Multiple perturbed solutions near nondegenerate manifolds of solutions

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Abstract. The existence of multiple solutions for perturbed equations is shown near a manifold of solutions of an unperturbed equation via the Nielsen fixed point theory.

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1. Introduction

Let us consider

$$F_\varepsilon(x) = 0\,, \tag{$(1)_\varepsilon$}$$

where $F \in C^1(\mathbb{R} \times H, X)$, $F_\varepsilon = F(\varepsilon, \cdot)$ and $x \in H$, H, X are Hilbert spaces. We assume that $F_0(\mathcal{M}) = 0$ for a C^2-smooth, compact manifold $\mathcal{M}$. We are interested in the existence of multiple solutions of $(1)_\varepsilon$ for $\varepsilon \neq 0$ small near $\mathcal{M}$. We shall assume that $\mathcal{M}$ is a nondegenerate manifold of $(1)_0$, i.e. $D_xF_0(m)$ is a Fredholm operator of index 0 and $\ker D_xF_0(m) = T_m\mathcal{M}$, the tangent space of $\mathcal{M}$ at m, for any $m \in \mathcal{M}$. ($\ker B$ and $\mathrm{im}\, B$ denote the kernel and range of B, respectively.)

There are several results concerning this problem. If F_ε has a gradient form, i.e. $F_\varepsilon = \mathrm{grad}\, f_\varepsilon$ for a function f_ε, then it holds

Theorem 1.1 ([2],[7]). *Let $(1)_\varepsilon$ have a gradient form and let $\mathcal{M}$ be a compact, nondegenerate manifold for $(1)_0$. Then $(1)_\varepsilon$ has at least Cat $(\mathcal{M})$ (the Ljusternik-Schnirelman category of $\mathcal{M}$) of solutions near $\mathcal{M}$ for each $\varepsilon \neq 0$ small.*

Since $\mathcal{M}$ is C^2-smooth, there is a local coordinate system near $\mathcal{M}$ [2], i.e. each z near $\mathcal{M}$ can be uniquely expressed as $z = m + v$, $m \in \mathcal{M}$ and $z - m \in (T_m\mathcal{M})^\perp$. Moreover, since $\mathcal{M}$ is nondegenerate then there is a continuous family $\{(P_m, Q_m)\}_{m\in\mathcal{M}}$ of orthogonal projections such that

$$\mathrm{im}\; Q_m = \mathrm{im}\; D_xF_0(m) \;\text{ and }\; I = P_m + Q_m,\, \forall m \in \mathcal{M}\,.$$

Then $\Sigma = \bigcup_{m\in\mathcal{M}} \mathrm{im}\; P_m$ is a finite-dimensional vector bundle over $\mathcal{M}$ (see [4]). Now we are ready to state a result for a general case.

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Theorem 1.2 ([2]). *Let $\mathcal{M}$ be a compact, nondegenerate manifold of* $(1)_0$. *Then* $(1)_\varepsilon$ *has a solution near $\mathcal{M}$ for $\varepsilon \neq 0$ small provided that $\mathcal{X}(\Sigma) \neq 0$ (the Euler number of Σ* [10]).

Remark 1.3. If F_0 has a gradient form, then $D_x F_0$ is self-adjoint. Thus Σ is equivalent to the tangent bundle $T\mathcal{M}$ and $\mathcal{X}(T\mathcal{M}) = \chi(\mathcal{M})$ (the Poincaré-Euler characteristic of $\mathcal{M}$ [10]).

We have the following result according to Theorem 1.2 and Remark 1.3.

Theorem 1.4. *Let $\mathcal{M}$ be a compact, nondegenerate manifold of* $(1)_0$. *Assume F_0 has a gradient form and $\chi(\mathcal{M}) \neq 0$. Then* $(1)_\varepsilon$ *has a solution near $\mathcal{M}$ for any $\varepsilon \neq 0$ small.*

The purpose of this paper is to give a multiplicity result of solutions for $(1)_\varepsilon$. The basic role will play the Nielsen fixed point theory as in [1], [3]. Our approach is similar to these papers where retraction techniques of Nielsen fixed point theory are developed to produce lower bounds for the number of solutions. We shall generalize some results of [1] to the above problem $(1)_\varepsilon$. Finally, we note that F_0 possesses usually the manifold $\mathcal{M}$ provided that F_0 is invariant under a continuous, compact group of symmetries [6].

The plan of this paper is as follows. In Sections 2–4, we present multiplicity results based on the Nielsen fixed point theory for $(1)_\varepsilon$ and its modifications. We refer the reader to [5] for more details about this theory. Section 5 is devoted to an example for the illustration of possible applications of abstract results.

2. A multiple result

First we need the following

Definition 2.1 ([1]). *Let $r\colon W \to A$ be a map and let W, A be subsets of X such that $A \subset W$. If $r(a) = a$ for each $a \in A$ then r is called a retraction of W to A and A is called a retract of W.*

Definition 2.2 ([1]). *Let X be a normed space with a norm $|\cdot|$. Suppose that $T\colon A \to X$ is a map and $A \subset X$. Let W be a subset that retracts onto a subset A of itself by a retraction $r\colon W \to A$. We shall say that T is μ-retractible onto A with a retraction r if it holds*

$$\{x \in X \mid \text{there exists } a \in T(A) \text{ such that } |x - a| < \mu\} \subset W$$

and

$$\text{if } y \in W \setminus A,\ r(y) = x \text{ then } |y - T(x)| > \mu\,.$$

We shall say that T is retractible onto A with a retraction r if $T(A) \subset W$ and if $y \in W \setminus A$, $r(y) = x$ then $y \neq T(x)$.

We see that if T is μ-retractible onto A then any perturbation of T with an amount μ is still retractible onto A. The main advantage of this Definition 2.2 is the following: If T is retractible onto A with a retraction $r\colon W \to A$ then the map $r \circ T\colon A \to A$ has a fixed point $x \in A$ if and only if $T(x) = x$.

Now we suppose that there is an isomorphism J_m from im P_m onto $T_m\mathcal{M}$ and the dependence on m is continuous. (P_m is defined above.) Moreover, we assume that $\mathcal{M}$ is embedded into a finite dimensional space $\mathbb{R}^k$. We shall denote by Dom the domain of definition.

Theorem 2.3. *Let $\mathcal{M}$ be a compact, nondegenerate manifold of $(1)_0$. Assume the existence of a compact, locally contractible subset $S \subset \mathcal{M}$ such that the map*

$$\Pi(m) = m + J_m P_m D_\varepsilon F_0(m)$$

is μ-retractible onto S with a retraction ϕ, where ϕ: Dom $\phi \subset \mathbb{R}^k \to S$, $\Pi\colon \mathcal{M} \to \mathbb{R}^k$.

Then $(1)_\varepsilon$ has at least $N(\phi \circ \Pi)$ (the Nielsen number of $\phi \circ \Pi\colon S \to S$) solutions near $\mathcal{M}$ for any $\varepsilon \neq 0$ small.

Proof: Let us modify $(1)_\varepsilon$ in the following way

$$\begin{aligned} &Q_m F_\varepsilon(m+v) = 0,\; v \in (T_m\mathcal{M})^\perp \\ &P_m F_\varepsilon(m+v) = 0\,. \end{aligned} \tag{2.1}$$

Since $\mathcal{M}$ is nondegenerate for $(1)_0$, we have that

$$C(m) = D_v Q_m F_0(m) = Q_m D_v F_0(m)$$

is invertible. Thus we can solve $v = v(m,\varepsilon)$ from the first equation of (2.1) for ε small and $v(m,0) = 0$. Note that v is C^1-smooth in ε. We have to solve the bifurcation equation

$$Q(m,\varepsilon) = P_m F_\varepsilon(m + v(m,\varepsilon)) = 0\,. \tag{2.2}$$

It holds $Q(m,0) = 0$ and $D_\varepsilon Q(m,0) = P_m D_\varepsilon F_0(m) + P_m D_x F_0(m) \cdot D_\varepsilon v(m,0) = P_m D_\varepsilon F_0(m)$. Thus we consider $B(m,\varepsilon) = m + J_m Q(m,\varepsilon)/\varepsilon = m$ instead of $Q(m,\varepsilon) = 0$ for $\varepsilon \neq 0$ small. But $B(m,\varepsilon)$ is near to Π for ε small. Using the μ-retractibility of Π, we have that $B(\cdot,\varepsilon)$ is retractible onto S with the retraction ϕ. Thus fixed points of $\phi \circ B(\cdot,\varepsilon)$ on S are precisely fixed points of $B(\cdot,\varepsilon)$. Since $B(\cdot,\varepsilon)$ is homotopic to Π for $\varepsilon \neq 0$ small, we have $N(\phi \circ B(\cdot,\varepsilon)) = N(\phi \circ \Pi)$. Hence $B(\cdot,\varepsilon)$ has at least $N(\phi \circ \Pi)$ fixed points in S for any $\varepsilon \neq 0$ small. The proof is finished. □

Remark 2.4. By following the proof of Theorem 2.3 we see that the compactness of $\mathcal{M}$ can be dropped. Moreover, H, X can be only Banach spaces for some cases. Indeed, we assume that H, X are Hilbert spaces only for the existence of a local coordinate system near $\mathcal{M}$, and for the existence of projections $\{P_m\}_{m\in\mathcal{M}}$ mentioned in Introduction. For specific cases, it is possible to construct explicitly both such a system of coordinates and those projections. For instance, if $\mathcal{M}$ is a closed, linear subspace of H.

3. A generalization

Suppose that $H \subset X$ is compactly embedded into X. Let us consider $(1)_\varepsilon$ in the form

$$F_\varepsilon(x) = G(x) - \varepsilon \cdot T(x)\,, \tag{$(2)_\varepsilon$}$$

where $G \in C^1(H, X)$, $T \in C^0(X, X)$. We still assume that $\mathcal{M}$ is a nondegenerate manifold of $F_0 = G$. The difference between $(2)_\varepsilon$ and the above problem is the following: We have a general, but a C^1-smooth perturbation of F_0 in $(1)_\varepsilon$. On the other hand, $(2)_\varepsilon$ is a continuous, but a special perturbation of F_0.

Theorem 3.1. *The assertion of Theorem* 2.3 *remains true for* $(2)_\varepsilon$ *with*

$$\Pi(m) = m + J_m P_m T(m)\,.$$

Proof: We modify $(2)_\varepsilon$ as $(1)_\varepsilon$ in the following way

$$\begin{aligned} Q_m G(m+v) &= \varepsilon \cdot Q_m T(m+v) \\ P_m G(m+v) &= \varepsilon \cdot P_m T(m+v)\,. \end{aligned} \tag{3.1}$$

Since $\mathcal{M}$ is nondegenerate for G, we have that

$$v \to Q_m G(m+v)$$

is invertible near $v = 0$. Thus (3.1) has the form

$$\begin{aligned} v &= \varepsilon \cdot K(m, v, \varepsilon) \\ P_m G(m+v) &= \varepsilon \cdot P_m T(m+v)\,, \end{aligned}$$

and

$$\begin{aligned} v &= \varepsilon \cdot K(m, v, \varepsilon) \\ P_m G(m + \varepsilon \cdot K(m, v, \varepsilon)) &= \varepsilon \cdot P_m T(m + \varepsilon \cdot K(m, v, \varepsilon))\,. \end{aligned} \tag{3.2}$$

Since $P_m G(m) = 0$ and $P_m DG(m) = 0$ we have

$$\begin{aligned} P_m G(m + \varepsilon \cdot K(m, v, \varepsilon)) &= P_m(G(m + \varepsilon \cdot K(m, v, \varepsilon)) - G(m)) = \\ &= P_m DG(m) \varepsilon K(m, v, \varepsilon) + \varepsilon \cdot o(1) = \\ &= \varepsilon \cdot o(1) \quad \text{as } \varepsilon \to 0\,. \end{aligned}$$

Then (3.2) has the form

$$\begin{aligned} v &= \varepsilon \cdot K(m, v, \varepsilon) \\ 0 &= \varepsilon \cdot o(1) + \varepsilon P_m T(m + \varepsilon \cdot K(m, v, \varepsilon)) \end{aligned}$$

i.e.

$$
\begin{aligned}
&v = \varepsilon \cdot K(m, v, \varepsilon) \\
&m = m + J_m P_m T(m + \varepsilon \cdot K(m, v, \varepsilon)) + o(1) = R(\varepsilon, m, v)\,.
\end{aligned}
\tag{3.3}
$$

Now we take

$$E = \{(m, v) \mid m \in S, v \in (T_m\mathcal{M})^{\perp},\ |v| \le \delta\}$$

for $\delta > 0$ small and

$$J(\varepsilon, m, v) = \big(\phi(R(\varepsilon, m, v)), \varepsilon \cdot K(m, v, \varepsilon)\big)$$

for ε small. It is clear that E is an ANR (Absolute Neighbourhood Retract) and $J\colon E \to E$ is a continuous, compact mapping. Using the μ-retractibility of Π we see as in the proof of Theorem 2.3 that fixed points of J on E are solutions of (3.3).

On the other hand,

$$N\big(J(\varepsilon, \cdot, \cdot)\big) = N\big(J(0, \cdot, \cdot)\big) = N(\phi \circ \Pi)\,.$$

This finishes the proof. □

4. A simple bifurcation theorem

Finally, we give a simple application to certain bifurcation problem. Let us consider

$$Lx = T(x, \lambda)\,, \tag{4.1}$$

where $L\colon X \to Y$ is a Fredholm, bounded, linear operator, $\lambda \in \mathbb{R}^n$, X, Y are Banach spaces, $T \in C^1(X \times \mathbb{R}^n, Y)$. Let $T(0, \cdot) = 0$. We are interested in the existence of nonzero solutions of (4.1). For this purpose, moreover we assume that $|QD_xT(0,\lambda)| \cdot |L^{-1}| < 1$, $\forall \lambda$ and codim im $L = n$. Here $Q\colon Y \to$ im L is a continuous projection and $L^{-1}\colon$ im $L \to X_1$, $\quad X_1 \oplus \ker L = X$.

We put

$$
\begin{aligned}
&\tilde{X} = X_1 \times \mathbb{R}^n, \\
&\tilde{L}(x_1, \lambda) = Lx_1, \quad \tilde{L}\colon \tilde{X} \to Y \\
&\tilde{T}(x_1, \lambda, x_2) = T(x_1 + x_2, \lambda), \quad \tilde{T}\colon \tilde{X} \times \ker L \to Y\,.
\end{aligned}
$$

Then we rewrite (4.1) as $\tilde{L}z = \tilde{T}(z, \varepsilon \cdot x_2)$, $z \in \tilde{X}$ and $x_2 \in \ker L$ is fixed. We consider ε as a parameter. Now $\mathcal{M} = \{(x_1, \lambda) \mid x_1 = 0\}$ and $\mathcal{M}$ is nondegenerate due to codim im $L = n$ and $|QD_xT(0,\cdot)| \cdot |L^{-1}| < 1$. Indeed, the linearization of the operator $\tilde{L} - \tilde{T}(\cdot, 0)$ at $\mathcal{M}$ is $\Big(\tilde{L} - D\tilde{T}\big((0,\lambda), 0\big)\Big)(x_1, \lambda_1) = Lx_1 - D_xT(0,\lambda)x_1$. We see that the equation $Lx_1 = D_xT(0,\lambda)x_1$ has only the solution $x_1 = 0$. Moreover

$$Q(\lambda)f = -(I - Q)D_xT(0,\lambda)\Big(\big(L - QD_xT(0,\lambda)\big)^{-1}Qf\Big) + Qf$$

is a projection of Y onto im $\left(\tilde{L} - D\tilde{T}\big((0,\lambda),0\big)\right)$.

Now we can apply Theorem 2.3 (see Remark 2.4) and obtain

$$\Pi(\lambda) = \lambda + JP(\lambda)D_xT(0,\lambda)x_2\,, \tag{4.2}$$

where

$$\begin{aligned}&P(\lambda) = I - Q(\lambda) = P\Big(D_xT(0,\lambda)\big(\big(L - QD_xT(0,\lambda)\big)^{-1}Q\big) + I\Big)\\ &P = I - Q\end{aligned}$$

and J is an isomorphism of im P onto $\mathbb{R}^n$. Moreover, the property of retractibility of Definition 2.2 is open. Hence if Π is retractible in the sense of Definition 2.2 for some $x_2 \in \ker L$ fixed, then Π is retractible also for any $z \in \ker L$ near to x_2. Summing up we have

Theorem 4.1. *Let L be a Fredholm, bounded, linear operator, $T \in C^1(X\times\mathbb{R}^n, Y)$, $T(0,\cdot) = 0$, $|QD_xT(0,\cdot)|\cdot|L^{-1}| < 1$ and* codim im $L = n$. *Here $L^{-1}\colon$ im $L \to X_1$, $X_1 \oplus \ker L = X$ and $Q\colon Y \to$ im L is a continuous projection. Let $S \subset \mathbb{R}^n$ be a locally contractible, compact subset.*

Assume Π defined by (4.2) is μ-retractible onto S with a retraction ϕ for some $x_2 \in \ker L$. Then for any $z \in \ker L$ sufficiently near to x_2, the equation (4.1) has at least $N(\phi\circ\Pi)$ branches of nontrivial solutions of the forms

$$\begin{aligned}&x_{1i}(\varepsilon) + \varepsilon\cdot z,\ \lambda_i(\varepsilon),\ x_{1i}(\varepsilon) \in X_1, \quad \varepsilon \neq 0 \text{ is small}\\ &i = 1,\dots,N(\phi\circ\Pi)\end{aligned}$$

bifurcating from the zero solution as $\varepsilon \to 0$.

Remark 4.2. We see that the dependence of (4.2) on x_2 is linear and the validity of Theorem 4.1 is caused partly by the nonlinearity of (4.2) in λ and partly by the relation codim im $L = n$. The variable x_2 is generally involved more sophistically (see [1]).

Remark 4.3. If $QD_xT(0,\cdot)x_2 = 0$ for some $x_2 \in \ker L$, then the formula (4.2) has the form

$$\Pi(\lambda) = \lambda + JD_xT(0,\lambda)x_2\,.$$

5. An Example

In this section, we give an example to illustrate the above abstract result. The most difficult task in verification of the above assumptions is the computation of Nielsen numbers for given mappings.

Example.

We shall apply Theorem 4.1. Consider

$$\begin{aligned}x_1' &= x_1\cdot f_1(x_1,x_2,t,\lambda)\\ x_2' &= x_2\cdot f_2(x_1,x_2,t,\lambda)\,,\end{aligned} \tag{5.1}$$

where f_1, $f_2 \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are C^1-smooth, 1-periodic in t. Assume $f_i(0,0,t,\lambda)$, $i = 1,2$ are independent of t.

We put

$$\begin{aligned}
&Y = \{z \in C^0(\mathbb{R}, \mathbb{R}^2) \mid z \text{ is 1-periodic}\}\\
&X = \{z \in Y \mid z \text{ is } C^1\text{-smooth}\}\\
&Lx = x',\ T(x,\lambda) = f(x,\lambda)\\
&f = (x_1 \cdot f_1, x_2 \cdot f_2),\ x = (x_1, x_2)\,.
\end{aligned}$$

We assume that f_1, f_2 are C^0-small. Then $D_x T(0,\lambda)$ is also small. We see that $\ker L = \{x = \text{constant}\}$ and $Pz = \int\limits_0^1 z(t)dt$, where we identify constant functions with real numbers. Moreover, it holds

$$QD_x T(0,\cdot)z = 0 \quad \forall z \in \ker L\,,$$

since $f_i(0,0,t,\lambda)$, $i = 1,2$ are independent of t. By Remark 4.3 the formula (4.2) has the following form for this case

$$\begin{aligned}
&\Pi(\lambda) = \lambda + \Big(c_1 \cdot f_1(0,0,t,\lambda), c_2 \cdot f_2(0,0,t,\lambda)\Big)\\
&c_1,\ c_2 \in \mathbb{R}\,.
\end{aligned}$$

Since f_i, $i = 1,2$ are C^0-small, we take $c_1 = c_2 = d$ sufficiently large, fixed and assume

$$\begin{aligned}
&f_i(0,0,t,\lambda) = \big(g_i(\lambda) - \lambda_i\big)/d, \quad i = 1,2\\
&\lambda = (\lambda_1, \lambda_2)\,.
\end{aligned}$$

Then

$$\Pi = g = (g_1, g_2)\,.$$

Moreover, let $S = A = \{\lambda \in \mathbb{R}^2 \mid 1/2 \le |\lambda| \le 1\}$ be the annulus with the retraction [1], [3]

$$\rho(\lambda) = \begin{cases} \lambda/2|\lambda|, & 0 < |\lambda| < 1/2\\ \lambda, & 1/2 \le |\lambda| \le 2\\ 2\lambda/|\lambda|, & 2 < |\lambda|\,. \end{cases}$$

Then we can construct the map g according to [3, p. 54] such that g is μ-retractible onto S with the retraction ρ for a $\mu > 0$ small. For instance, we can take $g(\lambda) = q(|\lambda|) \cdot \lambda^k$ (we identify $\mathbb{R}^2$ with $\mathbb{C}-$ the complex plane) satisfying

$$\begin{aligned}
&q(1/2)/2^k \ge 1/2 \quad + \mu,\\
&q(b) \cdot b^k > \mu \text{ for } b \ge 1/2,\\
&2^k \cdot q(2) \le 2 - \mu,\ k \in \mathcal{N} \setminus \{1\}\,.
\end{aligned} \tag{5.2}$$

We see that the above conditions for q are precisely the assumptions of [3, Proposition 1.5] for the map $g(\lambda) = q(|\lambda|) \cdot \lambda^k$. Finally, we know [3] that $N(\rho \circ g) = |\deg \lambda^k - 1| = k - 1$.

Summing up we obtain

Theorem 5.1. *Let f_1, $f_2 \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be C^1-smooth and 1-periodic in t. Assume*

(a) *f_i, $i = 1, 2$ are C^0-small, i.e. $|f_1|$, $|f_2| < K$ for a K sufficiently small;*
(b) *$f_i(0, 0, t, \lambda) = \big(g_i(\lambda) - \lambda_i\big)/d$, $i = 1, 2$, $\lambda = (\lambda_1, \lambda_2)$ with d large, fixed and $g(\lambda) = q\big(|\lambda|\big)\lambda^k$ (identifying $\mathbb{R}^2$ with $\mathbb{C}$) where q satisfies (5.2) and $g = (g_1, g_2)$.*

Then for any (c_1, c_2) near to (d, d), the equation (5.1) has at least $k - 1$ branches of nonzero, 1-periodic solutions of the forms $x_{1\varepsilon,i}$, $x_{2\varepsilon,i}$, $\lambda^i(\varepsilon)$, $i = 1, \dots, k - 1$ satisfying

$$\int_0^1 \Big(x_{1\varepsilon,i}(t), x_{2\varepsilon,i}(t)\Big)\,dt = \varepsilon \cdot (c_1, c_2)\,.$$

for any $\varepsilon \neq 0$ small and $i = 1, \dots, k - 1$. These solutions bifurcate from the zero one.

Finally, we note that the construction of functions f_1, f_2 in Example suggests an approach allowing the application of our abstract results to a broad variety of nonlinear equations. Hence the application of Theorems 2.3 and 3.1 is similar as in the above example in the framework of higher-dimensional Melnikov functions [8], [9]. Indeed, by following the proof of Theorem 5.1 together with the paper [9], we can derive ordinary differential equations possessing two-dimensional Melnikov functions, and those Melnikov functions are both μ-retractible onto the above annulus A with the retraction ρ and they are homotopic to the above map g. Those Melnikov functions predict the existence of multiple solutions of corresponding ordinary differential equations under small perturbations. As a matter of fact, this strategy has been already used in [1] and [9].

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