On variational approach to the Hamilton-Jacobi PDE

J. Chabrowski, Kewei Zhang

Abstract. In this paper we construct a minimizing sequence for the problem (1). In particular, we show that for any subsolution of the Hamilton-Jacobi equation $(*)$ there exists a minimizing sequence weakly convergent to this subsolution. The variational problem (1) arises from the theory of computer vision equations.

Keywords: Young measures, computer vision equations
Classification: 49R, 35E

Introduction.

The main purpose of this article is to construct a minimizing sequence for the problem

$$\inf_{u\in W^{1,\infty}(\Omega)} \int_\Omega |H(x,u,Du)|\,dx = 0, \tag{1}$$

where $H : \Omega \times \mathbb{R} \times \mathbb{R}_n \to \mathbb{R}_2$ is a continuous function and convex with respect to $P \in \mathbb{R}_2$ and Ω is a bounded domain in $\mathbb{R}_2$. We show that for any sub-solution $u \in C^1(\bar\Omega)$ of the Hamilton-Jacobi equation

$$H(x,u,Du) = 0 \ \text{ in } \ \Omega, \tag{$*$}$$

that is $H(x,u,Du) \le 0$ on Ω, there exists a sequence $u_j \in W^{1,\infty}(\Omega)$ such that $u_j \rightharpoonup u$ weak-$*$ in $W^{1,\infty}(\Omega)$, $u_j\mid_{\partial\Omega} = u\mid_{\partial\Omega}$ and

$$\lim_{j\to\infty} \int_\Omega |H(x,u_j,Du_j)|\,dx = 0.$$

In our earlier papers [CZ1] and [CZ2] we discussed a result of this nature for the eikonal equation, which arises from computer vision and for a system of two equations from photometric stereo.

For the eikonal equation

$$u_{x_1}^2 + u_{x_2}^2 = E(x) \ \text{ in } \ \Omega,$$

with $E \in C(\bar\Omega)$, $E(x) \ge 0$ and $E(x) \not\equiv 0$ on Ω, we showed that if $u \in C^1(\bar\Omega)$ and $|Du(x)|^2 \le E(x)$ on Ω, then there exists a sequence $\{u_j\}$, with $u_j\mid_{\partial\Omega} = u\mid_{\partial\Omega}$ and such that $u_j \rightharpoonup u$ weak-$*$ in $W^{1,\infty}(\Omega)$ and

$$\lim_{j\to\infty} \int_\Omega \big||Du_j(x)|^2 - E(x)\big|\,dx = 0.$$

with $E \in C(\bar{\Omega})$, $E(x) \geq 0$ and $E(x) \not\equiv 0$ on Ω, we showed that if $u \in C^1(\bar{\Omega})$ and $|Du(x)|^2 \leq E(x)$ on Ω, then there exists a sequence $\{u_j\}$, with $u_j \mid_{\partial\Omega} = u \mid_{\partial\Omega}$ and such that $u_j \rightharpoonup u$ weak-$*$ in $W^{1,\infty}(\Omega)$ and

$$\lim_{j\to\infty} \int_\Omega \left| |Du_j(x)|^2 - E(x) \right| dx = 0.$$

For the system of two equations arising from a photometric stereo

$$\frac{p_1^i u_{x_1} + p_2^i u_{x_2} - p_3^i}{|p^i| \sqrt{u_{x_1}^2 + u_{x_2}^2 + 1}} = E_i(x) \ \text{ in } \Omega,$$

$i = 1, 2$, where $p^i = (p_1^i, p_2^i, p_3^i)$, $i = 1, 2$, are linearly independent vectors and $E_i \in C(\bar{\Omega})$, $i = 1, 2$, we proved in [CZ2] that if u_1 and u_2 are two distinct solutions of this system and $u = \lambda u_1 + (1-\lambda) u_2$, with $0 < \lambda < 1$, then there exists a sequence $\{u_j\}$ in $W^{1,\infty}(\Omega)$ such that $u_j \mid_{\partial\Omega} = u \mid_{\partial\Omega}$, $u_j \rightharpoonup u$ weak-$*$ in $W^{1,\infty}(\Omega)$ and

$$\lim_{j\to} \int_\Omega \left(\left| f_1(Du_j) - E(x) \right| + \left| f_2(Du) - E_2(x) \right| \right) dx = 0.$$

Here we have used the notation

$$f_i(P) = \frac{P_1 p_1^i + P_2 p_2^i - p_3^i}{|p^i| \sqrt{|P|^2 + 1}},$$

and for this system we identify two solutions which differ by a constant.

Motivated by these results, we construct in this paper a minimizing sequence for the functional (1) corresponding to the Hamilton-Jacobi equation $(*)$ (see Theorem 2). As an immediate consequence we obtain the result from our earlier paper [CZ1]. The construction of a minimizing sequence for (1) which is presented in this paper is simpler than in the case of the eikonal equation [CZ1]. We point out here that a function which is a limit of a minimizing sequence must be a subsolution of the equation $(*)$. This is the result of the convexity condition imposed on $H(x, u, P)$. In our construction of a minimizing sequence, we essentially use the assumption that the level sets $H_{x,u} = \{P;\, H(x, u, P) \leq 0\}$ are bounded uniformly in (x, u). Therefore, we can call the problem (1) a variational problem of elliptic type. In Section 3 we discuss the same problem for the equation

$$(**) \qquad u_{x_1} u_{x_2} = E(x) \ \text{ in } \Omega$$

and in Section 4 for the equation

$$(***) \qquad u_{x_2}^2 - u_{x_1} = E(x) \ \text{ in } \Omega.$$

For the equation $(**)$ the corresponding function $H(x, p, q) = pq - E(x)$ is not convex. We show that every $C^1(\bar{\Omega})$-function is a candidate for a minimizer. In case of the equation $(***)$ the function $H(x,p,q) = q^2 - p - E(x)$ is convex and the situation is similar to that from Section 2, however, the level sets $\{(p,q) : q^2 - p - E(x) \leq 0\}$ are unbounded. Therefore, the construction of a minimizing sequence must be treated separately. Due to the nature of level sets of the function $H(x,p,q)$ for the equations $(**)$ and $(***)$ we call the variational problems associated with these equations, respectively, hyperbolic and parabolic.

To obtain some information on the structure of minimizing sequences we use Young measures. We point here that the use of Young measures is not an essential tool of this paper, however, they help to localize the oscillations of minimizing sequences which is a main focus in our construction. All three constructions described in this work follow the same pattern. For a given $u \in C^1(\bar{\Omega})$ we look for a minimizing sequence in the form $u_n(x) = u(x) + \phi_n(x)$, where $\phi_n \rightharpoonup 0$ weak-$*$ in $W^{1,\infty}(\Omega)$. Using Young measures we localize sets where oscillations of u_n should occur and then we construct ϕ_n in such a way that $Du(x)+D\phi_n(x)$ belongs to that set for sufficiently large number of values of x. We point out here that to construct ϕ_n we have used some ideas from [KS].

We emphasize that a good application of Theorem 2 is the variational problem for the eikonal equation. This equation and the equation $(**)$ are examples of image irradiance equations arising from computer vision. It is known that both equations may not have classical solutions. Therefore, our variational approach suggest that, especially in case of the eikonal equation, any sub-solution is a good candidate for a solution of the shape from shading problem (see [BCK1], [BCK2], [BU] and [DS]).

1. Preliminaries.

Let Ω be a bounded domain in $\mathbb{R}_2$ with a Lipschitz boundary $\partial\Omega$. By $W^{1,p}(\Omega)$, $1 \leq p < \infty$, we denote usual Sobolev spaces [AD]. Since $\partial\Omega$ is Lipschitz, the elements of $W^{1,p}(\Omega)$ have traces on $\partial\Omega$. For $x \in \mathbb{R}_2$ we write $x = (x_1, x_2)$. By $|A|$ we denote the Lebesgue measure of a set $A \subset \mathbb{R}_n$. Throughout this paper the gradient of a C^1-function $f : \Omega \to \mathbb{R}$ is denoted by Df.

For a given Banach space X, the weak convergence is denoted by "$\rightharpoonup$" and the strong convergence by "$\to$".

To examine the structure of minimizing sequences, in particular the nature of oscillations, we need the following result on Young measures (see [BA], [BL], [EV] or [TA]).

Theorem 1. *Let $\{z_j\}$ be bounded sequence in $L^1(\Omega, \mathbb{R}_s)$. Then there exist a subsequence $\{z_\nu\}$ of $\{z_j\}$ and a family $\{\nu_x\}$, $x \in \Omega$, of probability measures on $\mathbb{R}_s$, such that for any measurable subset $A \subset \mathbb{R}_2$*

$$f(\cdot, z_\nu) \rightharpoonup \langle \nu_x, f(x,\cdot)\rangle \ \text{ in } L^1(A)$$

for every Carathéodory function $f : \Omega \times \mathbb{R}_s \to \mathbb{R}$ such that $f(\cdot, z_\nu)$ is sequentially relatively compact in $L^1(A)$.

Here $\langle \nu_x, f(x,\cdot)\rangle$ denotes the expected value of $f(x,\cdot)$. We recall that a function $f : \Omega \times \mathbb{R}_s \to \mathbb{R}$ satisfies the Carathéodory condition if $f(x,\cdot) : \mathbb{R}_s \to \mathbb{R}$ is continuous on $\mathbb{R}_s$ for a.e. $x \in \Omega$ and $f(\cdot, p) : \Omega \to \mathbb{R}$ is measurable on Ω for every $p \in \mathbb{R}_s$.

2. Minimizing sequences for the Hamilton-Jacobi functional.

Throughout this section we assume that the function $H(x,u,P):\bar\Omega\times\times\mathbb{R}_2\to\mathbb{R}$ is continuous, convex in $P\in\mathbb{R}_2$ and coercive, that is,

$$H(x,u,P)\ge|P|^q-C \tag{2}$$

for all $(x,u,P)\in\bar\Omega\times\mathbb{R}\times\mathbb{R}_2$ and for some constants $1\le q<\infty$ and $C>0$.

Further, we assume that

$$H_{x,u}=\{P\in\mathbb{R}_2;\ H(x,u,P)<0\}\ne\varnothing$$

for every $(x,u)\in\Omega\times\mathbb{R}$ and moreover, we assume that all sets $H_{x,u}$ are contained in a disc $B(0,R)$.

We now define a functional $I: W^{1,\infty}(\Omega)\to\mathbb{R}$ by

$$I(u)=\int_\Omega|H(x,u(x),Du(x))|\,dx.$$

We commence with the following observation on minimizing sequences for the functional I.

Proposition 1. *Suppose that there exists a sequence $\{u_j\}$ in $W^{1,\infty}(\Omega)$ such that $u_j=u_1$ on $\partial\Omega$ for all $j\ge1$ and that*

$$\lim_{j\to\infty}I(u_j)=0.$$

Then up to a subsequence $u_j\rightharpoonup u$ in $W^{1,q}(\Omega)$ and

$$H(x,u(x),Du(x))\le 0\ \text{ a.e. on }\ \Omega.$$

Proof: It follows from (2) that $\{u_j\}$ is bounded in $W^{1,q}(\Omega)$. Therefore, there exists a subsequence, which we take as $\{u_j\}$ itself, such that $u_j\rightharpoonup u$ in $W^{1,q}(\Omega)$ and by the compactness of the imbedding $W^{1,2}(\Omega)\to L^q(\Omega)$ $u_j\to u$ a.e. on Ω. We denote by $CH(x,u,P)$ a lower convex envelope of $|H(x,u,P)|$ for each $(x,u)\in\Omega\times\mathbb{R}$, that is

$$CH(x,u,P)=\begin{cases}H(x,u,P) & \text{if } H(x,u,P)\ge0,\\ 0 & \text{if } H(x,u,P)<0.\end{cases}$$

Then a minimizing sequence for I is also minimizing sequence for a functional I_C given by (see [ET])

$$I_C(u)=\int_\Omega CH(x,u(x),Du(x))\,dx$$

and we have

$$\begin{aligned}0=\lim_{j\to\infty}\int_\Omega|H(x,u_j,Du_j)|\,dx&=\lim_{j\to\infty}\int_\Omega CH(x,u_j,Du_j)\,dx\\&=\int_\Omega CH(x,u(x),Du(x))\,dx.\end{aligned}$$

According to Theorem 1 there exists a family of Young measures $\{\nu_x\}$, $x \in \Omega$, such that

$$\lim_{j\to\infty}\int_\Omega |H(x,u_j,Du_j)|\,dx = \int_\Omega \langle \nu_x(\cdot), |H(x,u(x),\cdot)|\rangle\,dx,$$

so

$$0 = \int_\Omega \langle \nu_x(\cdot), |H(x,u(x),\cdot)|\rangle\,dx = \int_\Omega CH(x,u(x),Du(x))\,dx.$$

This implies that supp $\nu_x \subset \partial H_{x,u(x)}$ and since $CH(x,u,P) > 0$ on $\mathbb{R}_2 - \bar{H}_{x,u(x)}$, we have $H(x,u(x),Du(x)) \le 0$ a.e. on Ω. □

Remark. It is clear from the proof of Proposition 1, that the assumption (2) is only needed to obtain the boundedness in $W^{1,q}(\Omega)$ of a minimizing sequence. If we drop this assumption, then we must assume that a minimizing sequence is bounded, for example in $W^{1,\infty}(\Omega)$, and that $H(x,u,P)$ is positive for some (x,u,P). This will be required in Theorem 2. Such a situation occurs in problems discussed in Sections 3 and 4, where the corresponding functions $H(x,u,P)$ are not coercive. We point out here that the assumption (2) is not needed in the construction of a minimizing sequence for a given sub-solution (see Theorem 2 below).

The preceding result gives some information on the nature of minimizing sequences. Let $\phi_j = u_j - u$, then $\phi_j \rightharpoonup 0$ in $W^{1,q}(\Omega)$ and $\phi_j(x) \to 0$ a.e. on Ω. If $\{\bar{\nu}_x\}$, $x \in \Omega$, is a family of Young measures corresponding to $\{D\phi_j\}$, then

$$0 = \lim_{j\to\infty} I(u_j) = \int_\Omega \langle \bar{\nu}_x, |H(x,u(x),Du(x)+\lambda)|\rangle\,dx$$

and

$$\text{supp}\ \bar{\nu}_x \subset \{\lambda;\ |H(x,u(x),Du(x)+\lambda)| = 0\}.$$

This means that $Du(x) + \lambda \in \partial H_{x,u(x)}$. Hence, we shall construct a minimizing sequence $\{\phi_j\}$ in such a way that $Du(x) + D\phi_j(x) \in \partial H_{x,u(x)}$ for large number of x.

We are now in a position to establish the main result of this section.

Theorem 2. *Let $u \in C^1(\bar{\Omega})$ and suppose that $H(x,u(x),Du(x)) \le 0$ on Ω. Then there exists a sequence $\{u_n\}$ in $W^{1,\infty}(\Omega)$ such that $u_n \rightharpoonup u$ weak-$*$ in $W^{1,\infty}(\Omega)$, $u_n\,|_{\partial\Omega} = u\,|_{\partial\Omega}$ and $\lim_{n\to\infty} I(u_n) = 0$.*

Proof: Let

$$M = \max\bigl(\max_\Omega |u(x)|, \max_\Omega |Du(x)|\bigr)$$

and

$$K = \max\{|\ H(x,p,P)\ |; x \in \Omega,\ |p| \le M+1,\ |P| \le M+2R\}.$$

We look for a minimizing sequence $\{u_n\}$ of the form $u_n = u + \phi_n$, with $\phi_n \in \overset{\circ}{W}{}^{1,\infty}(\Omega)$, $\phi_n \rightharpoonup 0$ weak-$*$ in $W^{1,\infty}(\Omega)$. We commence by approximating Ω by a sequence of unions of squares

$$H_j = \bigcup_{k=1}^{I_j} D_k^j \ \text{ with }\ H_j \subset \Omega \ \text{ and }\ \lim_{j\to\infty} |\Omega - H_j| = 0.$$

We assume that the edges of D_k^j with the length $d(D_k^j) = \frac{1}{2^j}$ are parallel to the coordinate axes.
For each integer $n \geq 1$ we can find an integer j_n such that

$$|\Omega - H_{j_n}| \leq \frac{1}{4Kn}, \tag{4}$$

$$|u(x) - u(x_k^{j_n})| \leq \frac{1}{j_n} \text{ and } |Du(x) - Du(x_k^{j_n})| \leq \frac{1}{j_n} \tag{5}$$

for all $x \in D_k^{j_n}$, $k = 1, \dots, j_n$, where $x_k^{j_n}$ denotes the center of the square $D_k^{j_n}$. We may also assume, due to the uniform continuity of $|H(x,p,P)|$ on $\bar{\Omega} \times (|p| \leq M+1) \times (|P| \leq M+2R)$, that

$$\big||H(x,p,P)| - |H(x_k^{j_n}, \bar{p}, \bar{P})|\big| \leq \frac{1}{20n|\Omega|} \tag{6}$$

for all $x \in D_k^{j_n}$, $k = 1, \dots, j_n$, $|p - \bar{p}| \leq \frac{1}{j_n}$ and $|P - \bar{P}| \leq \frac{1}{j_n}$, with $|p|, |\bar{p}| \leq M+1$ and $|P|, |\bar{P}| \leq M + 2R$.
We now proceed to the construction of the sequence ϕ_n locally on $D_k^{j_n}$. We distinguish three cases:

$$H(x_k^{j_n}, u(x_k^{j_n}), Du(x_k^{j_n})) < 0 \text{ and } Du(x_k^{j_n}) \neq 0, \tag{a}$$

$$H(x_k^{j_n}, u(x_k^{j_n}), Du(x_k^{j_n})) < 0 \text{ and } Du(x_k^{j_n}) = 0, \tag{b}$$

$$H(x_k^{j_n}, u(x_k^{j_n}), Du(x_k^{j_n})) = 0. \tag{c}$$

Case (a).

Let $\ell_{D(x_k^{j_n})}$ be a straight line passing through $Du(x_k^{j_n})$ and orthogonal to $Du(x_k^{j_n})$. Since $Du(x_k^{j_n}) \in \operatorname{Int} H_{x_k^{j_n}, u(x_k^{j_n})}$ and $H_{x_k^{j_n}, u(x_k^{j_n})}$ is convex and bounded, $\ell_{Du(x_k^{j_n})}$ intersects $\partial H_{x_k^{j_n}, u(x_k^{j_n})}$ at two opposite points P_{k,j_n} and P'_{k,j_n}. Let us set

$$\frac{|P_{k,j_n} - Du(x_k^{j_n})|}{|P_{k,j_n} - P'_{k,j_n}|} = \lambda,$$

then

$$\frac{|P'_{k,j_n} - Du(x_k^{j_n})|}{|P_{k,j_n} - P'_{k,j_n}|} = 1 - \lambda$$

and

$$|P_{k,j_n} - Du(x_k^{j_n})| + |P'_{k,j_n} - Du(x_k^{j_n})| = |P_{k,j_n} - P'_{k,j_n}|.$$

Using these notations we can write P_{k,j_n} and P'_{k,j_n} as

$$P_{k,j_n} = \lambda \frac{\big(u_{x_2}(x_k^{j_n}), -u_{x_1}(x_k^{j_n})\big)}{|Du(x_k^{j_n})|} |P_{k,j_n} - P'_{k,j_n}| + Du(x_k^{j_n})$$

$$P'_{k,j_n} = -(1-\lambda) \frac{\big(u_{x_2}(x_k^{j_n}), -u_{x_1}(x_k^{j_n})\big)}{|Du(x_k^{j_n})|} |P_{k,j_n} - P'_{k,j_n}| + Du(x_k^{j_n}).$$

To simplify our notations we set

$$\overrightarrow{n} = \frac{\big(u_{x_2}(x_k^{j_n}), -u_{x_1}(x_k^{j_n})\big)}{|Du(x_k^{j_n})|} |P_{k,j_n} - P'_{k,j_n}|$$

and define a function $\psi_k^{j_n}$ by

$$\psi_k^{j_n}(x) = \begin{cases} \lambda \overrightarrow{n} x & \text{for } 0 \le \overrightarrow{n} x \le 1-\lambda, \\ -(1-\lambda)(\overrightarrow{n} x - 1) & \text{for } 1-\lambda \le \overrightarrow{n} x \le 1. \end{cases}$$

We see that $\psi_k^{j_n}(x) = 0$ on the lines $\overrightarrow{n} x = 0$ and $\overrightarrow{n} x = 1$. We extend $\psi_k^{j_n}$ periodically into $\mathbb{R}_2$ and this extended function is denoted again by $\psi_k^{j_n}$. We now set for every integer $m \ge 1$

$$\psi_{k,m}^{j_n}(x) = \frac{1}{m} \psi_k^{j_n}(mx)$$

and denote by $g_{k,m}^{j_n}$ the restriction of $\psi_{k,m}^{j_n}$ to the square $D_k^{j_n}$, that is,

$$g_{k,m}^{j_n}(x) = \psi_{k,m}^{j_n}(x) \mid_{D_k^{j_n}} .$$

One verifies easily that

$$|g_{k,m}^{j_n}(x)| \le \frac{1}{m} \text{ and } |Dg_{k,m}^{j_n}(x)| \le 2R \tag{7}$$

for all $x \in D_k^{j_n}$. We assume that m is sufficiently large to ensure that

$$\frac{1}{2^{j_n}} - 2\|g_{k,m_n}^{j_n}\|_{L^\infty(D_k^{j_n})} > 0 \tag{8}$$

for fixed n. Let $E_{k,m}^{j_n}$ be a square contained in $D_k^{j_n}$, with edges parallel to the coordinate axes and of length

$$\frac{1}{2^{j_n}} - 2\|g_{h,m}^{j_n}\|_{L^\infty(D_k^{j_n})}$$

and such that
$$\operatorname{dist}\left(\partial D_k^{j_n}, E_{k,m}^{j_n}\right) = \|g_{k,m}^{j_n}\|_{L^\infty(D_k^{j_n})}.$$

In the next step we define a function $h_{k,m}^{j_n}$ on $\partial D_k^{j_n} \cup E_{k,m}^{j_n}$ by

$$h_{k,m}^{j_n}(x) = \begin{cases} 0 & \text{for } x \in \partial D_k^{j_n}, \\ g_{k,m}^{j_n}(x) & \text{for } x \in E_{k,m}^{j_n}. \end{cases}$$

The function $h_{k,m}^{j_n}$ is Lipschitz on its domain of definition and its Lipschitz constant does not exceed $\max(1, 2R)$. Let $\phi_{k,m}^{j_n}$ be a Lipschitz extension of $h_{k,m}^{j_n}$ into $\bar{D}_k^{j_n}$ and define $\phi_{k,m}^{j_n}$ to be 0 outside $D_k^{j_n}$. We now choose m_n such that $m_n > n$ and $m_n > 2^{j_n+1}$ and

$$|D_k^{j_n} - E_{k,m_n}^{j_n}| \leq \frac{1}{20Kn|\Omega|}. \tag{9}$$

According to (7) the choice $m_n > 2^{j_n+1}$ ensures that (8) holds.

Case (b).

In this case $0 = Du(x_k^{j_n}) \in \operatorname{Int} H_{x_k^{j_n}, u(x_k^{j_n})}$ and we choose a line passing through 0. This line intersects $\partial H_{x_k^{j_n}, u(x_k^{j_n})}$ at two opposite points P_{k,j_n} and P'_{k,j_n}. Setting

$$\lambda = \frac{|P_{k,j_n}|}{|P'_{k,j_n} - P_{k,j_n}|} \quad \text{and} \quad 1 - \lambda = \frac{|P'_{k,j_n}|}{|P'_{k,j_n} - P_{k,j_n}|},$$

we then have

$$\begin{aligned} P_{k,j_n} &= \lambda |P'_{k,j_n} - P_{k,j_n}| \frac{P_{k,j_n}}{|P_{k,j_n}|}, \\ P'_{k,j_n} &= -(1-\lambda)|P'_{k,j_n} - P_{k,j_n}| \frac{P_{k,j_n}}{|P_{k,j_n}|}. \end{aligned}$$

Let
$$\overrightarrow{n_1} = |P'_{k,j_n} - P_{k,j_n}| \frac{P_k, j_n}{|P_{k,j_n}|}$$

and set
$$\xi_{k,j_n}(x) = \begin{cases} \lambda \overrightarrow{n_1} x & \text{for } 0 \leq \overrightarrow{n_1} x \leq 1 - \lambda, \\ -(1-\lambda)(\overrightarrow{n_1} x - 1) & \text{for } 1 - \lambda \leq \overrightarrow{n_1} x \leq 1. \end{cases}$$

For this function we repeat the preceding construction leading to a function $\bar{\phi}_{k,m_n}^{j_n}$ vanishing on $\mathbb{R}_2 - D_k^{j_n}$ and for which the pair $\left(D_k^{j_n}, E_{k,m_n}^{j_n}\right)$ satisfies (9).

Case (c).

We set

$$\phi_k^{j_n}(x) = 0 \quad \text{on} \quad D_k^{j_n}$$

and extend it by 0 outside $D_k^{j_n}$.

We can now define a minimizing sequence $\{u_n\}$ by $u_n(x) = u(x) + \phi_n(x)$, where

$$\phi_n(x) = \sum_{k=1}^{I_{j_n}} \beta_k^n(x)$$

with

$$\beta_k^n(x) = \begin{cases} \phi_{k,m_n}^{j_n}(x) & \text{if (a) holds,} \\ \bar{\phi}_{k,m_n}^{j_n}(x) & \text{if (b) holds,} \\ 0 & \text{if (c) holds.} \end{cases}$$

To show that $\lim_{n\to\infty} I(u_n) = 0$ we decompose H_{j_n} into the union $H_{j_n} = H_{j_n}^1 \cup H_{j_n}^2 \cup H_{j_n}^3$, where $H_{j_n}^1$, $H_{j_n}^2$ and $H_{j_n}^3$ are the unions of squares $D_k^{j_n}$ with centers $x_k^{j_n}$ satisfying (a), (b) and (c), respectively. We then have using (9)

$$\begin{aligned} I(u_n) &= \int_{H_{j_n}} |\, H(x, u_n, Du_n) \,|\; dx + \int_{\Omega - H_{j_n}} |\, H(x, u_n, Du_n) \,|\; dx \\ &\leq \frac{1}{4n} + \sum_{i=1}^{3} \int_{H_{j_n}^i} |\, H(x, u_n, Du_n) \,|\; dx = \frac{1}{4n} + J_1 + J_2 + J_3. \end{aligned}$$

To estimate J_1 we write

$$\begin{aligned} J_1 &= \\ &\sum_{D_k^{j_n} \in H_{j_n}^1} \left[\int_{D_k^{j_n} - E_{k,m_n}^{j_n}} |H(x, u_n, Du_n)|\, dx + \int_{E_{k,m_n}^{j_n}} |H(x, u_n, Du_n)|\, dx \right] \\ &= J_1^1 + J_1^2. \end{aligned}$$

According to (9) we have

$$|J_1^1| \leq \frac{1}{20n}. \tag{10}$$

Since $Du(x_k^{j_n}) + D\phi_n(x) \in \partial H_{x_k^{j_n}, u(x_k^{j_n})}$ we have

$$
\begin{aligned}
J_1^2 = &\sum_{D_k^{j_n} \in H_{j_n}^1} \int_{E_{k,m_n}^{j_n}} |H(x, u_n, Du_n)|\,dx \\
= &\sum_{D_k^{j_n} \in H_{j_n}^1} \left\{ \int_{E_{k,m_n}^{j_n}} \Big[|H(x, u(x) + \phi_n(x), Du(x) + D\phi_n(x))| \right. \\
&- |H(x_k^{j_n}, u(x) + \phi_n(x), Du(x) + D\phi_n(x))| \Big]\,dx \\
&+ \int_{E_{k,m_n}^{j_n}} \Big[|H(x_k^{j_n}, u(x) + \phi_n(x), Du(x) + D\phi_n(x))| \\
&- |H(x_k^{j_n}, u(x_k^{j_n}) + \phi_n(x), Du(x) + D\phi_n(x))| \Big]\,dx \\
&+ \int_{E_{k,m_n}^{j_n}} \Big[|H(x_k^{j_n}, u(x_k^{j_n}) + \phi_n(x), Du(x)\phi_n(x))| \\
&- |H(x_k^{j_n}, u(x_k^{j_n}) + \phi_n(x), Du(x_k^{j_n}) + D\phi_n(x))| \Big]\,dx \\
&+ \int_{E_{k,m_n}^{j_n}} \Big[|H(x_k^{j_n}, u(x_k^{j_n}) + \phi_n(x), Du(x_k^{j_n}) + D\phi_n(x))| \\
&\left. - \;|H(x_k^{j_n}, u(x_k^{j_n}), Du(x_k^{j_n}) + D\phi_n(x))| \Big]\,dx \right\} = H_1 + H_2 + H_3 + H_4.
\end{aligned}
$$

By virtue of (5) and (6) we have

$$
| H_i | \leq \frac{| H_{j_n}^1 |}{20n|\Omega|} \leq \frac{1}{20n}, \quad i = 1, 2, 3. \tag{11}
$$

Since $|\phi_n(x)| \leq \frac{1}{j_n}$, we find by (5) and (6) that

$$
|H_4| \leq \frac{1}{20n}. \tag{12}
$$

Combining (10)–(12) we obtain $J_1 \leq \frac{1}{4n}$. In a similar way we deduce the estimates $J_i \leq \frac{1}{4n}$, $i = 2, 3$. Consequently, we have

$$
I(u_n) \leq \frac{1}{n}.
$$

It follows from our construction of ϕ_n that

$$
\|\phi_n\|_{L^\infty(\Omega)} \leq \frac{1}{n} \text{ and } \|D\phi_n\|_{L^\infty(\Omega)} \leq 2R.
$$

Therefore, we may assume that $\phi \rightharpoonup 0$ weak-$*$ in $\overset{\circ}{W}{}^{1,\infty}(\Omega)$ and this completes the proof. □

3. A variational approach to the hyperbolic problem.

In Section 2 the minimizing sequence has been constructed for a function $u \in C^1(\bar{\Omega})$ subject to some constraint, namely, u must be a sub-solution of the equation $(*)$. This is the result of the convexity assumption. In this section we show that for the equation $(**)$ (see Introduction) any function in $C^1(\bar{\Omega})$ is limit of a minimizing sequence. In this case $H(x,u,P) = P_1P_2$ and the condition (2) does not hold. The equation $(**)$ is one of the image irradiance equations arising from the computer vision (see [BR]).

With the equation $(**)$ we associate a functional $J : W^{1,\infty}(\Omega) \to \mathbb{R}$ given by

$$J(u) = \int_\Omega |u_{x_1}u_{x_2} - E(x)|\, dx.$$

To get some insight into the structure of minimizing sequences, let us consider a sequence $\{u_j\}$ such that

$$\lim_{j\to\infty} J(u_j) = 0.$$

If $\{\nu_x\}$, $x \in \Omega$, is a family of probability measures (Young measures) corresponding to $\{Du_j\}$, then up to a subsequence we have

$$0 = \lim_{j\to\infty} J(u_j) = \int_\Omega \langle \nu_x, |\lambda_1\lambda_2 - E(x)|\rangle\, dx,$$

which means that supp $\nu_x \subset K_x$, where $K_x = \{(p,q);\, pq = E(x)\}$. We now observe that the lower convex envelope of $|\lambda_1\lambda_2 - E(x)|$ is identically equal to 0. These two observations suggest that it is possible to construct a minimizing sequence with oscillations occurring on K_x and convergent to given function from $C^1(\bar{\Omega})$ which is not subject to any additional conditions.

Theorem 3. *Let $u \in C^1(\bar{\Omega})$. Then there exists a sequence $\{u_j\}$ in $W^{1,\infty}(\Omega)$ such that $u_j \rightharpoonup u$ weak-$*$ in $W^{1,\infty}(\Omega)$, $u_j\,|_{\partial\Omega} = u\,|_{\partial\Omega}$ and $\lim_{j\to\infty} J(u_j) = 0$.*

Proof: We approximate Ω by a sequence of unions of squares

$$H_{jn} = \bigcup_{k=1}^{I_{jn}} D_k^{jn}$$

with the properties described in Theorem 2. The center of D_k^{jn} is denoted again by x_k^{jn}.

Let

$$M = \max\big(\|u_{x_1}\|_{L^\infty(\Omega)}, \|u_{x_2}\|_{L^\infty(\Omega)}, \|E\|_{L^\infty(\Omega)}\big)$$

and set

$$M_1 = M^2 + 2M \operatorname{diam}\Omega + \big(\operatorname{diam}\Omega\big)^2 + M.$$

For each integer $n \geq 1$ we can find an integer j_n such that

$$
\begin{aligned}
|u_{x_1}(x)u_{x_2}(x) - u_{x_1}(x_k^{j_n})u_{x_2}(x_k^{j_n})| &\leq \frac{1}{39n[|\Omega|(1+2\,\mathrm{diam}\ \Omega)]},\\
|u_{x_i}(x) - u_{x_i}(x_k^{j_n})| &\leq \frac{1}{39n[|\Omega|(1+2\,\mathrm{diam}\ \Omega)]},\ i = 1,2,
\end{aligned} \tag{14}
$$

$$
|\ E(x) - E(x_k^{j_n})\ | \leq \frac{1}{39n|\Omega|} \tag{15}
$$

for each $x \in D_k^{j_n}$ and

$$
|\ \Omega - H_{j_n}\ | \leq \frac{1}{13nM_1}. \tag{16}
$$

In order to construct a minimizing sequence we distinguish the following cases:

(i) $E(x_k^{j_n}) > 0$,

(ii) $E(x_k^{j_n}) < 0$,

(iii) $E(x_k^{j_n}) = 0$ and $u_{x_1}(x_k^{j_n})u_{x_2}(x_k^{j_n}) = 0$,

and

(iv) $E(x_k^{j_n}) = 0$ and $u_{x_1}(x_k^{j_n})u_{x_2}(x_k^{j_n}) \neq 0$.

Case (i).

We decompose $\mathbb{R}_2$ in the following way:

$$
\begin{aligned}
\mathbb{R}_2 &= \{(p,q);\ pq > E(x_k^{j_n}),\ p<0, q<0\} \cup \{(p,q);\ pq > E(x_k^{j_n}),\ p>0, q>0\}\\
&\cup \{(p,q);\ pq < E(x_k^{j_n})\} \cup \{(p,q);\ pq = E(x_k^{j_n})\}\\
&= A^1_{x_k^{j_n}} \cup A^2_{x_k^{j_n}} \cup A^2_{x_k^{j_n}} \cup K^+_{x_k^{j_n}}.
\end{aligned}
$$

To construct our minimizing sequence on $D_k^{j_n}$ we first consider the case $x_k^{j_n} \in A^1_{x_k^{j_n}}$. Let $\ell_{Du(x_k^{j_n})}$ be a straight line passing through $Du(x_k^{j_n})$ of the form

$$
x_1 + x_2 = u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n}).
$$

The line $\ell_{Du(x_k^{j_n})}$ intersects the branch of the hyperbola $K^+_{x_k^{j_n}}$ lying in the third quadrant at two points

$$
\begin{aligned}
\bar{x}_1 &= \frac{1}{2}\left[u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n}) + \sqrt{D(x_k^{j_n})}\right],\\
\bar{x}_2 &= \frac{1}{2}\left[u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n}) - \sqrt{D(x_k^{j_n})}\right]
\end{aligned}
$$

and

$$\tilde{x}_1 = \frac{1}{2}\left[u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n}) - \sqrt{D(x_k^{j_n})}\right],$$
$$\tilde{x}_2 = \frac{1}{2}\left[u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n}) + \sqrt{D(x_k^{j_n})}\right],$$

where

$$D(x_k^{j_n}) = \left(u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n})\right)^2 - 4E(x_k^{j_n}).$$

We now define two vectors $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$ pointing into the opposite directions and lying on $\ell_{Du(x_k^{j_n})}$ by

$$\overrightarrow{n}_1 = \left(\bar{x}_1, \bar{x}_2\right) - Du(x_k^{j_n}) \text{ and } \overrightarrow{n}_2 = \left(\tilde{x}_1, \tilde{x}_2\right) - Du(x_k^{j_n}).$$

The vectors $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$ can be written as

$$\overrightarrow{n}_1 = \alpha\overrightarrow{e} \text{ and } \overrightarrow{n}_2 = -\beta\overrightarrow{e},$$

where $\overrightarrow{e}$ is a unit vector and $\alpha > 0$ and $\beta > 0$ are constants. We now define a function $\phi_k^{j_n}$ by

$$\phi_k^{j_n}(x) = \begin{cases} \alpha\overrightarrow{e}x & \text{for } 0 \le \overrightarrow{e}x \le \beta, \\ \beta(\alpha + \beta - \overrightarrow{e}x) & \text{for } \beta \le \overrightarrow{e}x \le \alpha + \beta. \end{cases}$$

The function $\phi_k^{j_n}$ vanishes on the lines $\overrightarrow{e}x = 0$ and $\overrightarrow{e}x = \alpha + \beta$. We extend $\phi_k^{j_n}$ periodically into $\mathbb{R}_2$ and denote the extended function again by $\phi_k^{j_n}$. For every integer $m \ge 1$ we set

$$\phi_{k,m}^{j_n}(x) = \frac{1}{m}\phi_k^{j_n}(mx)$$

and let

$$h_{k,m}^{j_n}(x) = \phi_{k,m}^{j_n}(x)\mid_{D_k^{j_n}}.$$

It is obvious that

$$\|h_{k,m}^{j_n}\|_{L^\infty(D_k^{j_n})} \le \frac{\beta(\alpha+\beta)}{m} \text{ and } \|Dh_{k,m}^{j_n}\|_{L^\infty(D_k^{j_n})} \le \max(\alpha, \beta). \tag{17}$$

We now proceed as in the proof of Theorem 2. First, we assume that m is sufficiently large to ensure that

$$\frac{1}{2^{j_n}} - 2\|h_{k,m}^{j_n}\|_{L^\infty(D_k^{j_n})} > 0$$

for fixed m. The integer m will be chosen later. Let $E_{k,m}^{j_n}$ be a square contained in $D_k^{j_n}$, with edges parallel to the coordinate axes and of length

$$\frac{1}{2^{j_n}} - 2\|h_{k,m}^{j_n}\|_{L^\infty(D_k^{j_n})} > 0$$

and such that $\operatorname{dist}\left(\partial D_k^{j_n}, E_{k,m}^{j_n}\right) = \|h_{k,m}^{j_n}\|_{L^\infty(D_k^{j_n})}$. Finally, we define a function $g_{k,m}^{j_n}$ on $\partial D_k^{j_n} \cup E_{k,m}^{j_n}$ by

$$g_{k,m}^{j_n}(x) = \begin{cases} 0 & \text{for } x \in D_k^{j_n}, \\ h_{k,m}^{j_n}(x) & \text{for } x \in E_{k,m}^{j_n}. \end{cases}$$

The function $g_{k,m}^{j_n}$ is Lipschitz and its Lipschitz constant does not exceed $\max(1, \alpha, \beta)$. Let $\varphi_{k,m}^{j_n}$ be a Lipschitz extension of $g_{k,m}^{j_n}$ into $D_k^{j_n}$ and we extend $\varphi_{k,m}^{j_n}$ to all of $\mathbb{R}_2$, by setting $\varphi_{k,m}^{j_n}(x) = 0$ in $\mathbb{R}_2 - D_k^{j_n}$. We now choose m_n such that

$$m_n > \frac{n}{\alpha_1} \ \text{ and } \ m_n \geq 2^{j_n+1}\alpha_1,$$

where $\alpha_1 = \max(\alpha\beta, \beta(\alpha+\beta))$ and such that

$$| D_k^{j_n} - E_{k,m_n}^{j_n} | \leq \frac{1}{39n|\Omega|M_1}. \tag{18}$$

The function $\varphi_{k,m_n}^{j_n}$ corresponding to the pair $(D_k^{j_n}, E_{k,m_n}^{j_n})$ will be denoted by $F_{k,n}^1$, that is,

$$F_{k,n}^1(x) = \varphi_{k,m_n}^{j_n}(x).$$

A similar construction can be carried out if $x_k^{j_n} \in A^2_{x_k^{j_n}}$ and as a result we obtain a function $F_{k,n}^2(x)$ vanishing outside $D_k^{j_n}$ and with the corresponding pair $(D_k^{j_n}, E_{k,m_n}^{j_n})$ satisfying (18).
If $x_k^{j_n} \in A^3_{x_k^{j_n}}$ we take the straight line $\ell_{Du(x_k^{j_n})}$ of the form

$$x_2 - x_1 = u_{x_2}(x_k^{j_n}) - u_{x_1}(x_k^{j_n}).$$

The line $\ell_{Du(x_k^{j_n})}$ intersects both branches of the hyperbola $K^+_{x_k^{j_n}}$ at the points

$$\bar{x}_1 = \frac{1}{2}\left[u_{x_1}(x_k^{j_n}) - u_{x_2}(x_k^{j_n}) + \sqrt{D_1(x_k^{j_n})}\right],$$
$$\bar{x}_2 = \frac{1}{2}\left[u_{x_2}(x_k^{j_n}) - u_{x_1}(x_k^{j_n}) + \sqrt{D_1(x_k^{j_n})}\right]$$

and

$$\tilde{x}_1 = \frac{1}{2}\left[u_{x_1}(x_k^{j_n}) - u_{x_2}(x_k^{j_n}) - \sqrt{D_1(x_k^{j_n})}\right],$$
$$\tilde{x}_2 = \frac{1}{2}\left[u_{x_2}(x_k^{j_n}) - u_{x_1}(x_k^{j_n}) - \sqrt{D_1(x_k^{j_n})}\right],$$

where $D_1(x_k^{j_n}) = (u_{x_2}(x_k^{j_n}) - u_{x_1}(x_k^{j_n}))^2 + 4E(x_k^{j_n})$. Repeating the construction from the preceding step, we obtain a function $F_{k,n}^3(x)$ vanishing outside $D_k^{j_n}$ and with $(D_k^{j_n}, E_{k,m_n}^{j_n})$ satisfying (18).
If $x_k^{j_n} \in K^+_{x_k^{j_n}}$ we set $F_{k,n}^4(x) \equiv 0$ on $\bar{\Omega}$.

Case (ii).
The construction follows that of Case (i) with one difference: in the present case we decompose $\mathbb{R}_2$ as

$$\begin{aligned}\mathbb{R}_2 = \{pq < E(x_k^{j_n}),\, p<0, q>0\} \cup \{pq < E(x_k^{j_n}),\, p>0, q<0\}\\ \cup\,\{pq > E(x_k^{j_n})\} \cup \{pq = E(x_k^{j_n})\} = B^1_{x_k^{j_n}} \cup B^2_{x_k^{j_n}} \cup B^3_{x_k^{j_n}} \cup K^-_{x_k^{j_n}}.\end{aligned}$$

If $x_k^{j_n}$ belongs to one of the sets $B^1_{x_k^{j_n}}$, $B^2_{x_k^{j_n}}$ and $B^3_{x_k^{j_n}}$, then as in Case (i) we define functions $G_{k,n}^1$, $G_{k,n}^2$ and $G_{k,n}^3$, respectively, vanishing outside $D_k^{j_n}$ and with $(D_k^{j_n}, E_{k,m_n}^{j_n})$ satisfying (18).
If $x_k^{j_n} \in K^-_{x_k^{j_n}}$ we set $G_{k,n}^4(x) \equiv 0$ on $\bar{\Omega}$.

Case (iii).
In this case we define $K_{k,n}(x) \equiv 0$ on $\bar{\Omega}$.

Case (iv).
We split this case into four subcases:

(a) $u_{x_1}(x_k^{j_n}) > 0,\ u_{x_2}(x_k^{j_n}) > 0$,
(b) $u_{x_1}(x_k^{j_n}) < 0,\ u_{x_2}(x_k^{j_n}) > 0$,
(c) $u_{x_1}(x_k^{j_n}) < 0,\ u_{x_2}(x_k^{j_n}) < 0$,
(d) $u_{x_1}(x_k^{j_n}) > 0,\ u_{x_2}(x_k^{j_n}) < 0$.

Since all these cases can be treated in a similar way, we only consider the case (a). The straight line

$$x_1 + x_2 = u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n})$$

intersects the coordinate axes at the points

$$\big(u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n}), 0\big) \text{ and } \big(0, u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n})\big),$$

respectively. We now define two vectors $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$ lying on this line and pointing into opposite directions

$$\overrightarrow{n}_1 = \big(u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n}), 0\big) - Du(x_k^{j_n}) = \big(u_{x_2}(x_k^{j_n}), -u_{x_2}(x_k^{j_n})\big)$$

and

$$\overrightarrow{n}_2 = \big(0, u_{x_1}(x_k^{j_n}) + u_{x_2}(x_k^{j_n})\big) - Du(x_k^{j_n}) = \big(-u_{x_1}(x_k^{j_n}), u_{x_1}(x_k^{j_n})\big).$$

We write $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$ as $\overrightarrow{n}_1 = \alpha\overrightarrow{e}$ and $\overrightarrow{n}_2 = -\beta\overrightarrow{e}$, where $\overrightarrow{e}$ is a unit vector and $\alpha > 0$, $\beta > 0$ are constants. Using the vectors $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$ we define a function $\phi_k^{j_n}(x)$ as in Case (i). Following the construction from Case (i) we arrive at a function $K^1_{k,n}(x)$ vanishing outside $D_k^{j_n}$ and with $(D_k^{j_n}, E_{k,m_n}^{j_n})$ satisfying (18). In the remaining cases (b), (c) and (d) we repeat the above construction, always with a straight line passing through $Du(x_k^{j_n})$ and intersecting coordinate axes. As a result we obtain functions $K^2_{x_k^{j_n}}(x)$, $K^3_{k,n}(x)$ and $K^4_{k,n}(x)$ vanishing outside $D_k^{j_n}$ and with $(D_k^{j_n}, E_{k,m_n}^{j_n})$ satisfying (18).

To define the minimizing sequence we decompose H_{j_n} as

$$H_{j_n} = \bigcup_{s=1}^4 H^+_{s,j_n} \cup \bigcup_{s=1}^4 H^-_{s,j_n} \cup H^\circ_{j_n} \cup \bigcup_{s=1}^4 \widetilde{H}_{s,j_n},$$

where

> H^+_{s,j_n} is a collection of rectangles $D_k^{j_n}$ with centers belonging to $A^s_{x_k^{j_n}}$ ($s = 1, 2, 3$),
> H^+_{4,j_n} is a collection of rectangles $D_k^{j_n}$ with centers belonging to $K^+_{x_k^{j_n}}$,
> H^-_{s,j_n} is a collection of rectangles $D_k^{j_n}$ with centers belonging to $K^-_{x_k^{j_n}}$,
> $H^\circ_{j_n}$ is a collection of rectangles $D_k^{j_n}$ with centers $x_k^{j_n}$ satisfying $E(x_k^{j_n}) = 0$ and $u_{x_1}(x_k^{j_n})u_{x_2}(x_k^{j_n}) = 0$ (Case (iii)),

and

> $\widetilde{H}_{s,j_n}$ ($s = 1, 2, 3, 4$) is a collection of rectangles $D_k^{j_n}$ with centers $x_k^{j_n}$ satisfying (a), (b), (c) and (d), respectively (Case (iv)).

It follows from our construction that

$$Du(x_k^{j_n}) + DF^s_{k,n}(x) \in K^+_{x_k^{j_n}} \quad \text{for} \quad s = 1, 2, 3 \quad \text{and} \quad x_k^{j_n} \in A^s_{x_k^{j_n}}, \tag{19}$$

$$Du(x_k^{j_n}) + DG^s_{k,n}(x) \in K^-_{x_k^{j_n}} \quad \text{for} \quad s = 1, 2, 3 \quad \text{and} \quad x_k^{j_n} \in B^s_{x_k^{j_n}}, \tag{20}$$

and for $x_k^{j_n} \in \widetilde{H}_{s,j_n}$, $s = 1, 2, 3, 4$ we have

$$\left(u_{x_1}(x_k^{j_n}) + \frac{\partial K^s_{k,n}(x)}{\partial x_1}\right)\left(u_{x_2}(x_k^{j_n}) + \frac{\partial K^s_{k,n}(x)}{\partial x_2}\right) = 0, \quad s = 1, 2, 3, 4. \tag{21}$$

We are now in a convenient position to define the minimizing sequence

$$u_n(x) = u(x) + \phi_n(x),$$

where

$$\phi_n(x) = \sum_{k=1}^{I_{j_n}} \beta_k^n(x)$$

and

$$\beta_k^n(x) = \begin{cases} F_{k,n}^s(x) & \text{if } D_k^{j_n} \in H_{s,j_n}^+,\ s = 1,2,3, \\ G_{k,n}^s(x) & \text{if } D_k^{j_n} \in H_{s,j_n}^-,\ s = 1,2,3, \\ K_{k,n}^s(x) & \text{if } D_k^{j_n} \in \widetilde{H}_{s,j_n},\ s = 1,2,3,4, \\ 0 & \text{if } D_k^{j_n} \in H_{j_n}^\circ \cup H_{4,j_n}^+ \cup H_{4,j_n}^-. \end{cases}$$

To show that $\lim_{n\to\infty} J(u_n) = 0$ we write

$$\begin{aligned} J(u_n) = &\int_{\Omega - H_{j_n}} \Big| \left(u_{x_1}(x) + \frac{\partial \phi_n(x)}{\partial x_1} \right) \left(u_{x_2}(x) + \frac{\partial \phi_n(x)}{\partial x_2} \right) - E(x) \Big| \, dx \\ &+ \int_{H_{j_n}} \Big| \left(u_{x_1}(x) + \frac{\partial \phi_n(x)}{\partial x_1} \right) \left(u_{x_2}(x) + \frac{\partial \phi_n(x)}{\partial x_2} \right) - E(x) \Big| \, dx = I_1 + I_2. \end{aligned}$$

It follows from (16) that

$$|I_1| \le \frac{1}{13n}. \tag{22}$$

Setting

$$U_n(x) = \left(u_{x_1}(x) + \frac{\partial \phi_n(x)}{\partial x_1} \right) \left(u_{x_2}(x) + \frac{\partial \phi_n(x)}{\partial x_2} \right)$$

we write for I_2

$$\begin{aligned} I_2 = &\sum_{s=1}^{3} \int_{H_{s,j_n}^+} |U_n(x) - E(x)| \, dx + \sum_{s=1}^{3} \int_{H_{j_n}^-} |U_n(x) - E(x)| \, dx \\ &+ \sum_{s=1}^{4} \int_{\widetilde{H}_{s,j_n}} |U_n(x) - E(x)| \, dx + \int_{H_{j_n}^\circ} |u_{x_1}(x) u_{x_2}(x) - E(x)| \, dx \\ &+ \int_{H_{4,j_n}^+} |u_{x_1}(x) u_{x_2}(x) - E(x)| \, dx + \int_{H_{4,j_n}^-} |u_{x_1}(x) u_{x_2}(x) - E(x)| \, dx \\ &= \sum_{s=1}^{3} I_s^+ + \sum_{s=1}^{3} I_s^- + \sum_{s=1}^{4} \tilde{I}_s + I^\circ + I_4^+ + I_4^-. \end{aligned} \tag{23}$$

First we estimate $\sum_{s=1}^{3} I_s^+$ as follows

$$\sum_{s=1}^{3} I_s^+ \le \sum_{s=1}^{3} \sum_{D_k^{j_n}\in H_{s,j_n}^+} \Big[\int_{D_k^{j_n}-E_{k,m_n}^{j_n}} |U_n(x)-E(x)|\,dx + \int_{E_{k,m_n}^{j_n}} \Big|U_n(x)$$
$$- \left(u_{x_1}(x_k^{j_n}) + \frac{\partial\phi_n(x)}{\partial x_1}\right)\left(u_{x_2}(x_k^{j_n}) + \frac{\partial\phi_n(x)}{\partial x_2}\right)\Big|\,dx$$
$$+ \int_{E_{k,m_n}^{j_n}} \Big|\left(u_{x_1}(x_k^{j_n}) + \frac{\partial\phi_n(x)}{\partial x_1}\right)\left(u_{x_2}(x_k^{j_n}) + \frac{\partial\phi_n(x)}{\partial x_2}\right) - E(x_k^{j_n})\Big|\,dx$$
$$+ \int_{E_{k,m_n}^{j_n}} |\,E(x_k^{j_n}) - E(x)\,|\,\,dx\Big] = i_1 + i_2 + i_3 + i_4.$$

By (18) we have

$$i_1 \le \frac{|H_{j_n}|M_1}{39n|\Omega|M_1} < \frac{1}{39n}.$$

It follows from (14) and (15) that

$$i_k \le \frac{1}{39n}, \quad k = 2, 4.$$

We note that by (19) $i_3 = 0$ and consequently

$$\sum_{s=1}^{3} I_s^+ \le \frac{1}{13n}. \tag{24}$$

Similarly, using (14), (15), (18) and (20) we obtain

$$\sum_{s=1}^{3} I_s^- \le \frac{1}{13n}. \tag{25}$$

To estimate $\sum_{s=1}^{4} \tilde{I}_s$ we split the integration over $D_k^{j_n} \in \widetilde{H}_{s,j_n}$ in the same way as for $\sum_{s=1}^{3} I_s^+$ and arrive, using (14), (15), (18) and (21) at the estimate

$$\sum_{s=1}^{4} \tilde{I}_s \le \frac{1}{13n}.$$

To estimate I° we observe that

$$I^\circ = \int_{H_{j_n}^\circ} |u_{x_1}(x)u_{x_2}(x) - E(x)|\,dx \tag{26}$$
$$= \sum_{D_k^{j_n}\in H_{j_n}^\circ} \Big[\int_{D_k^{j_n}-E_{k,m_n}^{j_n}} |u_{x_1}(x)u_{x_2}(x) - E(x)|\,dx$$
$$+ \int_{E_{k,m_n}^{j_n}} |u_{x_1}(x)u_{x_2}(x) - E(x)|\,dx\Big] < \frac{1}{39n} + j,$$

now using the fact that $E(x_k^{j_n}) = 0$ and $u_{x_1}(x_k^{j_n})u_{x_2}(x_k^{j_n}) = 0$ we get by (14) and (15)

$$
\begin{aligned}
j = \sum_{D_k^{j_n} \in H_{j_n}^{\circ}} \Bigg[\int_{E_{k,m_n}^{j_n}} |u_{x_1}(x)u_{x_2}(x) - u_{x_1}(x_k^{j_n})u_{x_2}(x_k^{j_n})|\,dx \\
+ \int_{E_{k,m_n}^{j_n}} |E(x_k^{j_n}) - E(x)|\,dx \Bigg] \le \frac{2}{39n}.
\end{aligned} \tag{27}
$$

Analogously, since $Du(x_k^{j_n}) \in K^+_{x_k^{j_n}} \cup K^-_{x_k^{j_n}}$ for $D_k^{j_n} \in H^+_{4,j_n} \cup H^-_{4,j_n}$ we arrive at the estimate

$$
I_4^+ \le \frac{1}{13n} \quad \text{and} \quad I_4^- \le \frac{1}{13n}. \tag{28}
$$

Combining (22)-(28) we get $J(u_n) \le \frac{1}{n}$. It is clear that the sequence $\{u_n\}$ has also the remaining properties asserted in our theorem. □

4. The parabolic variational problem.

In this section we briefly discuss, for completeness, the parabolic case, that is the equation $(***)$. Setting for a fixed $x \in \Omega$ $H(x,p,q) = q^2 - p - E(x)$, we see that the lower envelope $CH(x,p,q)$ of $\mid H(x,p,q) \mid$ in (p,q) is given by

$$
CH(x,p,q) = \begin{cases} 0 & \text{for } q^2 - p - E(x) < 0 \\ q^2 - p - E(x) & \text{for } q^2 - p - E(x) \ge 0. \end{cases}
$$

Let us define a functional $F : W^{1,\infty}(\Omega) \to \mathbb{R}$ by

$$
F(u) = \int_{\Omega} |u_{x_2}^2 - u_{x_1} - E(x)|\,dx.
$$

We shall show that any candidate u for a minimizer of the problem

$$
\inf_{u \in W^{2,\infty}(\Omega)} F(u) = 0 \tag{29}
$$

must be a sub-solution of the equation $(***)$, that is,

$$
u_{x_2}(x)^2 - u_{x_1}(x) - E(x) \le 0 \ \text{ on } \Omega. \tag{30}
$$

This is due to the fact that $H(x,p,q)$ is convex. Indeed, suppose that $\{u_j\}$ is a bounded and minimizing sequence in $W^{1,\infty}(\Omega)$ for the problem (30). We may assume that $u_j \rightharpoonup u$ in $W^{1,\infty}(\Omega)$ and we have

$$
0 = \lim_{j \to \infty} F(u_j) = \int_{\Omega} CH(x, u_{x_1}, u_{x_2})\,dx.
$$

Since $CH(x, u_{x_1}, u_{x_2}) > 0$ on $\{x;\, u_{x_1}(x)^2 - u_{x_1}(x) > 0\}$ we get (30). The inequality (30) is obviously necessary condition for u to be a minimizer of the problem (29).

We now set $u_j(x) = u(x) + \phi_j(x)$, where $\phi_j \rightharpoonup 0$ in $W^{1,\infty}(\Omega)$. Denoting by $\{\nu_x\}$, $x \in \Omega$, a family of Young measures associated with $\{D\phi_j\}$ we get

$$0 = \int_\Omega \langle \nu_x(\cdot), |\big(u_{x_1}(x) + \lambda_2\big)^2 - \big(u_{x_1}(x) + \lambda_1\big) - E(x)|\rangle\, dx$$

and consequently

$$\operatorname{supp} \subset \Big\{(\lambda_1 \lambda_2);\ \big(u_{x_1}(x) - \lambda_2\big)^2 - \big(u_{x_1}(x) + \lambda_1\big) - E(x) = 0\Big\},$$

that is, $Du(x) + \lambda$ belongs to the parabola $\{(p,q);\, q^2 - p - E(x) = 0\}$. Consequently, we shall construct a minimizing sequence converging to u satisfying (30) in such a way that

$$Du(x) + D\phi_j(x) \in \{(p,q);\, q^2 - p - E(x) = 0\}$$

for sufficiently large number of values of x.

Theorem 4. *Let u be a function in $C^1(\bar{\Omega})$ satisfying* (30). *Then there exists a sequence $\{u_j\}$ in $W^{1,\infty}(\Omega)$ such that $u_j\,|_{\partial\Omega} = u\,|_{\partial\Omega}$, $u_j \rightharpoonup u$ weak-$*$ in $W^{1,\infty}(\Omega)$ and* $\lim_{j\to\infty} F(u_j) = 0$.

Proof: We construct a minimizing sequence in the form $u_n = u + \phi_n$, where $\phi_n \rightharpoonup 0$ weak-$*$ in $W^{1,\infty}(\Omega)$ and $\phi_n \in \overset{\circ}{W}{}^{1,\infty}(\Omega)$. The construction is similar to that given in the proofs of Theorems 2 and 3 and is even simpler.

Let $\{H_{j_n}\}$ be a sequence of unions of squares $D_k^{j_n}$, with centers at $x_k^{j_n}$, approximating Ω. We localize our construction to $D_k^{j_n}$. The integer j_n is obviously determined by the uniform continuity of u_{x_1}, u_{x_2} and E. We distinguish two cases:

(a) $u_{x_2}(x_k^{j_n})^2 - u_{x_1}(x_k^{j_n}) - E(x_k^{j_n}) < 0$

and

(b) $u_{x_2}(x_k^{j_n})^2 - u_{x_1}(x_k^{j_n}) - E(x_k^{j_n}) = 0$.

Case (a).

First, we find a vector $\overrightarrow{n} = (n_1, n_2)$ such that

$$\big(u_{x_2}(x_k^{j_n}) \pm n_2\big)^2 - \big(u_{x_1}(x_k^{j_n}) \pm n_1\big) = E(x_k^{j_n}).$$

This equation is satisfied by a vector $\overrightarrow{n}$ with coordinates given by

$$n_1 = 2u_{x_2}(x_k^{j_n})\sqrt{E(x_k^{j_n}) - u_{x_2}(x_k^{j_n})^2 + u_{x_1}(x_k^{j_n})},$$

and

$$n_2 = \sqrt{E(x_k^{j_n}) - u_{x_2}(x_k^{j_n})^2 + u_{x_1}(x_k^{j_n})}.$$

We now define a function $\phi_k^{j_n}(x)$ by

$$\phi_k^{j_n}(x) = \begin{cases} \overrightarrow{n}x & \text{for } 0 \le \overrightarrow{n}x \le \frac{1}{2}, \\ 1 - \overrightarrow{n}x & \text{for } \frac{1}{2} \le \overrightarrow{n}x \le 1, \end{cases}$$

which vanishes on the lines $\overrightarrow{n}x = 0$ and $\overrightarrow{n}x = 1$. We now repeat the construction from Theorems 2 and 3 leading to a function $\phi_{km_n}^{j_n}(x)$ vanishing on $\partial D_k^{j_n}$ and with $\operatorname{dist}(D_k^{j_n}, E_{k,m_n}^{j_n})$ sufficiently small.

Case (b).

We set $\phi_k^{j_n}(x) \equiv 0$ on Ω. We omit further details since the rest of the proof is now routine. □

Acknowledgement. The authors would like to express their gratitude to the unknown referee for careful examination of the paper and useful comments.

References

[DA] Adams R.A., *Sobolev spaces*, Academic Press, New York, 1975.
[BA] Ball J.M., *A version of the fundamental theorem for Young measures*, Lecture Notes in Physics, Springer Verlag **344** (1988), 207–215.
[BL] Berliochi H., Lasry J.M., *Intégrandes normales et mesures parametrées en calcul des variations*, Bull. Soc. Math. France **101** (1973), 129–184.
[BCK1] Brooks M.J., Chojnacki W., Kozera R., *Shading without shape*, Quarterly of Appl. Math., in press.
[BCK2] ———, *Circularly-symmetric eikonal equations and non-uniqueness in computer vision*, J. Math. Analysis and Appl., in press.
[BR] Anna R. Bruss, *The eikonal equation: some results applicable to computer vision*, J. Math. Phys. **23(5)** (1982), 890–896.
[CZ1] Chabrowski J., Zhang K., *On shape from shading problem*, Functional Analysis, Approximation Theory and Numerical Analysis, to appear in: Volume 1 dedicated to S. Banach, S. Ulam and A. Ostrowski, World Scientific Publishing, Singapore.
[CZ2] ———, *On variational approach to photometric stereo*, Ann. de l'Institut H. Poincaré, Analyse non linéaire, in press.
[DA] Dacorogna B., *Direct methods in the calculus of variations*, Applied Mathematical Sciences 78, Springer Verlag, Berlin - Heidelberg, 1989.
[DS] Deift P., Sylvester J., *Some remarks on the shape-from-shading problem in computer vision*, J. Math. Anal. Appl. **84(1)** (1982), 235–248.
[ET] Ekeland I., Temam R., *Analyse convexe et problèmes variationnels*, Dunod, Paris, 1974.
[EV] Lawrence C. Evans, *Weak convergence methods for nonlinear partial differential equations*, CBMS Regional Conference Series in Mathematics (74), Amer. Math. Soc.
[KG] Kohn R.V., Strang G., *Optimal design and relaxation of variational problems, I*, Comm. Pure and Applied Math. **39** (1986), 113–137.
[KO] Kozera R., *Existence and uniqueness in photometric stereo*, Applied Mathematics and Computation **44(1)** (1991), 1–103.
[TA] Tartar L., *Compensated compactness*, Heriot-Watt Symposium **4** (1978).

Department of Mathematics, The University of Queensland, St. Lucia 4067, Queensland, Australia

(*Received* September 23, 1992, *revised* March 23, 1993)