Equivalence and zero sets of certain maps in infinite dimensions

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Abstract. Equivalence and zero sets of certain maps on infinite dimensional spaces are studied using an approach similar to the deformation lemma from the singularity theory.

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1. Introduction

In this paper we shall use a singularity theory approach to study both right equivalance (see [1, p. 1038]) of certain two maps in Banach spaces, and zero sets of maps near their critical points. The method used in this paper is described in [1], where it was used in a proof of Tromba's Morse lemma. Using this method we obtain both a theorem which is a generalization of Kuiper's theorem [5], [6], and an infinite dimensional version of Theorem 1.3 of [2]. From the theorem in Section 2 it follows the splitting lemma [1].

The plan of the paper is as follows

1. Theorem 2.1 in Section 2 gives conditions under which two functions are related by a homeomorphism in some neighbourhood of a singular point.

2. Section 3 discusses the splitting lemma.

3. Section 4 deals with the infinite dimensional version of the Buchner, Marsden and Schecter theorem [2]. That theorem provides a relation between the zero set of a map near its singular point and the zero set of the first nonzero term of the Taylor expansion of that map at that singular point near that point.

2. The generalization of Kuiper's theorem

Theorem 2.1. *Let E be a Banach space. Let Q, $P\colon U \to \mathbb{R}$ be C^1-maps defined on a neighbourhood U of $0 \in E$ such that $Q(0) = P(0) = 0$ and DP, DQ are Lipschitz. Let A be a vector field defined on $U^+ = U \setminus \{0\}$ and $f\colon U \to \mathbb{R}$. We assume*

(1) $A \in C^1(U^+)$, $\| A(x) \| \leq 1$ *for any* $x \in U^+$*;*

(2) $DQ(x) \cdot A(x) \geq c \cdot f(x)$ *for some constant* $c > 0$, $x \in U^+$ *and* $\lim\limits_{x\to 0} \frac{|DP(x)|}{f(x)} = 0$*;*

(3) $f \in C^1(U^+)$, $f \in C^0(U)$, $f(0) = 0$, $f(x) > 0$ *for* $x \neq 0$, $f(t \cdot x) \leq K \cdot f(x)$ *for any* $0 \leq t \leq 1$ *and* $x \in U$, $K > 0$ *is constant.*

Then $Q + P$ is C^0-right equivalent to Q at 0.

We say that functions g, f defined on a neighbourhood of 0 with $g(0) = f(0) = 0$ are C^0-right equivalent if there is a homeomorphism r defined on a neighbourhood of 0 with $r(0) = 0$ such that $g(x) = h\big(r(x)\big)$.

Let us consider the initial value problem

$$
\begin{aligned}
y_t'(x) &= -P\big(y_t(x)\big) \cdot \bar{A}\big(y_t(x)\big) \\
y_0(x) &= x,
\end{aligned}
\tag{1}
$$

where $x \in U^+$, $y_t'(x) = \frac{d}{dt} y_t(x)$, $\bar{A}(x) = \frac{A(x)}{f(x)}$. Since P, $\bar{A} \in C^1$ there is a unique local solution of (1).

Lemma 2.2. *For any $T > 0$ there exists an open neighbourhood V_T of $0 \in E$ such that for $x \in V_T \setminus \{0\}$ the initial value problem* (1) *has a unique solution on the interval* $(-T, T)$.

Proof of Lemma 2.2: In the standard arguments we obtain

$$
\begin{aligned}
|P(x)| &\le \int_0^1 |DP(t \cdot x) \cdot x| dt \le \| x \| \cdot \int_0^1 |DP(t \cdot x)|\, dt \\
&\le \int_0^1 M_1 \cdot f(t \cdot x) \cdot \| x \| \ dt \le M_1 \int_0^1 K \cdot f(x) \cdot \| x \| \ dt \le M_2 \cdot f(x) \cdot \| x \|,
\end{aligned}
$$

where $M_2 = K \cdot M_1$, M_1 follows from the condition 2. Thus for a sufficiently small x we have

$$
|P(x)| \le M_2 \cdot \| x \| \cdot f(x), \tag{2}
$$

where M_2 is a positive constant. Hence from the assumption 1 and (2) we have for $x \neq 0$

$$
\begin{aligned}
\| y_t(x) \| &\le \int_0^t \| y_t'(x) \| \ ds + \| x \| \\
&\le \| x \| + \int_0^t \frac{\| P\big(y_s(x)\big) \cdot A\big(y_s(x)\big) \|}{f\big(y_s(x)\big)} \le \| x \| + \int_0^t M_2 \cdot \| y_s(x) \| \ ds.
\end{aligned}
$$

Using the Gronwall's lemma we have

$$
\| y_t(x) \| \le \| x \| \cdot e^{M_2 \cdot t} \le \| x \| \cdot e^{M_2 \cdot T} \le \| x \| \cdot M_4.
$$

By (2) it follows

$$\begin{aligned}\| x \| - \| y_s(x) \| &\leq \| y_s(x) - x \| \leq \| y_r'(x) \| \cdot |s| \\ &\leq T \cdot \frac{\| P\big(y_r(x)\big) \cdot A\big(y_r(x)\big) \|}{f\big(y_r(x)\big)} \leq T \cdot \| y_r(x) \| \cdot M_2\end{aligned}$$

for some $r \in (-T, T)$, and we obtain

$$\| x \| \leq \| y_s(x) \| + T \cdot \| y_r(x) \| \cdot M_2 \leq \| y_s(x) \| + M_2 \cdot T \cdot e^{M_2 \cdot T} \cdot \| x \| .$$

For a sufficiently small x we can find a small M_2 as well. Hence

$$\| x \| \leq \tilde{c} \cdot \| y_s(x) \|$$

for a constant $\tilde{c} > 0$. This finishes the proof, since

$$\| x \| / \tilde{c} \leq \| y_t(x) \| \leq M_4 \cdot \| x \|, \; \forall\, x \neq 0 \text{ small}, \; t \in [-T, T].$$

$\square$

Proof of Theorem 2.1: Consider the initial value problem

$$\begin{aligned}&\Big(DQ\big(y_t(x)\big) + h(t,x) \cdot DP\big(y_t(x)\big)\Big) \cdot \bar{A}\big(y_t(x)\big) = h'(t,x) \\ &h(0,x) = 0, \; x \neq 0 \\ &y_t(x) \text{ is the solution of (1),}\end{aligned} \tag{4}$$

where $x \in V_T$ and $T > 3/c$ is sufficiently large. Let us choose a small neighbourhood V_1 of 0 such that $V_1 \subset U$ and for $0 \neq x \in V_1$

$$\| DP\big(y_t(x)\big) \cdot \bar{A}\big(y_t(x)\big) \| < c/4.$$

Since $\lim\limits_{x \to 0} \frac{\|DP(x)\|}{f(x)} = 0$ and $\| y_t(x) \| \leq M_4 \cdot \| x \|$ we can find such V_1.
If $|h(t,x)| < 2$ for $t \in [0, T]$ then

$$\begin{aligned}h'(t,x) &= \Big(DP\big(y_t(x)\big) \cdot h(t,x) + DQ\big(y_t(x)\big)\Big) \cdot \bar{A}\big(y_t(x)\big) \\ &\geq -2 \cdot c/4 + c \geq c/2,\end{aligned}$$

for $x \in (V_T \setminus \{0\}) \cap V_1 = V_T^+$, and hence

$$h(T,x) \geq T \cdot c/2 > (3/c) \cdot c/2 = 3/2.$$

Since $h(0,x) = 0$ we obtain a C^0-map $t(x) \colon V_T^+ \to \mathbb{R}$ such that

$$h\big(t(x), x\big) = 1. \tag{+}$$

We put

$$H(x) = y_{t(x)}(x)$$

for any $x \in V_T^+$ and $H(0) = 0$. Since it holds

$$\| y_t(x) \| \leq M_4 \cdot \| x \| \ \forall x \neq 0 \text{ small}, \ t \in (-T, T)$$

from the proof of Lemma 2.2, the map H is continuous.

By the equations (4) and (1) we have

$$\frac{d}{dt}\Big(Q\big(y_t(x)\big) + h(t,x) \cdot P\big(y_t(x)\big)\Big) = 0$$

and using (+) we obtain

$$\begin{aligned} Q(x) &= Q\big(y_{t(x)}(x)\big) + h\big(t(x), x\big) \cdot P\big(y_{t(x)}(x)\big) \\ &= Q\big(y_{t(x)}(x)\big) + P\big(y_{t(x)}(x)\big). \end{aligned} \tag{5}$$

Lastly we show that H is a local homeomorphism. If we put

$$Q_1(x) = Q(x) + P(x) \text{ and } P_1(x) = -P(x)$$

then similarly as above we obtain maps $y_t^1(x) = y_{-t}(x)$ and $t^+(x)$. Hence $(Q_1 + P_1)\big(y_{-t^+(z)}(z)\big) = Q_1(z)$. We have

$$\begin{aligned} &Q\big(y_{-t^+(z)+t(x)}(x)\big) = Q\Big(y_{-t^+(z)}\big(y_{t(x)}(x)\big)\Big) = (Q_1 + P_1)\big(y_{-t^+(z)}(z)\big) = \\ &Q_1(z) = (Q + P)\big(y_{t(x)}(x)\big) = Q(x), \end{aligned}$$

where $z = y_{t(x)}(x)$. We have used the "flow" property of $y_t(x)$ at t in the previous equality. But

$$\frac{d}{dt}Q\big(y_t(x)\big) = -P\big(y_t(x)\big) \cdot DQ \cdot \bar{A}\big(y_t(x)\big).$$

According to the assumptions of Theorem 2.1, the map $w(t) = Q\big(y_t(x)\big)$ is monotone, and thus $t^+(z) = t(x)$ for $z = H(x)$. Hence

$$y_{-t^+(z)}(z) = y_{-t^+(z)}\big(y_{t(x)}(x)\big) = y_{-t^+(z)+t(x)}(x) = y_0(x) = x.$$

This implies $H^{-1}(x) = y_{-t^+(x)}(x)$. We obtain the conclusion of the proof. $\square$

Remark 2.3. If E is a Hilbert space and $f(x) = \| x \|^k$ where k is a natural number $(k \geq 2)$ then we have the Kuiper's theorem [5], [6].

Moreover, let $Q\colon U \to \mathbb{R}$ be a C^2-map defined on a neighbourhood U of $0 \in E$ such that $Q(0) = 0$. Assume

$$Q(t \cdot x) = t^\alpha \cdot Q(x) \quad \forall x \in E, \, t \geq 0$$
$$\| \operatorname{grad} Q(x) \| > c > 0 \quad \forall x, \, \| x \| = 1$$

for constants $\alpha > 1$, c. Then $Q + P$ is C^0-right equivalent to Q at 0 for any C^2-map $P\colon U \to \mathbb{R}$ such that $\lim\limits_{x \to o} \frac{|DP(x)|}{\|x\|^{\alpha-1}}$. Indeed, we take

$$A(x) = \operatorname{grad} Q(x) / \| \operatorname{grad} Q(x) \|, \quad f(x) = \| x \|^{\alpha - 1}.$$

3. The splitting lemma

We now briefly discuss the splitting lemma of Gromoll and Meyer [1].

Theorem 3.1. *Let E be a Banach space possessing a splitting $E = Y \oplus Z$, where Y, Z are Banach spaces. Let P, Q be C^0-smooth with a Lipschitz partial derivatives $D_y^1 P$, $D_y^1 Q$, defined on a neighbourhood U of $(0,0)$. Let $A(y,z)$ be a C^0-vector field on $U^+ = U \setminus \{(y,z) \mid y = 0\}$ and let $f\colon U \cap Y \to \mathbb{R}$ be a C^0-map such that*

(1) $A : U^+ \to Y$, $|A(y,z)| \leq 1$, *A is C^1-smooth by y;*
(2) $D_y Q(y,z) A(y,z) \geq c \cdot f(y)$ *for* $(y,z) \in U^+$, *where* $c > 0$ *and* $\lim_{x\to 0} \frac{|D_y P(y,z)|}{f(y)} = 0$ *uniformly with respect to a small z;*
(3) $f \in C^1(U^+ \cap Y)$, $f(0) = 0$, $f(y) > 0$ *if* $y \neq 0$ *and* $f(t \cdot y) \leq K \cdot f(y)$ *for any* $t \in [0,1]$, *where K is a positive constant.*

Then the function $Q(y,z) + P(0,z)$ is C^0-right equivalent to $Q(y,z) + P(y,z)$ at $(0,0)$ by a homeomorphism $H(y,z) = \big(h(y,z), z\big)$.

Proof: Applying Theorem 2.1 for the functions $Q_1(y,z) = Q(y,z) - Q(0,z)$, $P_1(y,z) = P(y,z) - P(0,z)$ uniformly with respect to a small z we obtain our result. $\square$

Splitting lemma. *Let H be a Hilbert space and $h\colon U \to \mathbb{R}$ a C^1-map, where U is a neighbourhood of 0. We assume that $h(0) = Dh(0) = 0$, $D^2 h(0)$ exists and $D^2 h(0) = \langle Bw_1, w_2\rangle$, where B is a Fredholm operator. Moreover we assume that h has a continuous partial derivative $D_y^2 h$ for $y \in Y \cap U$, where $H = Y \oplus Z$, $Y = \operatorname{im} B$, $Z = \ker B$.*

Then there is a homeomorphism $H(y,z) = \big(\bar h(y,z), z\big)$ such that

$$h\big(H(y,z)\big) = \frac{1}{2} \cdot \langle By, y\rangle + \tilde h(z),$$

where $(y,z) \in Y \oplus Z$ is small, $\tilde h$ is continuous, $\tilde h(0) = 0$.

Proof: We consider the equation $\nabla_y h(y,z) = 0$, where ∇_y is the partial gradient. The implicit function theorem guarantees that this equation uniquely defines a C^0-map $y(z)$ such that $\nabla_y h(y(z), z) = 0$. Let us put

$$h_1(y,z) = h\big(y + y(z), z\big) \text{ and } P(y,z) = h_1(y,z) - \frac{1}{2}\langle By, y\rangle$$
$$Q(y,z) = \frac{1}{2}\langle By, y\rangle,\ A(y,z) = By/ \parallel By \parallel,\ f(y) = \parallel y \parallel .$$

Since B is invertible on Y we obtain

$$D_y Q(y,z) \cdot \frac{By}{\parallel By \parallel} = \parallel By \parallel \geq c \cdot \parallel y \parallel$$

for some $c > 0$. Moreover

$$|D_y P(y,z)| \leq \int_0^1 \parallel D_y^2 P(t \cdot y, z) \parallel \cdot \parallel y \parallel \, dt$$

and from this we have

$$\lim_{y\to 0, z\to 0} \frac{|D_y P(y,z)|}{\| y \|} = 0.$$

Theorem 3.1 implies the assertion of the lemma. □

4. The infinite dimensional version of the Buchner, Marsden and Schecter theorem

We need the following definition.

Definition. *We say that an open set $S \subset H$ (H is a Hilbert space) has the property $\mathcal{B}$ if there exists a function $h\colon H \to \mathbb{R}$ such that*

(i) *h is a C^1-map, $0 \le h \le 1$;*
(ii) *$\operatorname{supp} h \subset S$, $\operatorname{supp} h \subset B_{\bar R}$ for some $\bar R > 0$ ($\operatorname{supp} h$ is the support of h), and $B_{\bar R}$ is the ball with the radius $\bar R$ at 0;*
(iii) *$\| \operatorname{grad} h \| \le \bar R$.*

Theorem 4.1. *Let g be a C^k-map $g\colon H \to \mathbb{R}$, ($k \ge 3$), $g(0) = Dg(0) = \cdots = D^{i-1}g(0) = 0$ ($2 \le i < k$) and Q be the i-form*

$$Q(x) = \frac{1}{i!} \cdot D^i g(0)(x \cdots x).$$

We assume that there exist an open set S and a number $r_0 > 0$ such that

(i) *S has the property $\mathcal{B}$ with a function h;*
(ii) *$P = \{x \mid \ \| x \| = 1,\ Q(x) = 0\} \subset \operatorname{Int}\{x \mid h(x) = 1\} = V$ $\operatorname{dist}(\bar V \setminus V, P) \ge r_0$;*
(iii) *$\| \operatorname{grad} Q(x) \| > r_0, \quad \forall x \in S$.*

Then there are neighbourhoods U_1, U_2 of the point 0 and a C^1-diffeomorphism $\tilde F$ such that

(a) *$\tilde F(Q^{-1}(0) \cap U_1) \subset g^{-1}(0) \cap U_2$;*
(b) *$\tilde F(0) = 0$, $D\tilde F(0) = I$.*

Moreover if we assume the condition

$$\text{(C)} \qquad \begin{aligned} &Q(y_n) \to 0 \text{ implies } \operatorname{dist}(y_n, P) \to 0 \\ &\text{for } \| y_n \| = 1 \text{ and } n \to \infty, \end{aligned}$$

then in (a) *we have the equality.*

Here Int A is the interior of the set A; dist (A, B) is the distance of the sets A, B.

Proof of Theorem 4.1: Let us put $N(x) = \frac{\operatorname{grad} Q(x)}{\|\operatorname{grad} Q(x)\|^2} \cdot h(x)$. By the assumptions of the theorem we have

$$\text{(6)} \qquad \begin{aligned} &N(x) \text{ is a } C^1\text{-map}, \| N(x) \| \le M,\ \| D_x N(x) \| \le M \\ &\text{for some } M > 0 \text{ and any } x \in H. \end{aligned}$$

We consider the following initial value problem

$$\text{(I)}\qquad \begin{aligned} &Y_t'(x,r) = \frac{d}{dt}Y_t(x,r) = h(x,r)\cdot N\big(Y_t(x,r)\big)\\ &Y_0(x,r) = x,\ r>0, \end{aligned}$$

where $h(x,r) = \bar{h}(x\cdot r)(r\cdot x,\cdots,r\cdot x)/r^i$, and $\bar{h}(x)(x,\cdots,x)$ we obtain by the Taylor's theorem

$$g(x) = Q(x) + \bar{h}(x)(x,\cdots,x),$$

where $\bar{h}$ is an i-linear C^{k-1}-map, $\bar{h}(0) = 0$.
Then there exist $\bar{M}$, $\tilde{r}_0 > 0$ such that

$$\text{(7)}\qquad |h(x,r)| \le \bar{M}\cdot|r|$$

for $|r| \le \tilde{r}_0$ and $\| x \| \le \bar{R}$. We can consider $\bar{R} \ge 3$.

Lemma 4.2. *There exist constants M_2, $r_1 > 0$ such that*

$$Y_t(x,r) \in B_{\bar{R}},\ \| Y_t(x,r) - x \| \le M_2\cdot|r|$$

for $\| x \| \le \bar{R}/2$, $|r| < r_1$ and $|t| < 2$.

Proof of Lemma 4.2: The assertion is a consequence of (6), (7). □

We put

$$V_1 = \{x\in V \mid \operatorname{dist}(x,P) < r_0/2\}.$$

Then V_1 is open and $P\subset V_1$.

Proposition 4.3. *If $x\notin V_1$, $\| x \| = 1$ then* dist $\big(x, Q^{-1}(0)\big) > r_0/4$.

Proof of Proposition 4.3: Let $y\in P$. We can assume that $\langle x,y\rangle \ge 0$, since $\pm y\in P$. Then we have for any $t\in\mathbb{R}$

$$\begin{aligned} &\| x - t\cdot y \|^2 = t^2 - 2t\langle x,y\rangle + 1 \ge 1 - \langle x,y\rangle^2\\ &= (1+\langle x,y\rangle)\cdot(1-\langle x,y\rangle) \ge 1 - \langle x,y\rangle\\ &= \| x-y \|^2/2 \ge r_0^2/8 > r_0^2/16. \end{aligned}$$

This completes the proof. □

As a consequence of Lemma 4.2 and Proposition 4.3 we obtain

Lemma 4.4. *There exists $\bar{r} > 0\,(\bar{r} < r_1, r_0)$ such that if $x\in V_1\cap\partial B_1$ then $Y_t(x,r)\in V$, and if $x\notin V_1$, $x\in\partial B_1$ then $Y_t(x,r)\notin Q^{-1}(0)$ for any t, $|t| < 2$ and r, $|r| < \bar{r}$.*

We put

$$F(x) = \| x \| \cdot Y_1\Big(x/\| x \|, \| x \|\Big)$$

for $x \neq 0$ and $F(0) = 0$. By Lemma 4.2 we have

$$DF(0) = I, \ (I = \text{Identity}). \tag{8}$$

From the equation (I) we obtain

$$X_t'(x,r) = D_x h(x,r) \cdot N\big(Y_t(x,r)\big) + h(x,r) \cdot D_x N\big(Y_t(x,r)\big) \cdot X_t(x,r)$$
$$X_0(x,r) = I,$$

where $X_t(x,r) = D_x Y_t(x,r)$. Since N satisfies (6) and $D_x h(x,r) \to 0$ uniformly with respect to x, $\| x \| \leq 2$ if $r \to 0$, applying the Gronwall's lemma we obtain

$$\big(X_1(x,r) - I\big) \to 0 \tag{9}$$

uniformly with respect to x, $\| x \| \leq 2$ if $r \to 0$.
We put

$$e(z,r) = Y_1(z,r) - z.$$

Then we have

$$F(x) = x + \| x \| \cdot e\Big(x / \| x \|, \| x \| \Big).$$

Hence

$$\begin{aligned} D_x F(x) v = {} & v + \langle x/ \| x \|, v \rangle \cdot e\Big(x/ \| x \|, \| x \| \Big) + \\ & + \frac{d}{dz} e\Big(x/ \| x \|, \| x \| \Big) \cdot \Big(v - \langle x / \| x \|, v \rangle \cdot x / \| x \| \Big) + \\ & + \langle x, v \rangle \cdot \frac{d}{dr} e\Big(x/ \| x \|, \| x \| \Big). \end{aligned}$$

By (8), (9) it follows

$$v - D_x F(x) v \to 0$$

uniformly with respect to v as $x \to 0$. Hence F is a local diffeomorphism at 0.
By Lemma 4.4 we have

$$\frac{d}{dt}\Big(Q(x) + t \cdot h(x,r) - Q\big(Y_t(x,r)\big)\Big) = h(x,r) - h(x,r) = 0$$

for $x \in V_1 \cap \partial B_1$, $r < \bar{r}$.

Hence for x such that $x / \| x \| \in V_1$ and $\| x \| < \bar{r}$, we have

$$g(x) = Q\big(F(x)\big).$$

On the other hand, Lemma 4.4 also implies

$$F(x) \notin Q^{-1}(0)$$

if $x / \| x \| \notin V_1$, $\| x \| < \bar{r}$.
Concerning the map $F^{-1} = \tilde{F}$ we obtain immediately the first assertion of the theorem.

To prove the last part of the theorem, assume $x \in g^{-1}(0) \cap U_2$ and $x \notin \tilde{F}\big(Q^{-1} \cap U_1\big)$. Then $g(x) = 0$, $F(x) \notin Q^{-1}(0)$. This implies $x / \| x \| \notin V_1$. On the other hand, $0 = g(x) = Q(x) + \bar{h}(x)(x, \cdots, x)$. Hence $0 = Q(x / \| x \|) + O(\| x \|)$. By (C) we have $|Q(y)| > \bar{c} > 0 \, \forall y \notin V_1$, $y \in \partial B_1$. We arrive at the contradiction for U_2 small.

Remark 4.5. 1. If $\| \operatorname{grad} Q(x) \| > c > 0$ for any x, $\| x \| = 1$ then we obtain again the Kuiper's lemma (see the assertion 2 of Theorem 4.6).

2. If H is a finite dimensional space then we have Theorem 1.3 from [2] for functions (see Remark 4.9).

Now we consider a map $g(x) = Q(x) + \tilde{h}(x)$, where $g\colon H_1 \to H_2$ is a map which has the same properties as in Theorem 4.1 where we considered the case $H_2 = \mathbb{R}$; H_1, H_2 are Hilbert spaces. But instead of the assumption (iii) of Theorem 4.1 we assume

$$\begin{aligned} &DQ(x) \text{ is surjective and } \| DQ(x)v \| > r_0 \text{ for any} \\ &x \in S \text{ and } v \text{ such that} \\ &\| v \| = 1 \text{ and } v \perp \ker DQ(x). \end{aligned} \tag{10}$$

By using (10) there exists $c > 0$ such that we can find for any $y \in S$ the linear mapping $B(y)\colon H_2 \to H_1$ satisfying $DQ(y) \cdot B(y) = I$ and $\| B(y) \| \leq c$, $\operatorname{im} B(y) = \big(\ker DQ(y)\big)^{\perp}$, $\| D_y B(y) \| \leq c$.

We put $N(x,r) = B(x) \cdot h(x,r) \cdot h(x)$, where $h(x,r)$ is defined as in the proof of Theorem 4.1. Then $DQ(x) \cdot N(x,r) = h(x,r) \cdot h(x)$ and we see that for the map $g\colon H_1 \to H_2$ possessing the above properties we obtain a similar theorem as Theorem 4.1. Indeed, we consider instead of (I) the following equation

$$\begin{aligned} Y_t'(x,r) &= N(x,r) \\ Y_0(x,r) &= x,\ r > 0, \end{aligned}$$

and we can repeat the above proof. We summarize our results in the following theorem.

Theorem 4.6. *Let H_1, H_2 be Hilbert spaces. Consider $g\colon H_1 \to H_2$ a C^k-map, $k \geq 3$ and $g(0) = Dg(0) = \cdots = D^{i-1}g(0) = 0$, $2 \leq i < k$. Let Q be the i-form*

$$Q(x) = \frac{1}{i!} \cdot D^i g(0)(x, \cdots, x).$$

We assume that there exist an open set S and a number $r_0 > 0$ such that

(i) *S has the property $\mathcal{B}$ with a function h;*
(ii) *$P = \{x \mid \ \| x \| = 1,\ Q(x) = 0\} \subset \operatorname{Int} \{x \mid h(x) = 1\} = V$ $\operatorname{dist}(\bar{V} \setminus V, P) \geq r_0$;*
(iii) *$\| DQ(x)v \| > r_0$, $DQ(x)$ is surjective for any $x \in S$ and v, $\| v \| = 1$, $v \perp \ker DQ(x)$.*

Then

1. *There are neighbourhoods U_1, U_2 of the point 0 and a C^1-diffeomorphism F such that*

(a) *$F\big(Q^{-1}(0) \cap U_1\big) \subset g^{-1}(0) \cap U_2$;*
(b) *$F(0) = 0$, $DF(0) = I$.*

Moreover if we assume the condition

$$\text{(C)} \qquad \begin{aligned} &Q(y_n) \to 0 \textit{ implies } \operatorname{dist}(y_n, P) \to 0 \\ &\textit{for any } \| y_n \| = 1 \textit{ and } n \to \infty. \end{aligned}$$

Then in (a) *we have the equality.*

2. *If the assumption* (iii) *is satisfied for any* x, $\| x \| = 1$, *i.e.* $\partial B_1 \subset S$ *in* (iii). *Then* $g\big(F(x)\big) = Q(x)$ *for any* $x \in U_1$. *For this case we do not assume the conditions* (i), (ii).

Proof: It remains to prove the statement 2. Since $Q(t \cdot y) = t^i \cdot Q(y)$ we have $DQ(t \cdot y) = t^{i-1} \cdot DQ(y)$. Thus we establish the assumptions (i), (ii) by taking

$$\begin{aligned} S &= \{t \cdot x \mid \ \| x \| = 1,\, t \in (1/2, 2)\}, \\ h(x) &= f(\| x \|^2), \end{aligned}$$

where $f\colon \mathbb{R} \to [0,1]$ is C^∞-smooth, supp $f \subset (1/4, 4)$ and

$$f(z) = 1 \, \forall z \in [9/16, 16/9].$$

□

Corollary 4.7. *Let* $g\colon H \to \mathbb{R}^k$ *be a* C^3*-map and* $g(0) = Dg(0) = 0$. *Let*

$$D^2 g(0)(u, v) = \Big((A_1 u, v), (A_2 u, v), \cdots, (A_k u, v)\Big),$$

where $A_i\colon H \to H$ *are continuous linear maps. If there exists* $r_0 > 0$ *such that*

$$|\det(A_i u, A_j u)| > r_0$$

for any $u \in H$ *such that* $\| u \| = 1$. *Then* g *is* C^1*-right equivalent to the map*

$$f(x) = \frac{1}{2}\Big((A_1 x, x), (A_2 x, x), \cdots, (A_k x, x)\Big).$$

Remark 4.8. This corollary generalizes the Morse-Palais lemma [1].

Remark 4.9. The condition (C) of Theorems 4.1–2 is always satisfied for finite dimensional cases. The assumptions (i), (ii) of Theorems 4.1–2 are satisfied for finite dimensional cases provided $P \subset S$. Indeed, by using the partion of unity theorem [4, p. 377], we can construct such a function h. On the other hand, the assumptions of these theorems implies $P \subset S$. For infinite dimensional cases, the last assumption of the definition of the property $\mathcal{B}$ is problematic by using the partion of unity theorem. The author does not know whether the condition

$$P \subset S,\ \operatorname{dist}(\bar{S} \setminus S, P) > c_0 > 0$$

will already imply the existence of such a function h. These conditions remind the well-known (P.S.) condition for variational problems [3].

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