

The sign condition on g used here is that there exists a function $\psi : R \to R$ such that $\limsup_{|u|\to\infty} \psi(u)/u = \alpha_0$ exists with $ug(x,u) \ge -\psi(|u|)$ for all $(x,u) \in I \times R$. We do not need any restriction on h except $h \in L^2(\Omega)$ and our proof is based on the use of Fourier series and Leray-Schauder's continuation theorem by finding an a priori bound for all possible solutions of the associated equations.

2. Preliminaries.

Let $Dom\, L = \{u \in H^{1.2}(\Omega) \cap H^{0.1}(\Omega) \mid u \text{ satisfies (1.2)}\}$ and define the linear operator

$$L : Dom\, L \subseteq C(\Omega) \to L^2(\Omega) \ \text{ by}$$

$$Lu = u_t - u_{xx}\,. \tag{2.1}$$

Then $Ker\, L = \{0\}$ and $Im\, L = L^2(\Omega)$. Hence $L^{-1} : L^2(\Omega) \to Dom\, L$ exists. Since we may expand function u and h satisfying (1.2) on Ω in the following form:

$$u(t,x) = \sum_{(l,m)\in Z\times Z^+} u_{lm} \exp(ilt) \sin(mx)$$

$$h(t,x) = \sum_{(l,m)\in Z\times Z^+} h_{lm} \exp(ilt) \sin(mx)$$

with $u_{lm} = u_{-l-m}$ and $h_{lm} = h_{-l-m}$ since u and h are real, the abstract actions of L and L^{-1} are determined as follows.

$$Lu(t,x) = \sum_{(l,m)\in Z\times Z^+} \left[li + m^2\right] u_{lm} \exp(ilt) \sin(mx) \tag{2.2}$$

and

$$L^{-1}u(t,x) = \sum_{(l,m)\in Z\times Z^+} \left[li + m^2\right]^{-1} h_{lm} \exp(ilt) \sin(mx). \tag{2.3}$$

Since

$$h_{lm} = \frac{1}{2\pi^2} \int\int_\Omega h(s,y) \exp(-ilt) \sin(mx)\, dt\, dx,$$

we can represent L^{-1} as a convolution product.

$$L^{-1}h(t,x) = \int\int_\Omega k(t,s,x,y) h(s,y)\, ds\, dy, \tag{2.4}$$

where

$$k(t,s,xy) = \frac{1}{2\pi^2} \sum_{(l,m)\in Z\times Z^+} \left[li + m^2\right] \exp\left[il(t-s)\right] \sin(mx) \sin(my).$$

Lemma. *The operator $L^{-1} : L^2(\Omega) \to C(\Omega)$ is a compact operator and $\|L^{-1}\|_\infty \leq C\|h\|_{L^2}$ for some constant $C > 0$ independently of h.*

Proof: Using (2.4) and the convergence $\sum_{(l,m)\in Z\times Z^+} [l^2 + m^4]^{-1}$ and Arzela-Ascoli theorem, we can prove our assertion.

$\square$

3. Main results.

Theorem. *Let $h \in L^2(\Omega)$ and suppose that (H_1) is satisfied and*

(H_2) for all $(x,u) \in I \times R$, $ug(x,u) \geq -\psi(|u|)$, where $\psi : R \to R$ is a function such that

$$\limsup_{|u|\to\infty} \psi(u)/u = \alpha_0, \quad \alpha_0 \in R,$$

exists, then the periodic-Dirichlet problem on Ω for the equation (1.1) *has at least one weak solution.*

Proof: Now we see that $L^{-1} : L^2(\Omega) \to C(\Omega)$ is continuous and compact operator as a mapping from $L^2(\Omega)$ into $C(\Omega)$. Define a substitution operator $N : C(\Omega) \to L^2(\Omega)$ by $Nu(t,x) = -g(x,u(t,x)) + h(t,x)$ for all $u \in C(\Omega)$ and $(t,x) \in \Omega$. These definitions have a meaning because $H^{1.2}(\Omega) \cap H^1(\Omega)$ are continuously imbedded in $C(\Omega)$. Then u is weak solution of the periodic-Dirichlet problem for (1.1) if and only if $u \in Dom\, L$ and satisfies

$$Lu = Nu, \qquad \text{or equivalently} \tag{3.1}$$

$$u = L^{-1}Nu. \tag{3.2}$$

If $u \in C(\Omega)$ solves the operator equation (3.2), then $u \in C(\Omega)$ is a weak solution to the periodic-Dirichlet problem. Since L^{-1} is compact, and N is continuous and maps bounded sets into set bounded sets, the composition $L^{-1}N : C(\Omega) \to C(\Omega)$ is continuous and compact. By using Leray-Schauder theory if all solutions u to the family of equations

$$u = \lambda L^{-1}Nu, \quad 0 \leq \lambda \leq 1 \tag{3.3}$$

are bounded in $C(\Omega)$ independently of $\lambda \in [0,1]$, then (3.1) has a solution.

If (u,λ) solves (3.3), then (u,λ) solves

$$Lu = \lambda Nu \tag{3.4}$$

and u is a weak solution to the periodic-Dirichlet problem of the equation

$$u_t - u_{xx} + \lambda g(x,u) = \lambda h(t,x) \ \text{ on } \ \Omega.$$

Thus the proof will be completed if we show that the solutions to (3.4) for $0 \leq \lambda \leq 1$ are bounded in $C(\Omega)$ independently of $\lambda \in [0,1]$. Since, if $\lambda = 0$, we have only the trivial solution $u \equiv 0$, it suffices to show our assertion for $0 \leq \lambda \leq 1$.

To this end, let (u,λ) be any solution to (3.4) with $0 \leq \lambda \leq 1$. By taking the inner product with u_t on both sides of (3.4), we obtain

$$\langle Lu, u_t\rangle + \lambda\langle g(x,u), u_t\rangle = \lambda\langle h, u_t\rangle.$$

Since $u \in Dom\, L$, there exists a sequence $u_n \in C^\infty(\Omega)$, u_n satisfy (1.2), such that the distributional derivatives u_t, u_x, u_{xx} can be obtained as limits in $L^2(\Omega)$ of the corresponding derivatives of u_n which tend to u in $H^{1.2}(\Omega)$. Now integration of these smooth functions, using the boundary conditions, shows that for each $n \in Z^+$, $\langle Lu_n, u_{nt}\rangle = \|u_{nt}\|_{L^2}$. Letting $n \to +\infty$, we obtain $\langle Lu, u_t\rangle = \|u_{nt}\|_{L^2}$.

Moreover, since for each $n \in Z^+$, the periodicity of $u_n(t,x)$ in t implies $\langle g(x,u_n), u_{nt}\rangle = 0$, we also have $\langle g(x,u), u_t\rangle = 0$. Thus

$$\|u_t\|_{L^2} \le \|h\|_{L^2}\,. \tag{3.5}$$

Next we prove that $\|u\|_{L^2} \le M_1$ for some $M_1 > 0$ independently of $[0,1]$. Since $\limsup_{|u|\to\infty} \psi(u)/u = \alpha_0$, for $\alpha \ge 0$ with $\alpha > \alpha_0$, there exists $r_0 > 0$ such that $\psi(u)/u \le \alpha$ with $|u| > r_0$. So $\psi(u) \le \alpha u$ for all $|u| \le r_0$. Thus

$$\int\int_{|u|>r_0} ug(x,u)\,dt\,dx \ge -2^{1/2}\alpha\pi\|u\|_{L^2}\,.$$

Since g is a Carathéodory function on $I \times R$, there exists a constant $M_{r_0} > 0$ such that $|g(x,u)| \le M_{r_0}$ for u with $|u| \le r_0$ for $x \in I$. Hence

$$\Big|\int\int_{|u|\le r_0} ug(x,u)\,dt\,dx\Big| \le 2\pi^2 r_0 M_{r_0}\,.$$

Thus

$$\begin{aligned}\langle g(x,u),u\rangle &= \int\int_{|u|>r_0} ug(x,u)\,dt\,dx + \int\int_{|u|\le r_0} ug(x,u)\,dt\,dx\\ &\ge -2^{1/2}\alpha\pi\|u\|_{L^2} - 2\pi^2 r_0 M_{r_0}\,.\end{aligned}$$

By taking the inner product with u on both sides of (3.4), we have, by the similar argument

$$\|u_{nx}\|^2_{L^2} + \lambda\langle g(x,u),u\rangle = \lambda\langle h,u\rangle$$

since

$$\langle u_t,u\rangle = 0 \quad\text{and}\quad \langle u_{xx},u\rangle = \|u_{nx}\|^2_{L^2}\,.$$

Thus

$$\begin{aligned}\|u_x\|^2_{L^2} &\le -\langle g(x,u),u\rangle + \langle h,u\rangle\\ &\le \big[2^{-1/2}\alpha\pi + \|h\|_{L^2}\big]\|u\|_{L^2} + 2\pi^2 r_0 M_{r_0}\,.\end{aligned} \tag{3.6}$$

But since $\|u\|_{L^2} \le c_1\|u_x\|_{L^2}$ for all $u \in Dom\,L$ and for some $c_1 > 0$,

$$\|u\|^2_{L^2} \le C_1^2\big[2^{-1/2}\alpha\pi + \|h\|_{L^2}\big]\|u\|_{L^2} + 2\pi^2 r_0 M_{r_0}\,.$$

Therefore, there exists a constant M_1 independently of $\lambda \in (0,1]$ such that

$$\|u\|_{L^2} \le M_1\,. \tag{3.7}$$

Hence, by (3.6), there exists a constant M_2 independently of $\lambda \in (0,1]$ such that

$$\|u\|_{L^2} \le M_2\,. \tag{3.8}$$

Hence, we have $u \in H^1(\Omega)$ and $\|u\|_{H^1} \le L_1$ for some constant independently of $\lambda \in (0,1]$. Since $L^2(\Omega) \subseteq L^q(\Omega)$ where $1 \le q \le 2$ and since $H^1(\Omega)$ is embedded in $L^q(\Omega)$ where $2 \le q \le \infty$ (see e.g. [2], [6]), $\|u\|_{L^q} \le L_2(q)$ where $L_2(q)$ may depend on L_1 and $q \ge 1$ but is independent of $\lambda \in (0,1]$.

Next, we will estimate the L^2-bound for $g(\cdot,u)$. For $|u| \le d_0+1$, $x \in I$, we have

$$|g(x,u)| \le \sup_{\substack{x\in I\\ |u|\le d_0+1}} |g(x,u)| \le M_3$$

for some $M_3 > 0$ since g is a Carathéodory function. For $|u| > d_0+1$, $x \in I$, we have, by (H_1),

$$|g(x,u)| = (1/|u|)|ug(x,u)| \le (1/d_0)\big[a(x)|u|^{p+1} + b(x)|u|\big].$$

Therefore, we have

$$\begin{aligned}|g(x,u)| &\le \sup_{\substack{x\in I\\ |u|\le d_0+1}} |g(x,u)| + \sup_{\substack{x\in I\\ |u|> d_0+1}} (1/|u|)|ug(x,u)|\\ &\le (1/d_0)\big[a(x)|u|^{p+1} + b(x)|u|\big] + M_3\,.\end{aligned}$$

Hence

$$\begin{aligned}\|g(\cdot,u)\|_{L^2}^2 \le M_4\|u\|_{L^{2p+2}}^{2p+2} + M_5\|u\|_{L^{p+2}}^{p+2}&\\ + M_6\|u\|_{L^{p+1}}^{p+1} + M_7\|u\|_{L^2}^2 + M_8\|u\|_{L^2} + M_9&\end{aligned}$$

for some appropriate constants M_4, M_9.

Since $\|u\|_{L^q} \le L_2(q)$ for $q \ge 1$, $\|g(\cdot,u)\|_{L^2} \le L_0$ for some L_0 independently of $\lambda \in [0,1]$.

So if (u,λ) is any solution to (3.3), then, by Lemma,

$$\|u\|_\infty \le \|L^{-1}Nu\|_\infty \le C\|g(\cdot,u)\|_{L^2} + \|h\|_{L^2} \le C(L_0 + \|h\|_{L^2})$$

and this completes our proof. □

Corollary. *For any $h \in L^2(\Omega)$ and $F \in L^\infty(I)$ the boundary value problem*

$$\begin{aligned}&u_t - u_{xx} + F(x)\,\mathrm{sgn}\,(u)|u|^p = h(t,x), \quad p > 0\\ &u(0,x) - u(2\pi,x) = u_t(0,x) - u_t(2\pi,x) = 0 \quad \textit{for all} \;\; x \in I\\ &u \mid \partial I = 0\end{aligned}$$

has at least one weak solution.

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