On the uniformly normal structure of Orlicz spaces with Orlicz norm*

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Abstract. We prove that in Orlicz spaces endowed with Orlicz norm the uniformly normal structure is equivalent to the reflexivity.

Keywords: Orlicz spaces, uniformly normal structure

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Closely related to the fixed point theory, the conceptions of normal structure and uniformly normal structure were introduced in Banach spaces [1], [2]. A Banach space X is said to have normal structure provided that for every closed bounded convex subset C of X containing more than one element, there is an element $p \in C$ such that $\sup\{\|p-x\| : x \in C\} < \operatorname{diam}(C)$, X is said to have uniformly normal structure provided that there is a constant $h < 1$ such that for all above C, there is a $p \in C$ with $\sup\{\|p-x\| : x \in C\} < h \operatorname{diam}(C)$.

In 1984, T. Landes found the criterion of normal structure for Orlicz sequence spaces equipped with Luxemburg norm, in light of his work it is easy to get it for Orlicz function spaces [3]. In recent years T. Wang, B. Wang [4] and S. Chen, Y. Duan [5] have investigated it for Orlicz norm. S. Chen and H. Sun recently get the criterion of uniformly normal structure for Orlicz spaces with Luxemburg norm [6]. In this paper we shall discuss it for Orlicz norm.

Let (G, Σ, μ) be a finite non-atomic measure space; $M(u)$ be an N-function and $N(v)$ be its complemented one $N(v) = \max\{u|v| - M(u) : \text{for } u \geq 0\}$; $R_M(x) = \int_G M(x(t))\, d\mu$ be the modular of an element $x(t)$; L_M be the Orlicz space generated by $M(u)$:

$$L_M = \{x(t) : R_M(\lambda x) < \infty, \quad \text{for some} \quad \lambda > 0\}$$

equipped with Orlicz norm

$$\|x\| = \inf_{k>0} \frac{1}{k}(1 + R_M(kx)) \quad (= \sup\{\int_G x(t)y(t)\, d\mu : y(t) \quad \text{with} \quad R_N(y) \leq 1\}),$$

where the infimum is attained, which forms a Banach space.

$M(u)$ is said to satisfy the Δ_2-condition ($M \in \Delta_2$) if for any $u_0 > 0$ and $H > 1$, there is $K > 1$ such that for all $u \geq u_0$, $M(Hu) \leq KM(u)$ [7].

We only discuss Orlicz function spaces because the result is the same in Orlicz sequence spaces. We first introduce several lemmas.

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Lemma 1. *If the Banach space X fails to have the uniformly normal structure, then for an arbitrary integer n and positive number $\varepsilon > 0$, there exist $x_1, \dots, x_{n+1} \in x$ such that*

$$\|x_j\| \leq 1,\ \|x_i - x_j\| \leq 1 \qquad 1 \leq i \leq j \leq n+1$$
$$\|x_{m+1} - \frac{1}{m}\sum_{i=1}^{m} x_i\| > 1 - \varepsilon \qquad m = 1, 2, \dots, n.$$

Proof: It is easy to deduce the result from the definition of the uniformly normal structure. □

Lemma 2. *The following statements are equivalent:*

(1) $M \in \Delta_2$,
(2) *for any $u_0 > 0$, any $\varepsilon > 0$, there is $\ell > 1$ such that* $M(\ell u) \leq (1+\varepsilon)M(u)$ *(for all $u \geq u_0$),*
(3) *for any $v_0 > 0$, any $0 < \alpha < 1$, there is $\delta > 0$ such that* $N(\alpha v) \leq \alpha(1-\delta)N(v)$ *(for all $v \geq v_0$).*

Proof: See [8]. □

Lemma 3. *Suppose $M \in \Delta_2$ and $N \in \Delta_2$, then for an arbitrary $\lambda_0 \in (0, \frac{1}{2})$ and $b > 0$, there exist $\delta > 0$ and $c > 1$ such that when $\lambda_0 \leq \lambda \leq 1 - \lambda_0$ and $|u| \geq b$, for either $uv < 0$ or $|u| \geq c|v|$ it holds*

$$M(\lambda u + (1-\lambda)v) \leq (1-\delta)[\lambda M(u) + (1-\lambda)M(v)].$$

Proof: Since $N \in \Delta_2$, for $b > 0$ and λ_0 there is δ', $0 < \delta' < 1$ such that

$$\frac{M((1-\lambda_0)u)}{(1-\lambda_0)M(u)} \leq 1 - \delta' \quad (\text{for all } \ |u| \geq \frac{\lambda_0}{1-\lambda_0} b).$$

Since $\frac{M(u)}{u}$ is a nondecreasing function, it follows that for all $\lambda \leq 1 - \lambda_0$

$$M(\lambda u) \leq (1-\delta')\lambda M(u) \quad (\text{for all } \ |u| \geq \frac{\lambda_0}{1-\lambda_0} b).$$

By $M \in \Delta_2$, there is $c > 1$ such that for all $|u| \geq b$

$$M((1 + \frac{1-\lambda_0}{c\lambda_0})u) \leq (1+\delta')M(u).$$

Now we shall discuss two cases.

(I) $uv < 0$ and $|u| \geq b$.

If $|\lambda u| \geq |(1-\lambda)v|$, we have

$$M(\lambda u + (1-\lambda)v) \leq M(\lambda u) \leq (1-\delta')\lambda M(u) \leq (1-\delta')(\lambda M(u) + (1-\lambda)M(v)).$$

If $\lambda|u| < |(1-\lambda)v|$, then $|v| \geq \frac{\lambda}{1-\lambda}|u| \geq \frac{\lambda_0}{1-\lambda_0}|u| \geq \frac{\lambda_0}{1-\lambda_0}b$, hence

$$\begin{aligned} M(\lambda u + (1-\lambda)v) &\leq M((1-\lambda)v) \\ &\leq (1-\delta')(1-\lambda)M(v) \leq (1-\delta')(\lambda M(u) + (1-\lambda)M(v)). \end{aligned}$$

(II) $|u| \geq c|v|$ and $|u| \geq b$.

$$\begin{aligned} M(\lambda u + (1-\lambda)v) &\leq M(\lambda(1+\frac{1-\lambda}{c\lambda})u) \leq (1-\delta')\lambda M((1+\frac{1-\lambda}{c\lambda})u) \\ &\leq (1-\delta')\lambda(1+\delta')M(u) = (1-\delta'^2)\lambda M(u) \\ &\leq (1-\delta'^2)(\lambda M(u) + (1-\lambda)M(v)). \end{aligned}$$

Setting $\delta = \delta'^2$, we get the required result. □

Let us come to the main result.

Theorem. *The Orlicz space L_M with Orlicz norm possesses uniformly normal structure if and only if L_M is reflexive, i.e. $M \in \Delta_2$ and $N \in \Delta_2$.*

Proof: Necessity. It is enough to notice that in the class of Banach spaces the uniformly normal structure implies the reflexivity [2].

Sufficiency. We shall prove it in five steps.

1. Find a finite set in which the distance of arbitrary two elements is near to one.

Denote $\overline{k} = \sup\{k_x : \frac{1}{2} \leq \|x\| \leq 1 \text{ where } \|x\| = \frac{1}{k}(1 + R_M(k_x x))\}$,
$\sigma = \inf\{R_M(x) : \frac{1}{2} \leq \|x\| \leq 1\}$.

By $M \in \Delta_2$ and $N \in \Delta_2$, it follows that $\overline{k} < \infty$ and $\sigma > 0$ [9].

Pick $a > 0$ with $M(2a)\mu G < \frac{\sigma}{4}$.

By $M \in \Delta_2$, it follows that there is $d > 0$ such that

$$M(2u) \leq dM(u), \quad |u| \geq a.$$

Pick $b > 0$ with $M(b)\mu G < \frac{\sigma}{8d}$.

Applying Lemma 3 to b and $\frac{1}{1+\overline{k}^2}$, we have that there exist $\delta > 0$ and $c > 1$ such that for all λ with $\frac{1}{1+\overline{k}^2} \leq \lambda \leq \frac{\overline{k}^2}{1+\overline{k}^2}$ and all u, v with $|u| \geq b$ such that either $|u| \geq c|v|$ or $uv < 0$, it holds

$$M(\lambda u + (1-\lambda)v) \leq (1-\delta)(\lambda M(u) + (1-\lambda)M(v)).$$

Pick a positive integer $p > 32dc^2\overline{k}^2/\sigma$ and $n = 4p$.

Suppose that L_M fails to have the uniformly normal structure. Then by Lemma 1, we deduce that for $0 < \varepsilon < \frac{\delta\sigma}{4n^2 d}$, there exist x_i $(i = 1, \ldots, n+1)$ with $\|x_i\| \leq 1$, $\|x_i - x_j\| \leq 1$ and $\|x^{m+1} - \frac{1}{m}\sum_{i=1}^m x_i\| \geq 1 - \varepsilon$ $(m = 1, 2, \ldots, n)$. Thus $\sum_{i=1}^m \|x_{m+1} - x_i\| > m(1-\varepsilon)$, hence $\|x_{m+1} - x_i\| > 1 - m\varepsilon > \frac{1}{2}(m+1 \neq i)$.

2. Establish the inequality $\sum_{s=1}^{2p} \int_{A_s} (M(v_s(t)) + M(v_{p+s}(t)))\, d\mu < \frac{\sigma}{4d}$ (the meaning of symbols will be given below).

Set $x_{n+1}(t) - x_i(t) = u_i(t)$ $i = 1, 2, \ldots, n$.

For each $t \in G$, rearrange $\{u_i(t)\}_{i=1}^n$ from the smallest to the largest and denote as $v_1(t) \le v_2(t) \le \cdots \le v_n(t)$. Set $v(t) = \frac{1}{2}(v_{2p}(t) + v_{2p+1}(t))$. Define

$$A = \{t \in G: \ \text{for at least } 2p \text{ `}i\text{'} \ : u_i(t)v(t) < 0 \ \text{ or } \ |u_i(t)| > \overline{k}c|v(t)| \ \text{ or } \ |u_i(t)| < |v(t)|/\overline{k}c\}.$$

When $t \in A$, for $s = 1, \ldots, 2p$,

$$(*) \qquad v_s(t)v_{2p+s}(t) < 0 \ \text{ or } \ |v_s(t)| > \overline{k}c|v_{2p+s}(t)| \ \text{ or } \ |v_s(t)| < |v_{2p+s}(t)|/\overline{k}c.$$

In fact, suppose that $(*)$ fails to hold for some s, $1 \le s \le 2p$. Since $\{v_s(t)\}_{s=1}^n$ is not decreasing with respect to $s, v_s(t), v_{s+1}(t), \ldots, v_{2p+s}(t)$ and also $v(t)$ have the same sign, assumed to be positive without loss of generality. Therefore, from $v(t) \ge v_s(t) \ge v_{2p+s}(t)/\overline{k}c \ge v(t)/\overline{k}c$, we derive

$$\frac{v(t)}{\overline{k}c} \le v_s \overbrace{(t) \le v_{s+1}(t) \le \cdots \le v_{2p+s}}^{2p+1}(t) \le \overline{k}cv(t).$$

Combined with the definition of A, we get $t \notin A$. Set

$$A_s = \{t \in A\colon \ \text{either} \ |v_s(t)| > b \ \text{ or } \ |v_{2p+s}(t)| > b\} \quad (s = 1, \ldots, 2p),$$

$$\frac{1}{k_i}(1 + R_M(k_i u_i)) = \|u_i\| \quad (i = 1, 2, \ldots, n), \quad k = n/(\sum_{i=1}^n \frac{1}{k_i}),$$

$$\prod_{\substack{\mathrm{j=1} \\ j \ne i}}^{\mathrm{n}} k_j / \sum_{i=1}^n \prod_{\substack{\mathrm{j=1} \\ j \ne i}}^{\mathrm{n}} k_j = \lambda_i = \frac{k}{nk_i}.$$

Notice that $\frac{1}{2} \le \|u_i\| \le 1$, so that $1 < k_i \le \overline{k}$ and $\frac{1}{1+(n-1)\overline{k}} \le \lambda_i \le \frac{\overline{k}}{n-1+\overline{k}}$.

Define k'_i and λ'_i as $k'_i(t) = k_j$ and $\lambda'_i(t) = \lambda_j$ if $v_i(t) = u_j(t)$. Notice that when $t \in A$, $v_i(t)v_{2p+i}(t) < 0$ or $|k'_i(t)v_i(t)| \ge |v_i(t)| \ge \overline{k}c|v_{2p+i}(t)| \ge c|k'_{2p+i}(t)v_{2p+i}(t)|$ or $|k'_{2p+i}(t)v_{2p+i}(t)| \ge |v_{2p+i}(t)| > \overline{k}c|v_i(t)| \ge c|k'_i(t)v_i(t)|$, we have

$$\begin{aligned}
\varepsilon = 1 - (1 - \varepsilon) &\ge \frac{1}{n}\sum_{i=1}^n \|x_{n+1} - x_i\| - \|x_{n+1} - \frac{1}{n}\sum_{i=1}^n x_i\| \\
&= \frac{1}{n}\sum_{i=1}^n \|u_i\| - \|\frac{1}{n}\sum_{i=1}^n u_i\| \ge \frac{1}{n}\sum_{i=1}^n \frac{1}{k_i}(1 + R_M(k_i u_i)) - \frac{1}{k}(1 + R_M(\frac{k}{n}\sum_{i=1}^n u_i)) \\
&= \frac{1}{k}(\sum_{i=1}^n \lambda_i R_M(k_i u_i) - R_M(\sum_{i=1}^n \lambda_i k_i u_i))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{k}\int_G [\sum_{i=1}^{n} \lambda_i M(k_i u_i(t)) - M(\sum_{i=1}^{n} \lambda_i k_i u_i(t))]\, d\mu \\
&= \frac{1}{k}\int_G \{\sum_{i=1}^{n} \lambda_i'(t) M(k_i'(t) v_i(t)) - M(\sum_{i=1}^{n} \lambda_i'(t) k_i'(t) v_i(t))\}\, d\mu \\
&\geq \frac{1}{k}\int_G \{\sum_{s=1}^{2p} [\lambda_s'(t) M(k_s'(t) v_s(t)) + \lambda_{2p+s}'(t) M(k_{2p+s}'(t) v_{2p+s}(t))] \\
&\quad - \sum_{s=1}^{2p} (\lambda_s'(t) + \lambda_{2p+s}'(t)) M(\frac{\lambda_s'(t)}{\lambda_s'(t) + \lambda_{2p+s}'(t)} k_s'(t) v_s(t) \\
&\quad + \frac{\lambda_{2p+s}'(t)}{\lambda_s'(t) + \lambda_{2p+s}'(t)} k_{2p+s}'(t) v_{2p+s}(t))\}\, d\mu \\
&= \frac{1}{k}\sum_{s=1}^{2p} \{\int_G [\lambda_s'(t) M(k_s'(t) v_s(t)) + \lambda_{2p+s}'(t) M(k_{2p+s}'(t) v_{2p+s}(t)) \\
&\quad - (\lambda_s'(t) + \lambda_{2p+s}'(t)) M(\frac{\lambda_s'(t)}{\lambda_s'(t) + \lambda_{2p+s}'(t)} k_s'(t) v_s(t) \\
&\quad + \frac{\lambda_{2p+s}'(t)}{\lambda_s'(t) + \lambda_{2p+s}'(t)} k_{2p+s}'(t) v_{2p+s}(t))]\, d\mu\} \\
&\geq \frac{1}{k}\sum_{s=1}^{2p} \{\int_{A_s} [\lambda_s'(t) M(k_s'(t) v_s(t)) + \lambda_{2p+s}'(t) M(k_{2p+s}'(t) v_{2p+s}(t)) \\
&\quad - (\lambda_s'(t) + \lambda_{2p+s}'(t)) M(\frac{\lambda_s'(t)}{\lambda_s'(t) + \lambda_{2p+s}'(t)} k_s'(t) v_s(t) \\
&\quad + \frac{\lambda_{2p+s}'(t)}{\lambda_s'(t) + \lambda_{2p+s}'(t)} k_{2p+s}'(t) v_{2p+s}(t))]\, d\mu\} \\
&\geq \frac{1}{k}\sum_{s=1}^{2p} \{\int_{A_s} [\lambda_s'(t) M(k_s'(t) v_s(t)) + \lambda_{2p+s}'(t) M(k_{2p+s}'(t) v_{2p+s}(t)) \\
&\quad - (1-\delta)(\lambda_s'(t) M(k_s'(t) v_s(t)) + \lambda_{2p+s}'(t) M(k_{2p+s}'(t) v_{2p+s}(t)))]\, d\mu\}
\end{aligned}
$$

which follows because of $\frac{1}{1+\overline{k}^2} \leq \frac{\lambda_i}{\lambda_i+\lambda_j} \leq \frac{\overline{k}^2}{1+\overline{k}^2}$.

Notice that $\lambda_i k_i = \frac{k}{n}$ and $k_i \geq 1$; we continuously have

$$
\begin{aligned}
\varepsilon &\geq \frac{\delta}{k}\sum_{s=1}^{2p} \{\int_{A_s} \lambda_s'(t) M(k_s'(t) v_s(t)) + \lambda_{2p+s}'(t) M(k_{2p+s}'(t) v_{2p+s}(t))\, d\mu\} \\
&\geq \frac{\delta}{n}\sum_{s=1}^{2p} \int_{A_s} [M(v_s(t)) + M(v_{2p+s}(t))]\, d\mu.
\end{aligned}
$$

From the choice of ε, we get

$$\sum_{s=1}^{2p}\int_{A_s}[M(v_s(t))+M(v_{2p+s}(t))]\,d\mu\le\frac{n\varepsilon}{\delta}<\frac{\sigma}{4d}\,.$$

3. Establish the inequality $R_M(\frac{x_2-x_1}{2}\chi_B)\ge\frac{3\sigma}{8d}$ where $B=G\setminus A$. By $\|x_2-x_1\|\ge\frac{1}{2}$, we derive $R_M(x_2-x_1)\ge\sigma$. Hence

$$\begin{aligned}\sigma\le R_M(x_2-x_1)&\le\int_{G(|x_2(t)-x_1(t)|\ge 2a)}M(x_2(t)-x_1(t))\,d\mu\\&\quad+\int_{G(|x_2(t)-x_1(t)|<2a)}M(x_2(t)-x_1(t))\,d\mu\\&\le dR_M(\frac{x_2-x_1}{2})+\frac{\sigma}{4}\,,\end{aligned}$$

so

$$R_M(\frac{x_2-x_1}{2})\ge\frac{3\sigma}{4d}\,.$$

Set $D'=\{t\in A\colon |u_1(t)|>b\}$, $D''=\{t\in A\colon |u_2(t)|>b\}$; we have

$$\begin{aligned}&\int_A M(\frac{x_2(t)-x_1(t)}{2})\,d\mu\le\frac{1}{2}\int_A[M(u_1(t))+M(u_2(t))]\,d\mu\\&\le\frac{1}{2}\int_{D'}M(u_1(t))\,d\mu+\frac{1}{2}\int_{D''}M(u_2(t))\,d\mu+\frac{\sigma}{8d}\\&\le\sum_{s=1}^{2p}\int_{A_s}[M(v_s(t))+M(v_{2p+s}(t))]\,d\mu+\frac{\sigma}{8d}<\frac{\sigma}{4d}+\frac{\sigma}{8d}=\frac{3\sigma}{8d}\,.\end{aligned}$$

Hence

$$R_M(\frac{x_2-x_1}{2}\chi_B)=R_M(\frac{x_2-x_1}{2})-R_M(\frac{x_2-x_1}{2}\chi_A)\ge\frac{3\sigma}{4d}-\frac{3\sigma}{8d}=\frac{3\sigma}{8d}\,.$$

4. Establish $\int_{\widetilde{B}}M(x'(t)-x_1(t))\,d\mu\ge\frac{3\sigma}{16d}$
(the meaning of symbols will be given below).

Split B into the following parts:

$B_4=\{t\in B: |x_4(t)-x_3(t)|\le\frac{\overline{k}c}{p}|v(t)|\}$,

$B_5=\{t\in B\setminus B_4: |x_5(t)-x_i(t)|\le\frac{\overline{k}c}{p}|v(t)|$ for some i, $3\le i<5\}$,

… …

$B_n=\{t\in B\setminus\bigcup_{j=4}^{n-1}B_j: |x_n(t)-x_i(t)|\le\frac{\overline{k}c}{p}|v(t)|$ for some i, $3\le i<n\}$.

There is $B = B_4 \cup B_5 \cup \cdots \cup B_n$. Indeed, if $t \in B \setminus \bigcup_{j=4}^{n} B_j$, it follows that

$$|x_i(t) - x_j(t)| = |u_i(t) - u_j(t)| \geq \overline{k}c|v(t)|/p \quad (i = 4, 5, \ldots, n;\ j = 3, \ldots, i-1).$$

While there are q 'i' with $u_i(t)v(t) < 0$, there are $4p - q - 2$ 'i' with $\{u_i(t)\}$ having the same sign as $v(t)$. Therefore there are $3p - q - 2$ 'i' satisfying $|u_i(t) - u_{i_0}(t)| > \overline{k}c|v(t)|$, where $u_{i_0}(t)$ is the smallest one with respect to the absolute value, so for such i, $|u_i(t)| > \overline{k}c|v(t)|$. Notice that for such t, there are $3p - q - 2 + q = 3p - 2 > 2p$ 'i' with $u_i(t)v(t) < 0$ or $|u_i(t)| > \overline{k}c|v(t)|$, thus we get $t \in A$, which contradicts the fact $t \in B$.

Define

$$x'(t) = \begin{cases} 0 & t \in A, \\ x_m(t) & t \in B_m \qquad m = 4, 5, \ldots, n, \end{cases}$$

then $x'(t)$ is μ-measurable, and we have

$$\frac{1}{2}[R_M((x' - x_1)\chi_B) + R_M((x' - x_2)\chi_B)] \geq R_M(\frac{x_2 - x_1}{2}\chi_B) \geq \frac{3\sigma}{8d}.$$

Without loss of generality, we assume that $R_M((x' - x_1)\chi_B) \geq \frac{3\sigma}{8d}$. Set

$$\widetilde{B} = \{t \in B : |x'(t) - x_1(t)| > \max(\frac{c^2\overline{k}^2}{p}|v(t)|, b)\}.$$

Notice that fact that $|v(t)| \leq \frac{2}{n}\sum_{i=1}^{n}|v_i(t)|$; indeed, when $|v_{2p}(t)| \leq |v_{2p+1}(t)|$, then $v_{2p+1}(t) > 0$, so

$$\begin{aligned} |v(t)| &\leq \frac{1}{2}(|v_{2p}(t)| + |v_{2p+1}(t)|) \leq |v_{2p+1}(t)| \\ &\leq \frac{|v_{2p+1}(t)| + \cdots + |v_n(t)|}{n/2} = \frac{2(|v_{2p+1}(t)| + \cdots + |v_n(t)|)}{n} \\ &\leq \frac{2(|v_1(t)| + \cdots + |v_n(t)|)}{n}. \end{aligned}$$

The argument is analogous to that when $|v_{2p}(t)| > |v_{2p+1}(t)|$. Thus we derive

$$\begin{aligned} &\int_{B\setminus\widetilde{B}} M(x'(t) - x_1(t))\,d\mu \leq M(b)\mu G + \int_G M(\frac{c^2\overline{k}^2}{p}v(t))\,d\mu \\ &\leq \frac{\sigma}{8d} + \int_G M(\frac{c^2\overline{k}^2}{p}\frac{2(|v_1(t)| + |v_2(t)| + \cdots + |v_n(t)|)}{n})\,d\mu \\ &\leq \frac{\sigma}{8d} + \frac{2c^2\overline{k}^2}{p}\int_G M(\frac{|u_1(t)| + |u_2(t)| + \cdots + |u_n(t)|}{n})\,d\mu \\ &\leq \frac{\sigma}{8d} + \frac{2c^2\overline{k}^2}{p} < \frac{\sigma}{8d} + \frac{\sigma}{16d} = \frac{3\sigma}{16d}, \end{aligned}$$

so

$$\int_{\widetilde{B}} M(x'(t)-x_1(t))\,d\mu \geq \int_B M(x'(t)-x_1(t))\,d\mu - \int_{B\setminus\widetilde{B}} M(x'-x_1)\,d\mu$$
$$\geq \frac{3\sigma}{8d} - \frac{3\sigma}{16d} = \frac{3\sigma}{16d}\,.$$

Set $\widetilde{B}_m = \widetilde{B}\cap B_m$. Then $\widetilde{B} = \bigcup_{m=4}^{n} \widetilde{B}_m$.

5. Prove $\int_{\widetilde{B}} M(x'(t)-x_1(t))\,d\mu < \frac{3\sigma}{16d}$; this implies a contradiction:

Split $\widetilde{B}_m$ precisely into the following parts $(m=4,5,\dots,n)$

$\widetilde{B}_m^3 = \{t\in \widetilde{B}_m : |x_3(t)-x_m(t)| \leq \frac{\overline{k}c}{p}|v(t)|\}$,

$\widetilde{B}_m^4 = \{t\in \widetilde{B}_m\setminus \widetilde{B}_m^3 : |x_4(t)-x_m(t)| \leq \frac{\overline{k}c}{p}|v(t)|\}$,

$\dots \quad \dots$

$\widetilde{B}_m^{m-1} = \{t\in \widetilde{B}_m\setminus \bigcup_{i=3}^{m-2}\widetilde{B}_m^i : |x_{m-1}(t)-x_m(t)| \leq \frac{\overline{k}c}{p}|v(t)|\}$.

Then $\widetilde{B}_m = \bigcup_{i=3}^{m-1}\widetilde{B}_m^i$.

Notice that for $t\in\widetilde{B}_m^i$,

$$(**)\qquad \begin{aligned} &|x_m(t)-x_1(t)| = |x'(t)-x_1(t)| \geq b,\\ &|x_m(t)-x_1(t)| = |x'(t)-x_1(t)| \geq \frac{\overline{k}^2c^2}{p}|v(t)| \geq \overline{k}c|x_m(t)-x_i(t)|.\end{aligned}$$

Define

$$k_m^i : \|x_m-x_i\| = \frac{1}{k_m^i}(1+R_M(k_m^i(x_m-x_i))) \qquad (i=1,\dots,m-1),$$
$$\widetilde{k}_m = (m-1)/(\sum_{j=1}^{m-1} 1/k_m^j) \qquad (m=4,\dots,n),$$
$$\lambda_m^i = \prod_{\substack{\text{j=1}\\ j\neq i}}^{\text{m-1}} k_m^j \Big/ \sum_{i=1}^{m-1}\prod_{\substack{\text{j=1}\\ j\neq i}}^{\text{m-1}} k_m^j = \widetilde{k}_m/(m-1)k_m^i.$$

For $t\in\widetilde{B}_m^i$, $k_m^i|x_m(t)-x_1(t)| \geq \overline{k}c|x_m(t)-x_i(t)| \geq c|k_m^i(x_m(t)-x_i(t))|$; we have

$$\varepsilon = 1-(1-\varepsilon) \geq \frac{1}{m-1}\sum_{i=1}^{m-1}\|x_m-x_i\| - \|x_m - \frac{1}{m-1}\sum_{i=1}^{m-1}x_i\|$$
$$\geq \frac{1}{m-1}\sum_{i=1}^{m-1}\frac{1}{k_m^i}(1+R_M(k_m^i(x_m-x_i))) - \frac{1}{\widetilde{k}_m}(1+R_M(\widetilde{k}_m\sum_{i=1}^{m-1}\frac{x_m-x_i}{m-1}))$$

$$
\begin{aligned}
&= \frac{1}{\widetilde{k}_m} \int_G [\sum_{i=1}^{m-1} \lambda_m^i M(k_m^i(x_m(t) - x_i(t))) - M(\sum_{i=1}^{m-1} \lambda_m^i k_m^i(x_m(t) - x_i(t)))]\, d\mu \\
&\geq \frac{1}{\widetilde{k}_m} \int_{\widetilde{B}_m} [\sum_{i=1}^{m-1} \lambda_m^i M(k_m^i(x_m(t) - x_i(t))) - M(\sum_{i=1}^{m-1} \lambda_m^i k_m^i(x_m(t) - x_i(t)))]\, d\mu \\
&= \frac{1}{\widetilde{k}_m} \sum_{j=3}^{m-1} \int_{\widetilde{B}_m^j} [\sum_{i=1}^{m-1} \lambda_m^i M(k_m^i(x_m(t) - x_i(t))) - M(\sum_{i=1}^{m-1} \lambda_m^i k_m^i(x_m(t) - x_i(t)))]\, d\mu \\
&\geq \frac{1}{\widetilde{k}_m} \sum_{j=3}^{m-1} \int_{\widetilde{B}_m^j} \{\sum_{i=1}^{m-1} \lambda_m^i M(k_m^i(x_m(t) - x_i(t))) - \sum\nolimits_{\substack{i=2\\ i\neq j}}^{\text{m-1}} \lambda_m^i M(k_m^i(x_m(t) - x_i(t))) \\
&\quad - (1-\delta)(\lambda_m^1 M(k_m^1(x_m(t) - x_1(t))) + \lambda_m^j M(k_m^j(x_m(t) - x_j(t))))\}\, d\mu,
\end{aligned}
$$

which follows for the same fact as in **2**. Continuing the computation, we have

$$
\begin{aligned}
\varepsilon &\geq \frac{\delta}{\widetilde{k}_m} \sum_{j=3}^{m-1} \int_{\widetilde{B}_m^j} [\lambda_m^1 M(k_m^1(x_m(t) - x_1(t))) + \lambda_m^j M(k_m^j(x_m(t) - x_j(t)))]\, d\mu \\
&\geq \frac{\delta}{m-1} \sum_{j=3}^{m-1} \int_{\widetilde{B}_m^j} M(x_m(t) - x_1(t))\, d\mu = \frac{\delta}{m-1} \int_{\widetilde{B}_m} M(x_m(t) - x_1(t))\, d\mu,
\end{aligned}
$$

hence

$$
\int_{\widetilde{B}_m} M(x_m(t) - x_1(t))\, d\mu \leq \frac{(m-1)\varepsilon}{\delta} \qquad (m = 4, 5, \ldots, n).
$$

We obtain

$$
\begin{aligned}
\int_{\widetilde{B}} M(x'(t) - x_1(t))\, d\mu &= \int_{\bigcup_{m=4}^n \widetilde{B}_m} M(x'(t) - x_1(t))\, d\mu \\
&= \sum_{m=4}^{n} \int_{\widetilde{B}_m} M(x_m(t) - x_1(t))\, d\mu \leq \frac{\varepsilon n^2}{2\delta} \leq \frac{\sigma}{8d} < \frac{3\sigma}{16d}
\end{aligned}
$$

which yields a contradiction to

$$
\int_{\widetilde{B}} M(x'(t) - x_1(t))\, d\mu \geq \frac{3\sigma}{16d},
$$

and the proof is completed. □

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References

[1] Belluce L.P., Kirk W.A., Steiner E.F., *Normal structure in Banach spaces*, Pacific J. Math. **26** (1968), 433–440.

[2] Bae J.S., *Reflexivity of a Banach space with a uniformly normal structure*, Proc. Amer. Math. Soc. **90** (1984), 269–270.

[3] Landes T., *Normal structure and weakly normal structure of Orlicz sequence spaces*, Trans. AMS **285** (1984), 523–533.

[4] Tingfu Wang, Baoxiang Wang, *Normal structure ad sum-property and LD property of Orlicz spaces*, J. Math. Res. Exp. **12** (1992), 477–478.

[5] Shutao Chen, Yanzheng Duan, *Normal structure and weakly normal structure of Orlicz spaces*, Comment. Math. Univ. Carolinae **32** (1991), 219–225.

[6] Shutao Chen, Huiying Sun, *Reflexive Orlicz spaces have uniformly normal structure*, to appear.

[7] Krasnoselskii M.A., Rutickii Ya.B., *Convex Functions and Orlicz Spaces*, Moscow, 1958.

[8] Wu C., Wang T., Chen S., Wang Y., *Geometry of Orlicz Spaces*, H.I.T. Press, Harbin, 1986.

[9] Shutao Chen, *Some rotundities of Orlicz spaces with Orlicz norm*, Bull. Polish Academy Sci. Math. **34** (1986), 585–596.

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