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Some remarks on the regularity of minimizers of integrals with anisotropic growth

Tilak Bhattacharya, Francesco Leonetti

Abstract. We prove higher integrability for minimizers of some integrals of the calculus of variations; such an improved integrability allows us to get existence of weak second derivatives.

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0. Introduction.

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$; u be such that $u : \Omega \to \mathbb{R}^N$, $N \geq 1$. Consider the integral functional

$$I(u) = \int_\Omega F(Du(x))\, dx, \tag{0.1}$$

where F satisfies an anisotropic growth condition, namely

$$a \sum_{i=1}^{n} |\xi_i|^{q_i} - b \leq F(\xi) \leq c \sum_{i=1}^{n} |\xi_i|^{q_i} + d, \tag{0.2}$$

$\forall\, \xi \in \mathbb{R}^{nN}$. Here a, b, c and d are positive constants and $1 \leq q_i$, $i = 1, \dots, n$. It is well known that the standard results of the isotropic case, i.e. $q_i = q$, $i = 1, \dots, n$, fail to hold if the q_i's are too far apart [10], [14], [15]. The main aim of this paper is to show that under some restrictions on the q_i's, an improved integrability result holds for minimizers u of (0.1) verifying (0.2) and some additional restrictions. The prototype for our work is the integral

$$I(u) = \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{n-1} |D_i u(x)|^2 + \frac{1}{p}(1 + |D_n u(x)|^2)^{p/2} \right) dx, \tag{0.3}$$

where $Du = (D_1 u, \dots, D_n u)$ and $1 < p < 2$, for which (0.2) holds with $q_1 = \cdots = q_{n-1} = 2$ and $q_n = p$. We have arranged our work as follows. In Section 1 we state the main result, Section 2 contains some preliminaries while Sections 3 and 4 deal with the proofs of the results of the paper.

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1. Notation and main results.

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, u be a vector-valued function, $u : \Omega \to \mathbb{R}^N$, $N \geq 1$; we consider integrals

$$\int_\Omega F(Du(x))\,dx, \tag{1.1}$$

based on (0.3). More precisely, we assume that $F : \mathbb{R}^{nN} \to \mathbb{R}$ is in $C^2(\mathbb{R}^{nN})$ and satisfies, for some positive constants c, m, M, p,

$$|F(\xi)| \leq c(1 + \sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^p); \tag{1.2}$$

$$|\frac{\partial F}{\partial \xi_i^\alpha}(\xi)| \leq c(1 + \sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^p)^{1/2} \quad \text{if } \ i = 1, \dots, n-1; \tag{1.3}$$

$$|\frac{\partial F}{\partial \xi_n^\alpha}(\xi)| \leq c(1 + \sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^p)^{1-1/p}; \tag{1.4}$$

and

$$\begin{aligned} m\Big(\sum_{i=1}^{n-1} |\lambda_i|^2 + (1 + |\xi_n|^2)^{(p-2)/2} |\lambda_n|^2\Big) &\leq \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 F}{\partial \xi_j^\beta \partial \xi_i^\alpha}(\xi) \lambda_i^\alpha \lambda_j^\beta \\ &\leq M\Big(\sum_{i=1}^{n-1} |\lambda_i|^2 + (1 + |\xi_n|^2)^{(p-2)/2} |\lambda_n|^2\Big), \end{aligned} \tag{1.5}$$

for every $\lambda, \xi \in \mathbb{R}^{nN}$. Here, $\lambda = \{\lambda_i^\alpha\}$, $\xi = \{\xi_i^\alpha\}$, $|\lambda_i|^2 = \sum_{\alpha=1}^N |\lambda_i^\alpha|^2$, etc. About p, we assume that

$$1 < p < 2. \tag{1.6}$$

We remark that the integrand of (0.3) satisfies (1.2), … ,(1.5). We say that u minimizes the integral (1.1) if $u : \Omega \to \mathbb{R}^N$, $u \in W^{1,p}(\Omega)$ with $D_i u \in L^2(\Omega)$, $i = 1, \dots, n-1$, and for every $\phi : \Omega \to \mathbb{R}^N$ with $\phi \in W_0^{1,p}(\Omega)$ and $D_i \phi \in L^2(\Omega)$, $i = 1, \dots, n-1$, we have

$$I(u) \leq I(u + \phi). \tag{1.7}$$

We have the following regularity results.

Theorem 1. *Let $u : \Omega \to \mathbb{R}^N$ satisfy $u \in W^{1,p}(\Omega) \cap L^2(\Omega)$ with $D_i u \in L^2(\Omega)$, $i = 1, \dots, n-1$, where*

$$1 < p < 2 \qquad \textit{if } \ n = 2, 3; \tag{1.8}$$

$$98/97 < p < 2 \qquad \textit{if } \ n = 4; \tag{1.9}$$

and

$$2 - 4/n < p < 2 \qquad \text{if } \; n \geq 5. \tag{1.10}$$

If F *satisfies* (1.2), ... , (1.5) *and* u *minimizes the integral* (1.1) *in the sense of* (1.7), *then*

$$D_n u \in L^2_{\mathrm{loc}}(\Omega). \tag{1.11}$$

This result of higher integrability implies the following improved differentiability.

Corollary 1. *Under the assumptions of Theorem* 1, *we obtain the existence of the weak second derivatives. Furthermore,*

$$D_i Du \in L^2_{\mathrm{loc}}(\Omega), \quad i = 1, \dots, n-1 \;\text{ and }\; D_n Du \in L^p_{\mathrm{loc}}(\Omega).$$

Remark 1. We prove Theorem 1 by employing a technique in [6]. The idea is to gain a fractional order derivative of Du thereby improving its integrability. Also see [4], [7], [13].

Remark 2. It is not clear to us whether the restriction $2 - 4/n < p$ is a consequence of the technique we have used. We are unable to prove or disprove Theorem 1 outside this range. It must be mentioned that the same restriction was arrived at in a slightly different context in the work [7].

Remark 3. It is to be noted that local boundedness of scalar valued minimizers has been proved without any restrictions on p from below [8], [9].

2. Preliminaries.

For a vector-valued function $f(x)$, define the difference

$$\tau_{s,h} f(x) = f(x + h e_s) - f(x),$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction, and $s = 1, 2, \dots, n$. For $x_0 \in \mathbb{R}^n$, let $B_R(x_0)$ be the ball centered at x_0 with radius R. We will often suppress x_0 whenever there is no danger of confusion. We now state several lemmas that are crucial to our work. In the following $f : \Omega \to \mathbb{R}^k$, $k \geq 1$; B_R, B_{2R} and B_{3R} are concentric balls.

Lemma 2.1. *If* $f, D_s f \in L^t(B_{3R})$ *with* $1 \leq t < \infty$ *then*

$$\int_{B_R} |\tau_{s,h} f(x)|^t \, dx \leq |h|^t \int_{B_{2R}} |D_s f(x)|^t \, dx,$$

for every h *with* $|h| < R$. (*See* [11, p. 45], [5, p. 28].)

Lemma 2.2. *Let $f \in L^t(B_{2R})$, $1 < t < \infty$; if there exists a positive constant C such that*

$$\int_{B_R} |\tau_{s,h} f(x)|^t \, dx \leq C|h|^t,$$

for every h with $|h| < R$, then there exists $D_s f \in L^t(B_R)$. (*See* [11, p. 45], [5, p. 26].)

Lemma 2.3. *If $f \in L^2(B_{3R})$ and for some $d \in (0,1)$ and $C > 0$*

$$\sum_{s=1}^{n} \int_{B_R} |\tau_{s,h} f(x)|^2 \, dx \leq C|h|^{2d},$$

for every h with $|h| < R$, then $f \in L^r(B_{R/4})$ for every $r < 2n/(n-2d)$.

Proof: The previous inequality tells us that $f \in W^{b,2}(B_{R/2})$ for every $b < d$, so we can apply the imbedding theorem for fractional Sobolev spaces [3, Chapter VII]. □

Lemma 2.4. *For every t with $1 \leq t < \infty$ there exists a positive constant C such that*

$$\int_{B_R} |\tau_{s,h} f(x)|^t \, dx \leq C \int_{B_{2R}} |f(x)|^t \, dx,$$

for every $f \in L^t(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \ldots, n$.

Lemma 2.5 (Anisotropic Sobolev imbedding theorem)**.** *If $q_i \geq 1$, $i = 1, \ldots, n$, we assume that $f \in W^{1,1}(Q)$ and $f, D_i f \in L^{q_i}(Q)$, $\forall\, i = 1, \ldots, n$, where $Q \subset \mathbb{R}^n$ is a cube with faces parallel to the coordinate planes. Define $\overline{q}$ by*

$$\frac{1}{\overline{q}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{q_i} \quad \text{and set} \quad \overline{q}^{\,*} = \begin{cases} n\overline{q}/(n-\overline{q}), & \text{if } \overline{q} < n, \\ \text{any number}, & \text{if } \overline{q} \geq n. \end{cases}$$

If $q_i < \overline{q}^{\,*}$, $\forall\, i = 1, \ldots, n$, then $f \in L^{\overline{q}^{\,*}}(Q)$. (*See* [16], [1].)

Now we state some basic inequalities.

Lemma 2.6. *For every $\gamma \in (-1/2, 0)$ we have*

$$1 \leq \frac{\int_0^1 (1 + |b + t(a-b)|^2)^\gamma \, dt}{(1 + |a|^2 + |b|^2)^\gamma} \leq \frac{8}{2\gamma + 1},$$

for all $a, b \in \mathbb{R}^k$. (*See* [2].)

Lemma 2.7. *For every $\gamma \in (-1/2, 0)$ we have*

$$(2\gamma + 1)|a - b| \leq \frac{|(1 + |a|^2)^\gamma a - (1 + |b|^2)^\gamma b|}{(1 + |a|^2 + |b|^2)^\gamma} \leq \frac{c(k)}{2\gamma + 1} |a - b|,$$

for all $a, b \in \mathbb{R}^k$. (*See* [2].)

3. Proof of Theorem 1.

Since u minimizes the integral (1.1) with growth conditions as in (1.2), ... ,(1.4), u solves the Euler equation,

$$\int_\Omega \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial F}{\partial \xi_i^\alpha}(Du(x)) D_i \phi^\alpha(x)\, dx = 0, \tag{3.1}$$

for all functions $\phi : \Omega \to \mathbb{R}^N$, with $\phi \in W_0^{1,p}(\Omega)$ and $D_1\phi, \dots, D_{n-1}\phi \in L^2(\Omega)$. Let $R > 0$ be such that $\overline{B_{3R}} \subset \Omega$ and let B_ϱ and B_R be concentric balls, $0 < \varrho < R \le 1$. Fix s, take $0 < |h| < R$ and let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a "cut off" function in $C_0^1(B_R)$ with

$$\eta \equiv 1 \ \text{ on } \ B_\varrho, \quad 0 \le \eta \le 1 \ \text{ and } \ |D\eta| \le c/(R-\varrho).$$

Using $\phi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$ in (3.1), via a standard reduction, we get the following Caccioppoli estimate, i.e. for some positive constants $C_0 = C_0(n, N, p, m, M)$,

$$\begin{aligned}
&\int_{B_\varrho} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u(x)|^2\, dx \\
&\qquad + \int_{B_\varrho} (1+|D_n u(x)|^2+|D_n u(x+he_s)|^2)^{(p-2)/2} |\tau_{s,h} D_n u(x)|^2\, dx \\
&\le \frac{C_0}{(R-\varrho)^2} \int_{B_R} \{1+(1+|D_n u(x)|^2+|D_n u(x+he_s)|^2)^{(p-2)/2}\} |\tau_{s,h} u(x)|^2\, dx \\
&\le \frac{2C_0}{(R-\varrho)^2} \int_{B_R} |\tau_{s,h} u(x)|^2\, dx,
\end{aligned} \tag{3.2}$$

where we have used the fact that $p < 2$. Set

$$\hat{V}(\xi) = |V(\xi_n)| + \sum_{i=1}^{n-1} |\xi_i|, \quad V(\xi_n) = (1+|\xi_n|^2)^{(p-2)/4} \xi_n, \quad \forall\, \xi \in \mathbb{R}^{nN}. \tag{3.3}$$

Clearly,

$$|\tau_{s,h} \hat{V}(Du)| \le |\tau_{s,h} V(D_n u)| + \sum_{i=1}^{n-1} |\tau_{s,h} D_i u| \tag{3.4}$$

and

$$\hat{V}(Du) \in L^r \ \text{ if and only if } \ \begin{cases} D_i u \in L^r, & i = 1, \dots, n-1, \\ D_n u \in L^{rp/2}. \end{cases} \tag{3.5}$$

Using Lemma 2.7 we find

$$\begin{aligned}
C_1 |\tau_{s,h} D_n u(x)| &\le \frac{|\tau_{s,h} V(D_n u(x))|}{(1+|D_n u(x)|^2+|D_n u(x+he_s)|^2)^{(p-2)/4}} \\
&\le C_2 |\tau_{s,h} D_n u(x)|,
\end{aligned} \tag{3.6}$$

where C_1, C_2 depend only on N and p. From (3.4), (3.6) and (3.2) we get

$$\begin{aligned} \int_{B_\varrho} |\tau_{s,h}\hat{V}(Du)|^2\,dx &\le C_3 \int_{B_\varrho} \sum_{i=1}^{n-1} |\tau_{s,h}D_iu|^2\,dx + C_3 \int_{B_\varrho} |\tau_{s,h}V(D_nu)|^2\,dx \\ &\le \frac{C_4}{(R-\varrho)^2}\int_{B_R} |\tau_{s,h}u|^2\,dx, \end{aligned} \tag{3.7}$$

for some positive constants $C_3 = C_3(n)$ and $C_4 = C_4(n, N, p, m, M)$. Recalling that $D_su \in L^2$ for $s = 1, \ldots, n-1$, we may use Lemma 2.1 in order to get

$$\int_{B_R} |\tau_{s,h}u|^2\,dx \le C_5|h|^2 \qquad \forall\, s = 1, \ldots, n-1, \qquad \forall\, h : |h| < R, \tag{3.8}$$

with C_5 independent of h. Since we do not know apriori that $D_nu \in L^2$, the integral corresponding to $s = n$ in (3.8) is dealt with as follows. We write

$$\int_{B_R} |\tau_{n,h}u|^2\,dx = \int_{B_R} |\tau_{n,h}u|^a|\tau_{n,h}u|^{2-a}\,dx, \tag{3.9}$$

where $0 < a < 2$ is to be chosen later. Let us first assume that

$$u, D_iu \in L^r_{\mathrm{loc}},\ 2 \le r,\ \forall\, i = 1, \ldots, n-1, \quad \text{and} \quad D_nu \in L^t_{\mathrm{loc}},\ p \le t < 2. \tag{3.10}$$

In order to apply the anisotropic Sobolev imbedding theorem contained in Lemma 2.5, let $\overline{r}$ be the harmonic mean of the numbers $q_i = r$, $i = 1, \ldots, n-1$ and $q_n = t$, i.e.

$$\overline{r} = \frac{nrt}{(n-1)t + r}\,. \tag{3.11}$$

Note that $\overline{r} < n$ if and only if $r < t(n-1)/(t-1)$; define $\overline{r}^*$ as

$$\overline{r}^* = \begin{cases} n\overline{r}/(n-\overline{r}), & \text{if } \ \overline{r} < n, \\ \text{any number} \quad > r, & \text{if } \ \overline{r} \ge n. \end{cases}$$

In either case, $\overline{r}^* > r$ and Lemma 2.5 yields

$$u \in L^{\overline{r}^*}_{\mathrm{loc}}\,. \tag{3.12}$$

Thus applying Hölder's inequality on (3.9), with exponents t/a, $t/(t-a)$, provided $0 < a < t$, it follows that

$$\int_{B_R} |\tau_{n,h}u|^2\,dx \le \Big(\int_{B_R} |\tau_{n,h}u|^t\,dx\Big)^{a/t}\Big(\int_{B_R} |\tau_{n,h}u|^{(2-a)t/(t-a)}\,dx\Big)^{(t-a)/t}. \tag{3.13}$$

Because of (3.10) we may use Lemma 2.2 in order to get

$$\left(\int_{B_R} |\tau_{n,h} u|^t \, dx\right)^{a/t} \le C_6 |h|^a \quad \forall\, h : |h| < R, \tag{3.14}$$

with C_6 independent of h. If

$$(2-a)t/(t-a) \le \overline{r}^*, \tag{3.15}$$

then we may use Lemma 2.4 in order to get

$$\left(\int_{B_R} |\tau_{n,h} u|^{(2-a)t/(t-a)} \, dx\right)^{(t-a)/t} \le C_7 \qquad \forall\, h : |h| < R, \tag{3.16}$$

with C_7 independent of h. The inequalities (3.14), (3.16) and (3.13) yield

$$\int_{B_R} |\tau_{n,h} u|^2 \, dx \le C_8 |h|^a \qquad \forall\, h : |h| < R, \tag{3.17}$$

with C_8 independent of h. Thus, noting that $a < 2$ and $R \le 1$, (3.8), (3.17) and (3.7) yield

$$\sum_{s=1}^{n} \int_{B_\varrho} |\tau_{s,h} \hat{V}(Du)|^2 \, dx \le C_9 |h|^a \qquad \forall\, h : |h| < R, \tag{3.18}$$

with C_9 independent of h. Straightforward computations in (3.15) yield that

$$\begin{cases} 0 < a \le \frac{r(tn+2(t-1))-2(n-1)t}{r(n+t-1)-(n-1)t}, & \text{if } \ \overline{r} < n, \\ a \ \text{ any number in } \ (0,t), & \text{if } \ \overline{r} \ge n. \end{cases} \tag{3.19}$$

Let us remark that, when $\overline{r} < n$,

$$0 < \frac{r(tn+2(t-1))-2(n-1)t}{r(n+t-1)-(n-1)t} < t. \tag{3.20}$$

Now via Lemma 2.3 we improve on integrability:

$$\hat{V}(Du) \in L^{\hat{r}}_{\mathrm{loc}} \quad \forall\, \hat{r} < 2n/(n-a).$$

This implies via (3.5) that $D_i u \in L^{\hat{r}}_{\mathrm{loc}}$, $i = 1, \ldots, n-1$ and $D_n u \in L^{\hat{r}p/2}_{\mathrm{loc}}$. Elementary computations from (3.12) yield

$$2n/(n-a) \le \overline{r}^*, \tag{3.21}$$

implying that $u \in L^{\hat{r}}_{\mathrm{loc}}$. Let us summarize as follows. If

$$u, D_i u \in L^r_{\mathrm{loc}}, \quad 2 \le r, \quad \forall\, i = 1, \ldots, n-1 \ \text{ and } \ D_n u \in L^t_{\mathrm{loc}}, \quad p \le t < 2,$$

then

$$u, D_i u \in L^{\hat{r}}_{\mathrm{loc}},\ \forall\, i = 1, \dots, n-1 \ \text{ and } \ D_n u \in L^{\hat{r}p/2}_{\mathrm{loc}},\ \forall\, \hat{r} < 2n/(n-a). \tag{3.22}$$

It is useful to remark that (3.22) continues to hold if (3.10) is replaced by a weaker condition, namely

$$u, D_i u \in L^{\tilde{r}}_{\mathrm{loc}},\ \forall\, \tilde{r} < r,\ \forall\, i = 1, \dots, n-1 \ \text{ and } \ D_n u \in L^{\tilde{t}}_{\mathrm{loc}},\ \forall\, \tilde{t} < t, \tag{3.23}$$

provided $2 < r$ and $p < t < 2$. Assuming that $\hat{r} > r$ and $\hat{r}p/2 > t$, we may improve upon $\overline{r}^*$ by using Lemma 2.5 and hence in turn improve on a. Thus the whole analysis behind higher integrability depends upon whether the above process leads to an augmented value of a at each stage of iteration. In what follows we show that this can actually be realized. Although some improvement in t is always possible we can show that t can be boosted to 2, i.e. $D_n u \in L^2_{\mathrm{loc}}$, for only a limited range of p. We now describe the iterative process that will be used to boost integrability. At each stage we will compare r to the initial values of $r = 2$ and $t = p$. In the following we have broken down the analysis into two steps. Also, we will firstly assume $n \geq 5$ and although the most of the analysis is valid for $n = 2, 3$ and 4, we treat these separately for better presentation.

Step 1. Since $u, D_i u \in L^2$, $i = 1, \dots, n-1$, and $D_n u \in L^p$, (3.10) holds with $r = 2$ and $t = p$; we insert the values $r = 2$ and $t = p$ into (3.11). Call $\overline{r}(0)$ the resulting expression, i.e.

$$\overline{r}(0) = \frac{2pn}{(n-1)p + 2}\,. \tag{3.24}$$

We remark that $\overline{r}(0) < n$ so that, by the first line of (3.19) with $r = 2$ and $t = p$, we choose $a(0)$ to be the maximum value allowed for a, that is

$$a(0) = \frac{2(3p-2)}{n(2-p) + (3p-2)}\,. \tag{3.25}$$

We set

$$\varepsilon(0) = \frac{2n}{n - a(0)} - 2 = \frac{4(3p-2)}{n^2(2-p) + (n-2)(3p-2)}\,. \tag{3.26}$$

From (3.22) we find

$$\begin{aligned} u, D_i u \in L^{\hat{r}}_{\mathrm{loc}}, \quad &\forall\, \hat{r} < 2 + \varepsilon(0) \quad \forall\, i = 1, \dots, n-1 \\ &\text{and} \quad D_n u \in L^{\hat{t}}_{\mathrm{loc}}, \quad \forall\, \hat{t} < p(1 + \varepsilon(0)/2). \end{aligned} \tag{3.27}$$

We now describe an intermediate stage in the iterative process.

Step 2. Let $\varepsilon > 0$, take $r(\varepsilon) = 2 + \varepsilon$, $t(\varepsilon) = p(1 + \varepsilon/2)$; assume that

$$u, D_i u \in L^{\tilde{r}}_{\mathrm{loc}},\ \forall \tilde{r} < r(\varepsilon),\ \forall i = 1, \ldots, n-1 \ \text{ and } \ D_n u \in L^{\tilde{t}}_{\mathrm{loc}},\ \forall \tilde{t} < t(\varepsilon). \tag{3.28}$$

We now split the discussion into three cases, namely (i) $0 < \varepsilon < 2(2-p)/p$, (ii) $\varepsilon = 2(2-p)/p$ and (iii) $\varepsilon > 2(2-p)/p$.

Case (i). We assume that

$$0 < \varepsilon < 2(2-p)/p. \tag{3.29}$$

Then $2 < r(\varepsilon)$ and $p < t(\varepsilon) < 2$. Clearly, (3.23) holds with $r = r(\varepsilon)$ and $t = t(\varepsilon)$. The improvements as in (3.22) are as follows. We insert $r = r(\varepsilon) = 2 + \varepsilon$ and $t = t(\varepsilon) = p(1 + \varepsilon/2)$ into (3.11); setting $\overline{r}(\varepsilon)$ as the resulting expression, we have

$$\overline{r}(\varepsilon) = \frac{2pn}{(n-1)p + 2}(1 + \varepsilon/2). \tag{3.30}$$

Note that, for $n \geq 3$, the condition (3.29) implies $\overline{r}(\varepsilon) < n$, so that we use the first line in (3.19) with $r = r(\varepsilon)$ and $t = t(\varepsilon)$. We choose $a(\varepsilon)$ to be the maximum value allowed for a, that is

$$a(\varepsilon) = \frac{2(3p-2) + (n+2)\varepsilon p}{n(2-p) + (3p-2) + \varepsilon p}\,. \tag{3.31}$$

Set

$$I(\varepsilon) = \frac{2n}{n - a(\varepsilon)} = \frac{4(3p-2) + 2(n+2)\varepsilon p}{n^2(2-p) + (n-2)(3p-2) - 2\varepsilon p} \tag{3.32}$$

and thus in (3.22) we get

$$\begin{aligned} &u, D_i u \in L^{\hat{r}}_{\mathrm{loc}}, \quad \forall \hat{r} < 2 + I(\varepsilon) \quad \forall i = 1, \ldots, n-1 \\ &\qquad\qquad \text{and} \ \ D_n u \in L^{\hat{t}}_{\mathrm{loc}}, \quad \forall \hat{t} < p(1 + I(\varepsilon)/2). \end{aligned} \tag{3.33}$$

Case (ii). We now assume

$$\varepsilon = 2(2-p)/p\,; \tag{3.34}$$

then the assumption (3.28) implies that, for every $\varepsilon' < \varepsilon = 2(2-p)/p$ we have

$$\begin{aligned} &u, D_i u \in L^{\tilde{r}}_{\mathrm{loc}}, \quad \forall \tilde{r} < r(\varepsilon') \quad \forall i = 1, \ldots, n-1 \\ &\qquad\qquad \text{and} \ \ D_n u \in L^{\tilde{t}}_{\mathrm{loc}}, \quad \forall \tilde{t} < t(\varepsilon'). \end{aligned} \tag{3.35}$$

Now $\varepsilon' < 2(2-p)/p$ so that we can apply the method in Case (i) with ε' instead of ε and we get (3.33), in particular,

$$D_n u \in L^{\hat{t}}_{\mathrm{loc}}, \quad \forall \hat{t} < p(1 + I(\varepsilon')/2), \quad \forall \varepsilon' < 2(2-p)/p. \tag{3.36}$$

As ε' approaches $2(2-p)/p$, $p(1 + I(\varepsilon')/2)$ goes to $p(1 + 2/(n-2))$ which is bigger than 2, provided $n \geq 3$ and $2 - 4/n < p$; then (3.36) implies

$$D_n u \in L^2_{\mathrm{loc}} \tag{3.37}$$

and Theorem 1 follows.

Case (iii). We assume that

$$\varepsilon > 2(2-p)/p. \tag{3.38}$$

Now $t(\varepsilon) = p(1+\varepsilon/2) > 2$, so that (3.28) implies (3.37) and the statement of Theorem 1 follows.

The preceding discussion indicates that (3.28) implies the result in Theorem 1, whenever $\varepsilon \geq 2(2-p)/p$. However, for $0 < \varepsilon < 2(2-p)/p$, we get only (3.33). This necessitates an iterative process where the new ε is given by $I(\varepsilon)$ as in (3.32). We now describe more precisely this process of bootstrapping ε. In (3.26), set

$$\varepsilon_0 = \varepsilon(0) = \frac{4(3p-2)}{n^2(2-p)+(n-2)(3p-2)}, \tag{3.26}$$

$$\varepsilon_{m+1} = I(\varepsilon_m) \ \text{ if } \ m \geq 0 \ \text{ and } \ 0 < \varepsilon_m < 2(2-p)/p. \tag{3.39}$$

We recall that the proof is achieved whenever, for some m, $\varepsilon_m \geq 2(2-p)/p$. We now prove that $m \to \varepsilon_m$ is strictly increasing. Set $a = 4(3p-2)$, $b = 2(n+2)p$, $c = n^2(2-p)+(n-2)(3p-2)$ and $d = 2p$; then

$$0 < \varepsilon_m < 2(2-p)/p \quad \Longrightarrow \quad c - d\varepsilon_m > 0 \tag{3.40}$$

and

$$\varepsilon_{m+1} = \frac{a+b\varepsilon_m}{c-d\varepsilon_m}. \tag{3.41}$$

Direct computations show that $\varepsilon_0 > 0$; moreover, if $\varepsilon_0 < 2(2-p)/p$, then

$$\varepsilon_1 - \varepsilon_0 = \frac{(b+d\varepsilon_0)\varepsilon_0}{c-d\varepsilon_0} > 0. \tag{3.42}$$

We are going to prove that

$$0 < \varepsilon_i < 2(2-p)/p,\ \forall\, i = 0,\dots,m \quad \Longrightarrow \quad \varepsilon_j < \varepsilon_{j+1},\ \forall\, j = 0,\dots,m. \tag{3.43}$$

Let us set

$$\varepsilon_j < \varepsilon_{j+1}\,; \tag{3.44(j)}$$

we prove (3.44(j)) recursively on j: if $j = 0$ then (3.44(j)) reduces to (3.42); let us assume that (3.44(j)) holds true and $0 \leq j \leq j+1 \leq m$, then

$$\varepsilon_{j+2} - \varepsilon_{j+1} = \frac{(ad+bc)(\varepsilon_{j+1}-\varepsilon_j)}{(c-d\varepsilon_{j+1})(c-d\varepsilon_j)}.$$

Since ε_j and ε_{j+1} are between 0 and $2(2-p)/p$, by (3.40) we have $(c-d\varepsilon_{j+1})(c-d\varepsilon_j) > 0$, so, using the recursive assumption (3.44(j)) we get $(\varepsilon_{j+1}-\varepsilon_j) > 0$ and (3.44(j+1)) holds true. (3.43) is completely proved.

Let us summarize as follows; if $n \geq 3$ and $\max\{1, 2-4/n\} < p < 2$ we have shown that either (a) for some m, $\varepsilon_m \geq 2(2-p)/p$ and Theorem 1 follows, or (b) for every m, $0 < \varepsilon_m < 2(2-p)/p$, also implying that ε_m is increasing. We now confine ourselves to the latter case. Set

$$L = \lim_{m\to\infty} \varepsilon_m.$$

Recall

$$0 < \varepsilon_m < 2(2-p)/p, \quad \forall\, m = 0, 1, \ldots \tag{3.45}$$

From (3.32)

$$I(n,p,\varepsilon) = \frac{4(3p-2)+2(n+2)\varepsilon p}{n^2(2-p)+(n-2)(3p-2)-2\varepsilon p}.$$

Moreover, for $1 \leq p < 2$

$$\frac{\partial I}{\partial p}(n,p,\varepsilon) > 0, \quad \frac{\partial I}{\partial \varepsilon}(n,p,\varepsilon) > 0 \quad \text{for } \ 0 < \varepsilon \leq 2(2-p)/p \tag{3.46}$$

and

$$\varepsilon \longrightarrow I(n,p,\varepsilon) \ \text{ is continuous in } \ (0, 2(2-p)/p]. \tag{3.47}$$

By (3.39) we can see that ε_m depends on n, p; it is easy to prove that

$$p \longrightarrow \varepsilon_0(n,p) \ \text{ is increasing in } \ [1,2).$$

By (3.46) and (3.47) we get

$$p \longrightarrow \varepsilon_m(n,p) \ \text{ increasing } \quad \Longrightarrow \quad p \longrightarrow \varepsilon_{m+1}(n,p) \ \text{ increasing,}$$

so that

$$p \longrightarrow \varepsilon_m(n,p) \ \text{ is increasing in } \ [1,2) \quad \forall\, m \geq 0. \tag{3.48}$$

Let us point out that L depends on n, p too:

$$L(n,p) = \lim_{m\to\infty} \varepsilon_m(n,p). \tag{3.49}$$

Because of (3.45) and (3.46) we have

$$0 < L(n,p) \leq 2(2-p)/p\,; \tag{3.50}$$

since I is continuous with respect to ε, passing to the limit in (3.39) we get

$$L(n,p) = I(n,p,L(n,p)). \tag{3.51}$$

Now (3.48) implies

$$p \longrightarrow L(n,p) \ \text{ is increasing in } \ [1,2). \tag{3.52}$$

We now treat the cases $n = 2$, $n = 3$, $n = 4$ and $n \geq 5$ separately.

Case A. Take $n \geq 5$.

From (1.10) and (3.52), we have

$$L(n, 2 - 4/n) \leq L(n, p). \tag{3.53}$$

Set $\hat{p} = 2 - 4/n$; then we have $2(2 - \hat{p})/\hat{p} = 4/(n-2)$; because of (1.10), (3.50) and (3.53) we get

$$0 < L(n, 2 - 4/n) \leq 4/(n-2). \tag{3.54}$$

Moreover

$$L(n, 2 - 4/n) = I(n, 2 - 4/n, L(n, 2 - 4/n)). \tag{3.55}$$

Solving the equation $y = I(n, 2 - 4/n, y)$, we find $y_1 = 4/(n-2) < n - 3 = y_2$, so that $L(n, 2 - 4/n) = 4/(n-2)$. Going back to (3.53),

$$4/(n-2) = L(n, 2 - 4/n) \leq L(n, p) \leq 2(2-p)/p < 4/(n-2), \tag{3.56}$$

where the last inequality holds as $y \to 2(2-y)/y$ is strictly decreasing and $2 - 4/n < p$. The inequalities in (3.56) imply that (3.45) does not hold and the Theorem follows when $n \geq 5$ (also see the discussion following (3.38)).

Case B. Let $n = 4$.

Solving the equation in (3.51),

$$pL^2 - (14 - 11p)L + (6p - 4) = 0, \tag{3.57}$$

it turns out that

$$L = \frac{(14 - 11p) \pm \sqrt{\Delta}}{2p}, \quad \Delta = (14 - 11p)^2 - 4p(6p - 4). \tag{3.58}$$

We have

$$\Delta < 0 \quad \text{if and only if} \quad 98/97 < p < 2. \tag{3.59}$$

We claim that, for $p \in (98/97, 2)$, $\varepsilon_m \geq 2(2-p)/p$ for some m. We argue by contradiction. If not, then $\varepsilon_m < 2(2-p)/p$ for every m, then $L = \lim_{m\to\infty} \varepsilon_m \in (0, 2(2-p)/p]$. Clearly, L solves (3.57), but by (3.58) L cannot be real. Hence Theorem 1 follows.

Case C. Now consider $n = 3$.

Again by (3.51),

$$pL^2 - 8(1-p)L + (6p - 4) = 0; \tag{3.60}$$

it turns out that

$$L = \frac{4(1-p) \pm \sqrt{\Delta_1}}{p}, \quad \Delta_1 = 16 - 28p + 10p^2. \tag{3.61}$$

We have

$$\Delta_1 < 0 \ \text{ if and only if } \ 4/5 < p < 2, \tag{3.62}$$

so that, if $1 < p < 2$, then, as in the case $n = 4$, for some $m \geq 0$ we must have $\varepsilon_m \geq 2(2-p)/p$.

Case D. Lastly, we treat n=2.

Computing $\varepsilon(0)$ from (3.26)

$$\varepsilon(0) = (3p-2)/(2-p). \tag{3.63}$$

We have

$$\begin{aligned} -3+\sqrt{17} < p < 2 \quad &\Longrightarrow \quad \varepsilon(0) > 2(2-p)/p, \\ 1 < p \leq -3+\sqrt{17} \quad &\Longrightarrow \quad 0 < \varepsilon(0) \leq 2(2-p)/p. \end{aligned}$$

In the case $-3+\sqrt{17} < p < 2$ the proof is finished. Let us consider the case $1 < p \leq -3+\sqrt{17}$. The inequality (3.27) allows us to start from (3.28) (see Step 2) with any ε satisfying $0 < \varepsilon < \varepsilon(0)$. Since $(2-p)/p < \varepsilon(0) \leq 2(2-p)/p$, we may select ε such that $(2-p)/p < \varepsilon < 2(2-p)/p$. Clearly, (3.29) holds and we have $\overline{r}(\varepsilon) \geq 2 = n$. By (3.19), $a(\varepsilon)$ can be chosen to be in $(0, p(\varepsilon))$ and we get as in (3.33),

$$D_n u \in L^{\hat{t}}_{\mathrm{loc}} \quad \forall \hat{t} < 2p/(2-a(\varepsilon)). \tag{3.33}$$

Since

$$\lim_{a(\varepsilon)\to p(\varepsilon)} \frac{2p}{2-a(\varepsilon)} = \frac{2p}{2-p(\varepsilon)} > 2,$$

we can select $a(\varepsilon)$ so that $2 < 2p/(2-a(\varepsilon))$, then (3.33) implies that $D_n u \in L^2_{\mathrm{loc}}$ and the proof is finished in the case $1 < p \leq -3+\sqrt{17}$, too.

The theorem is completely proved. □

4. Proof of Corollary 1.

As in the proof of Theorem 1, we start from the Euler equation and we arrive at (3.7): for some positive constant $C_{10} = C_{10}(n, N, p, m, M)$ we have

$$\int_{B_\varrho} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \, dx + \int_{B_\varrho} |\tau_{s,h} V(D_n u)|^2 \, dx \leq \frac{C_{10}}{(R-\varrho)^2} \int_{B_R} |\tau_{s,h} u|^2 \, dx. \tag{3.7}$$

In Theorem 1 we have proved higher integrability of $D_n u$ so that

$$D_1 u, \ldots, D_{n-1} u, D_n u \in L^2_{\mathrm{loc}} \tag{4.1}$$

and we can apply Lemma 2.1 with $t = 2$ for $s = n$ too, compare with (3.8),

$$\int\limits_{B_R} |\tau_{s,h} u|^2 \, dx \leq |h|^2 \int_{B_{2R}} |D_s u|^2 \, dx \qquad \forall\, s = 1, \ldots, n-1, n, \quad \forall\, h : |h| < R. \tag{4.2}$$

We put together (3.7) and (4.2): for some positive constant C_{11} independent of h we have

$$\int\limits_{B_\varrho} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \, dx + \int\limits_{B_\varrho} |\tau_{s,h} V(D_n u)|^2 \, dx \leq C_{11} |h|^2 \tag{4.3}$$

$$\forall\, s = 1, \ldots, n-1, n, \quad \forall\, h : |h| < R.$$

We apply Lemma 2.2 in order to get

$$\exists\, D_s D_i u \in L^2_{\mathrm{loc}} \quad \exists\, D_s(V(D_n u)) \in L^2_{\mathrm{loc}} \qquad \forall\, s = 1, \dots, n-1, n, \quad \forall\, i = 1, \dots, n-1. \tag{4.4}$$

In order to prove existence of $D_n D_n u$, we use (3.6), Hölder's inequality, Lemma 2.4 and (4.3); thus, for some constants C_{12} and C_{13}, independent of h, we have

$$\begin{aligned} &\int_{B_\varrho} |\tau_{s,h} D_n u|^p \, dx \\ &\le C_{12} \int_{B_\varrho} \left(1 + |D_n u(x)|^2 + |D_n u(x + he_s)|^2\right)^{(2-p)p/4} |\tau_{s,h} V(D_n u(x))|^p \, dx \\ &\le C_{12} \left(\int_{B_\varrho} \left(1 + |D_n u(x)|^2 + |D_n u(x + he_s)|^2\right)^{p/2} dx \right)^{(2-p)/2} \\ &\qquad \left(\int_{B_\varrho} |\tau_{s,h} V(D_n u(x))|^2 \, dx \right)^{p/2} \\ &\le C_{13} |h|^p \qquad \forall\, s = 1, \dots, n, \quad \forall\, h : |h| < R. \end{aligned} \tag{4.5}$$

Inequality (4.5) with $s = n$ allows us to apply Lemma 2.2:

$$\exists D_n D_n u \in L^p_{\mathrm{loc}}(\Omega). \tag{4.6}$$

This ends the proof. □

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Indian Statistical Institute, 7SJS Sansanwal Marg, New Delhi 110 016, India

Dipartimento di Matematica, Università di L'Aquila, via Vetoio, 67010 Coppito, L'Aquila, Italy

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