F_σ-absorbing sequences in hyperspaces of subcontinua

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Abstract. Let $\mathcal{D}$ denote a true dimension function, i.e., a dimension function such that $\mathcal{D}(\mathbb{R}^n) = n$ for all n. For a space X, we denote the hyperspace consisting of all compact connected, non-empty subsets by $C(X)$. If X is a countable infinite product of non-degenerate Peano continua, then the sequence $(\mathcal{D}_{\geq n}(C(X)))_{n=2}^{\infty}$ is F_σ-absorbing in $C(X)$. As a consequence, there is a homeomorphism $h : C(X) \to Q^\infty$ such that for all n, $h[\{A \in C(X) : \mathcal{D}(A) \geq n+1\}] = B^n \times Q \times Q \times \dots$, where B denotes the pseudo boundary of the Hilbert cube Q. It follows that if X is a countable infinite product of non-degenerate Peano continua then $\mathcal{D}_{\geq n}(C(X))$ is an F_σ-absorber (capset) for $C(X)$, for every $n \geq 2$.

Let dim denote covering dimension. It is known that there is an example of an everywhere infinite dimensional Peano continuum X that contains arbitrary large n-cubes, such that for every $k \in \mathbb{N}$, the sequence $(\dim_{\geq n}(C(X^k)))_{n=2}^{\infty}$ is not F_σ-absorbing in $C(X^k)$. So our result is in some sense the best possible.

Keywords: Hilbert cube, absorbing system, F_σ, $F_{\sigma\delta}$, capset, Peano continuum, hyperspace, hyperspace of subcontinua, covering dimension, cohomological dimension

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1. Introduction.

If X is a compact metric space then 2^X denotes the hyperspace of all nonempty closed subsets of X topologized by the Hausdorff metric. The subspace of 2^X consisting of all connected sets is denoted by $C(X)$. For $k \in \{0, 1, \dots, \infty\}$ and a dimension function $\mathcal{D}$, let $\mathcal{D}_{\geq k}(2^X)$ denote the subspace of 2^X consisting of all at least k-dimensional elements of 2^X. We define $\mathcal{D}_k(2^X)$, $\mathcal{D}_{\geq k}(C(X))$ and $\mathcal{D}_k(C(X))$ in the same way. Let Q denote the Hilbert cube and dim covering dimension. In [10] it is proved that there exists a homeomorphism $f : 2^Q \to Q^\infty$ such that for every $k \geq 0$,

$$f[\dim_{\geq k}(2^Q)] = \underbrace{B \times \dots \times B}_{k \text{ times}} \times Q \times Q \times \dots, \tag{1}$$

where B is the pseudo-boundary of Q. As a consequence,

$$f[\dim_\infty(2^Q)] = B^\infty. \tag{2}$$

The proof of (1) is based in an essential way on the "convex" structure of Q as well as on the technique of absorbing systems. There are results generalizing (1) and (2). In [6] it is proved that in (2) one can replace the Hilbert cube by a countable infinite product of non-degenerate Peano continua. In the same paper an example is constructed of an everywhere infinite dimensional Peano continuum X such that $\dim_\infty(2^{X^k})$ is not homeomorphic to B^∞ for all $k \in \mathbb{N}$. In [7] is showed that it is also possible to replace the Hilbert cube by a countable infinite product of non-degenerated Peano continua in (1). It is also sometimes possible to replace the covering dimension by another dimension function. In [9] it is proved that this is possible in (1) for any true dimension function, in particular the cohomological dimension $\dim_G$ for an Abelian group G. Dobrowolski and Rubin have also a generalization for hyperspaces of subcontinua. In [9] they prove that for any true dimension function $\mathcal{D}$ there is a homeomorphism $f : C(Q) \to Q^\infty$ such that for every $k \geq 2$,

$$f[\mathcal{D}_{\geq k+1}(C(Q))] = \underbrace{B \times \cdots \times B}_{k \text{ times}} \times Q \times Q \times \ldots . \tag{3}$$

Since we know that for a Peano continuum X that has no free arcs, the hyperspace $C(X)$ is a Hilbert cube, it is natural to wonder whether we can replace the Hilbert cube in (3) by other spaces. In this paper we show that for a space X that is a countable infinite product of non-degenerate Peano continua and $\mathcal{D}$ a true dimension function, the sequence $\big(\mathcal{D}_{\geq n}(C(X))\big)_{n=2}^\infty$ is F_σ-absorbing in $C(X)$. As a consequence we find that for such a space X and $n \geq 2$, the set $\mathcal{D}_{\geq n}(C(X))$ is a capset for $C(X)$. This is nice because there are not many natural capsets known for $C(X)$.

The example presented in [6] of an everywhere infinite dimensional space X such that for no $k \in \mathbb{N}$, the set $\dim_\infty(2^{X^k})$ is homeomorphic to B^∞, also works for $C(X^k)$, i.e., for no $k \in \mathbb{N}$, the set $\dim_\infty(C(X^k))$ is homeomorphic to B^∞. This implies that for no $k \in \mathbb{N}$ the sequence $\big(\dim_{\geq n}(C(X^k))\big)_{n=2}^\infty$ is F_σ-absorbing.

A combination of some ideas in this paper and [7] can be used to show that in (1) the Hilbert cube can be replaced by a countable infinite product of non-degenerate Peano continua and (at the same time) the covering dimension by an arbitrary true dimension function. See also the Remark at the end of this paper.

2. Terminology.

All spaces under discussion are separable and metrizable. For any space X we let d denote an admissible metric on X, i.e., a metric that generates the topology. All our metrics are bounded by 1. We use dim to denote the covering dimension. Following the terminology in [9] we say that a dimension function $\mathcal{D}$ is *true* if $\mathcal{D}(\mathbb{R}^n) = n$ for all n. Examples of true dimension functions are dim and the cohomological dimension $\dim_G$ for any Abelian group G. In this paper $\mathcal{D}$ denotes an arbitrary true dimension function.

As usual I denotes the interval $[0, 1]$ and Q the Hilbert cube $\prod_{i=1}^\infty [-1, 1]_i$ with metric $d(x, y) = \sum_{i=1}^n 2^{-(i+1)}|x_i - y_i|$. In addition s is the *pseudo-interior* of Q, i.e., $\{x \in Q : (\forall i \in \mathbb{N})(-1 < x_i < 1)\}$. The complement B of s in Q is called the *pseudo-boundary* of Q. Any space that is homeomorphic to Q is called a *Hilbert cube*. If X is a set then the identity function on X will be denoted by 1_X.

Let A be a closed subset of a space X. We say that A is a *Z-set* provided that every map $f : Q \to X$ can be approximated arbitrarily closely by a map

$g : Q \to X \setminus A$. The collection of all Z-sets in X will be denoted by $\mathcal{Z}(X)$. A countable union of Z-sets is called a *σZ-set*. A *Z-embedding* is an embedding the range of which is a Z-set.

Let $\mathcal{M}$ be a class of spaces that is topological and closed hereditary.

Let Γ be an ordered set. An *$\mathcal{M}_\Gamma$-system* in a space X is an order preserving indexed collection (with respect to inclusion) $(A_\gamma)_{\gamma\in\Gamma}$ of subsets of X such that $A_\gamma \in \mathcal{M}$ for every γ. Let $\mathfrak{S} = (S_\gamma)_{\gamma\in\Gamma}$ be an order preserving indexed collection of subsets of a space X. The following definitions can be found in [10] (see also [8]). For more information (such as historical comments), see [10].

Definition. The system $\mathfrak{S}$ is called *strongly $\mathcal{M}_\Gamma$-universal* in X if for every $\mathcal{M}_\Gamma$-system $(A_\gamma)_{\gamma\in\Gamma}$ in Q, every map $f : Q \to X$ that restricts to a Z-embedding on some compact subset K of Q, can be approximated arbitrarily closely by a Z-embedding $g : Q \to X$ such that $g \mid K = f \mid K$ while moreover for every $\gamma \in \Gamma$ we have $g^{-1}(S_\gamma) \setminus K = A_\gamma \setminus K$.

Definition. The system $\mathfrak{S}$ is called *$\mathcal{M}_\Gamma$-absorbing* in X if:

(1) $\mathfrak{S}$ is an $\mathcal{M}_\Gamma$-system;
(2) $\bigcup_{\gamma\in\Gamma} X_\gamma \subseteq \bigcup_{i=1}^{\infty} A_i$, where each A_i is a compact Z-set in X;
(3) $\mathfrak{S}$ is strongly $\mathcal{M}_\Gamma$-universal in X.

In this paper we only use absorbing sets ($\Gamma = \{pt\}$) and absorbing sequences ($\Gamma = \mathbb{N}$ with the inverted order).

Theorem 2.1 ([10]). *Let X be a Hilbert cube and let $\mathcal{A} = (A_\gamma)_{\gamma\in\Gamma}$ and $\mathcal{B} = (B_\gamma)_{\gamma\in\Gamma}$ be $\mathcal{M}$-absorbing systems for X. Then there is a homeomorphism $h : X \to X$ with $h[A_\gamma] = B_\gamma$ for every γ. Moreover, h can be chosen arbitrarily close to the identity.*

□

There are three absorbers that are important in the present paper. The first one is an absorber for the class consisting of all finite-dimensional compacta. Such absorbers were first constructed by Anderson and Bessaga and Pełczyński [1] and were called *fd-capsets* by Anderson. A basic example of an fd-capset in Q is

$$\{x \in Q : (\exists N \in \mathbb{N})(\forall n \geq N)(x_n = 0)\}.$$

For details see [1]. The second one is an absorber for the class of all (sigma) compacta. Again, such absorbers were first constructed by Anderson and Bessaga and Pełczyński: they were called *capsets* by Anderson. A basic example of a capset in Q is B. For details see [1]. The third one is an absorbing sequence for the Borel class F_σ. A basic example of an F_σ-absorbing sequence is the sequence $(B^n \times Q \times Q \times \dots)_{n=1}^{\infty}$ in the Hilbert cube Q^∞. For details see [10].

Definition. A subset A of a space X is *locally homotopy negligible in X* if for every map $f : M \to X$, where M is any ANR, and for every open cover $\mathcal{U}$ of X there exists a homotopy $H : M \times I \to X$ such that $\{H(\{x\} \times I)\}_{x\in M}$ refines $\mathcal{U}$, $H_0 = f$ and $H\big[M \times (0,1]\big] \subseteq X \setminus A$.

If A is an $\mathcal{M}$-absorber in X and if $\mathcal{M}$ contains the class of all finite-dimensional compacta then both A and $X \setminus A$ are locally homotopy negligible in X ([8, Corollary 4.3]). So it follows in particular that if E is one of the absorbers for Q mentioned above, then there is a homotopy $H : Q \times I \to Q$ such that $H_0 = 1_Q$ and $H\big[Q \times (0,1]\big] \subseteq E$. This can also be seen by noting that on the one hand for the basic examples of absorbers such homotopies exist and that on the other hand every absorber is "equivalent" to its own model (Theorem 2.1).

Let X be a non-degenerate Peano continuum. The subspace of 2^X consisting of all finite non-empty subsets of X of cardinality at most n is denoted by $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ denotes $\bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$.

We will need the following result.

Theorem 2.2 ([5]). *Let X be a non-degenerate Peano continuum. Then* $\dim_{\geq 1}(2^X)$ *is a capset (F_σ-absorber) for 2^X.*

□

Let X be a Peano continuum. It follows from [2] that X admits a so-called convex metric d. For such a metric it is known (and easy to prove) that the function $H : 2^X \times I \to 2^X$ defined by $H(A,t) = \{x \in X : d(x,A) \leq t\}$ is continuous. *From now on we assume that all metrics on Peano continua are convex.*

On a product space $X = \prod_{n=1}^{\infty} X_n$ we will always use the admissible metric d on X given by

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n) \qquad (x, y \in X)$$

where for every n, d_n is a metric on X_n that is bounded by 1. We also let for every m, π_m be the projection $X \to X_m$ and Π_m the projection $X \to X_1 \times \cdots \times X_m$. Note that if for two points $x, y \in \prod_{n=1}^{\infty} X_n$ we have that $\pi_i(x) = \pi_i(y)$ for all $i \leq N$, then their distance is $\leq 2^{-N}$.

On the hyperspace 2^X consisting of all non-empty closed subsets of X, the Hausdorff metric is given by

$$d_H(E,F) = \inf\{\varepsilon > 0 : E \subseteq B(F,\varepsilon) \quad \text{and} \quad F \subseteq B(E,\varepsilon)\}$$

where d is a metric on X and $B(C,\delta)$ consists of all points $x \in X$ such that $d(x,C) < \delta$. On $C(X)$ we will use the same metric.

We will denote the distance between functions $f, g : X \to Y$ by

$$\hat{d}(f,g) = \sup\{d(f(x), g(x)) : x \in X\} \in [0,\infty],$$

where d is an admissible metric for Y.

3. The set $\mathcal{D}_{\geq 2}(C(X))$ is a σZ-set in $C(X)$.

The main result in this section is that for a Peano continuum X and $n \geq 2$, the set $\mathcal{D}_{\geq n}(C(X))$ is a σZ-set in $C(X)$. This is a step in the proof that the sequence

$(\mathcal{D}_{\geq n}(C(X)))_{n=2}^{\infty}$ is F_σ-absorbing in $C(X)$ if X is a countable infinite product of Peano continua. The maps that we construct in this section are somewhat more complicated than necessary to derive the main result of this section. The reason for that is that we will need these more complicated maps later on.

Let X be a Hilbert cube and $(B_i)_{i=1}^{\infty}$ a tower in X. We say that $(B_i)_{i=1}^{\infty}$ has the *mapping approximation property* if, for every $\varepsilon > 0$, every $k \in \mathbb{N}$ and every map $f : I^k \times I \to X$ such that $f[I^k \times \{0\}] \subseteq B_i$ for some i, there exist a $j \in \mathbb{N}$ and a map $g : I^k \times I \to B_j$ with $\hat{d}(f, g) < \varepsilon$ and $g \mid I^k \times \{0\} = f \mid I^k \times \{0\}$.

We say that $(B_i)_{i=1}^{\infty}$ has the *deformation property* if there exists a deformation $h : X \times I \to X$ with $h \mid X \times \{0\} = 1_X$ and such that for each $t > 0$, $h[X \times [t, 1]] \subseteq B_i$ for certain i.

The following theorem is a reformulation of a result in [3, the proof of Theorem 4.6].

Theorem 3.1. *Let $(B_i)_{i=1}^{\infty}$ be a tower of compacta in a Hilbert cube X such that $(B_i)_{i=1}^{\infty}$ has the mapping approximation property. Then $(B_i)_{i=1}^{\infty}$ has the deformation property.*

□

We will need the following well-known lemma.

Lemma 3.2 ([4])**.** *For each $n \geq 1$, there exists a map $r : B^{n+1} \to \mathcal{F}_3(S^n)$ such that $r(b) = \{b\}$ for all $b \in S^n$.*

□

The following Theorem is essentially [4, Lemmas 4.4. and 4.6]. It is clear that a modification of the proofs given there can be used to prove the following result. Since we are dealing in this paper with an infinite product of Peano continua we will give an easier proof for this situation.

Theorem 3.3. *Let X be a countable infinite product of non-degenerate Peano continua. Then there exists a sequence $(\Gamma_i)_{i=1}^{\infty}$ of (connected) graphs in X such that:*

(1) *for all $i \in \mathbb{N}$, $\Gamma_i \subseteq \Gamma_{i+1}$,*
(2) $\overline{\bigcup_{i=1}^{\infty} \Gamma_i} = X$,
(3) *for all $x, y \in \Gamma_i$, there exists a path γ in Γ_{i+1} between x and y with* $\operatorname{diam}(\gamma) < d(x, y) + \frac{1}{i}$,
(4) *the tower $(\mathcal{F}_i(\Gamma_i))_{i=1}^{\infty}$ has the mapping approximation property,*
(5) *the projection $\Pi_m(\Gamma_n)$ of Γ_n on $X_1 \times \cdots \times X_m$ is a finite graph for all $m \geq n + 1$,*
(6) *there is a point $(c_1, c_2, \dots) \in X$ such that $\pi_m(\Gamma_n) = \{c_m\}$ for all $m \geq n+1$.*

Proof: Choose for every $n \in \mathbb{N}$ an arc $Y_n \subseteq X_n$ and a homeomorphism $\beta_n : I \to Y_n$. (We use here that the factors of X are non-degenerate.) We put $c_n = \beta_n(0)$. Let

$$\Sigma_n = \{x \in X : \pi_m(x) = c_m \text{ for all } m > n\}.$$

Observe that $\bigcup_{n=1}^{\infty} \Sigma_n$ is dense in X. Choose a countable, dense subset $\{x_1, x_2, \dots\}$ of X such that $\{x_1, \dots, x_n\} \subseteq \Sigma_n$ for all n. We will construct the graphs Γ_i in such a way that $\{x_i, \dots, x_i\} \subseteq \Gamma_i$ for every i. This will take care of (2).

Let Γ_1 be the point x_1. Let $\tilde{\Gamma}_2$ be an arc in $X_1 \times X_2$ containing the points $(\pi_1(x_1), \pi_2(x_1))$ and $(\pi_1(x_2), \pi_2(x_2))$. Define $\Gamma_2 = \tilde{\Gamma}_2 \times \{c_3\} \times \{c_4\} \times \dots$. Note that Γ_1 and Γ_2 are as required.

Assume $\Gamma_1, \dots, \Gamma_n$ have been chosen such that:

(i) for all $1 \le i < n$, $\Gamma_i \subseteq \Gamma_{i+1}$,

(ii) for all $1 \le i \le n$, $\{x_1, \dots, x_i\} \subseteq \Gamma_i$,

(iii) for all $1 \le i < n$ and $x, y \in \Gamma_i$, there exists a path γ between x and y in Γ_{i+1} such that $\operatorname{diam}(\gamma) < d(x, y) + \frac{1}{i}$,

(iv) for all $1 \le i \le n$, $\Gamma_i \subseteq \Sigma_i$.

Let T be a triangulation of Γ_n with $\operatorname{mesh}(T) < \frac{1}{7n}$. For each pair of distinct vertices v, w of T, choose an arc α between v and w in X with $\operatorname{diam}(\alpha) < d(v, w) + \frac{1}{14n}$. We may assume that these arcs are contained in Σ_n.

> This can be seen as follows. Let $\tilde{v}$ and $\tilde{w}$ be the projections of v and w on $X_1 \times \dots \times X_n$. Choose an arc $\tilde{\alpha}$ between $\tilde{v}$ and $\tilde{w}$ with small diameter. Now let α between v and w be the arc $\alpha = \tilde{\alpha} \times \{c_{n+1}\} \times \{c_{n+2}\} \times \dots$

We wish to adjoin these arcs $\alpha_1, \dots, \alpha_n$ to Γ_n, as a part of the procedure for constructing a graph Γ_{n+1} satisfying the condition (iii) of the inductive requirements. Of course, $\Gamma_n \cup \alpha_1 \cup \dots \cup \alpha_k$ may fail to be a graph. To obtain a graph we must partially reroute the paths given by $\{\alpha_1, \dots, \alpha_k\}$.

Consider the arc α_1. Let $\psi_1 : I \to \alpha_1$ be a homeomorphism and let

$$\begin{aligned}\Gamma_n^1 = {} & \Gamma_n \cup \\ & \{(\pi_1(\psi_1(t)), \pi_2(\psi_1(t)), \dots, \pi_n(\psi_1(t)), \beta_n(\frac{1}{n}t(1-t)), c_{n+2}, c_{n+3}, \dots) : t \in I\}.\end{aligned}$$

In order to make the arc α_1 disjoint from points in Γ_n, we replace α_1 by a close arc α_1^*, having the same boundary points as α_1. We may assume that the length of the arc α_1^* differs less than $\frac{1}{14n}$ from the diameter of the arc α_1. This is so because we can assume that $\beta_n(\frac{1}{n}t(1-t))$ is close to c_n. Now we want to add the arc α_2. We might get into trouble by adding it in the same way as the arc α_1. Therefore we have to proceed with a little more care. Let $\psi_2 : I \to \alpha_2$ be a homeomorphism. It is obvious that we can define a continuous function $\phi : I \to I$ such that for points $\psi(t)$ that are not on the boundary of α_2 we have

$$(\pi_1(\psi_2(t)), \pi_2(\psi_2(t)), \dots, \pi_n(\psi_2(t)), \beta_n(\phi(t) \cdot t \cdot (1-t)), c_{n+2}, c_{n+3}, \dots) \notin \Gamma_n^1.$$

Now it is clear how to define Γ_n^2; put

$$\begin{aligned}\Gamma_n^2 = {} & \Gamma_n^1 \cup \\ & \{(\pi_1(\psi_2(t)), \pi_2(\psi_2(t)), \dots, \pi_n(\psi_2(t)), \beta_n(\frac{1}{n}\phi(t) \cdot t \cdot (1-t)), c_{n+2}, c_{n+3}, \dots) : t \in I\}.\end{aligned}$$

Note that we defined the map ϕ in such a way that the "image" α_2^* of α_2 does not intersect Γ_n^1, in other than the points on the boundary. We again assume that $\phi(t)$ is such a small function that the diameters of α_2 and α_2^* differ at most $\frac{1}{14n}$.

Continuing this procedure, we obtain a graph Γ_n^k in which we added one by one the arcs α_i^* that we derived from α_i. We claim that this graph satisfies the condition (iii) of the inductive requirements.

> Choose arbitrary points $x, y \in \Gamma_n$, Let v be a vertex of a simplex in T containing x and let w be a vertex of a simplex in T containing y. Then there are arcs η_x between x and v and η_y between y and w in Γ_n such that diam (η_x) and diam (η_y) are at most $\frac{3}{7n}$. For some j, α_j is an arc between v and w and so is the added arc α_j^*. We constructed α_j^* in such a way that diam $(\alpha_j^*) < d(v,w) + \frac{3}{7n}$. Thus $\gamma = \eta_x \cup \alpha_j^* \cup \eta_y$ provides a path between x and y in Γ_n^k with diam $(\gamma) < \frac{3}{7n} + \frac{3}{7n} + d(v,w) + \frac{1}{7n} \leq d(x,y) + \frac{1}{n}$.

Since X is path-connected, the graph Γ_n^k may be extended, if necessary, to produce a graph Γ_{n+1} containing x_{n+1}. It is clear that this extension of the graph is possible in Σ_{n+1}.

Now we will prove the mapping approximation property for the tower $(\mathcal{F}_i(\Gamma_i))_{i=1}^\infty$. Let $k \in \mathbb{N}$, $\varepsilon > 0$ and let $f : I^k \times I \to 2^X$ be a map such that $f[I^k \times \{0\}] \subseteq \mathcal{F}_i(\Gamma_i)$ for certain i. Let R be a triangulation of I^k such that diam $(\tau) < \frac{1}{24}\varepsilon$ for every $\tau \in R$. Let $\mathcal{E} = \{x_1, x_2, \dots\}$ be a countable dense subset of $\bigcup_{n=1}^\infty \Gamma_n$ as the above. We will define g on the vertices $R^{(0)}$ of R and extend g inductively over the higher dimensional simplices in R. Define $g : R^{(0)} \to \mathcal{E} \cup \mathcal{F}_i(\Gamma_i)$ such that $g \mid R^{(0)} \cap (I^k \times \{0\}) = f$ and $d(f(v), g(v)) < \frac{1}{24}\varepsilon$ for every vertex v of R. Since $R^{(0)}$ is finite and $\mathcal{E} \subseteq \bigcup_{n=1}^\infty \mathcal{F}_n(\Gamma_n)$, we may assume $g[R^{(0)}] \subseteq \mathcal{F}_j(\Gamma_j)$ for some $j > i$ such that $\frac{1}{j} < \frac{1}{24}\varepsilon$. Now consider a 1-simplex of $I^k \times (0,1]$ with endpoints v_1 and v_2 and barycenter σ. Put $g(\sigma) = \{g(v_1), g_(v_2)\}$. Since $d(g(v_1), g(v_2)) < \frac{1}{8}\varepsilon$, it follows that there exist paths in $\mathcal{F}_{2j}(\Gamma_{j+1})$ between $g(\sigma)$ and $g(v_i)$ with diameter $< \frac{1}{6}\varepsilon$. Using such paths we obtain a map $g : R^{(1)} \to \mathcal{F}_{2j}(\Gamma_{j+1})$ of the 1-skeleton with $g \mid I^k \times \{0\} = f$ and

$$\begin{aligned} d(f(p), g(p)) &\leq \operatorname{diam}(f(\sigma)) + d(f(v), g(v)) + \operatorname{diam}(g(\sigma)) \\ &< \frac{1}{24}\varepsilon + \frac{1}{24}\varepsilon + \frac{1}{6}\varepsilon \\ &= \frac{1}{4}\varepsilon \end{aligned}$$

where σ is some 1-simplex with vertex v containing p.

Using Lemma 3.2, we inductively extend g over the higher dimensional simplices in R, thereby obtaining a map $g : I^k \times I \to \mathcal{F}_{2j\cdot 3^k}(\Gamma_{j+1}) \subseteq \mathcal{F}_{2j\cdot 3^k}(\Gamma_{2j\cdot 3^k})$ with $g \mid I^k \times \{0\} = f$ and $\hat{d}(f,g) < \varepsilon$. □

Lemma 3.4. *There is a homotopy $R_n : \Gamma_n \times I \to \Gamma_{n+1}$ such that*

(1) $R_n(x,0) = \{x\}$ *for all* $x \in \Gamma_n$,
(2) $d_H(\{x\}, R_n(x,t)) \le t + \frac{2}{7n}$,
(3) *if* $x, y \in \Gamma_n$ *and* $d(x,y) = t$ *then* $R_n(x, t+\frac{4}{7n})$ *is a connected subset of* Γ_{n+1} *containing both* x *and* y.

PROOF: Fix $n \in \mathbb{N}$. Let $\{\sigma_1, \dots, \sigma_p\}$ be a triangulation of Γ_n in maximal 1-simplexes with diameter at most $\frac{1}{7n}$, as in Theorem 3.3. We assume that the triangulation of Γ_{n+1} has the same vertex set, i.e., we assume that the added arcs α_j^* contain no points of the vertex set different from the boundary points of α_j. Let $\{v_i, w_i\}$ be the vertices of the simplex σ_i. For every simplex σ_i, let $r_i : [0, d(v_i, w_i)] \to \sigma_i$ be a homeomorphism. We first define for every $x \in \Gamma_n$ a map $s_x : \{\mathit{vertices}\} \to [0, \infty)$. We say that a sequence $v_0, \dots . v_m$ provides a path between v_0 and v_m in Γ_{n+1} if for every $1 \le p \le m$ the set $\{v_{p-1}, v_p\}$ is the vertex set of some simplex in Γ_{n+1}. There are two situations to consider. First assume that x is a vertex point of the given triangulation of Γ_{n+1}. Then

$$\begin{aligned} s_x(v) = \min\{\sum_{k=1}^{m} d(v_k, v_{k+1}) : \quad &\text{where } \ x = v_1, v_2, \dots, v_{k+1} = v \\ &\text{is a path between } x \text{ and } v \text{ in the graph } \Gamma_{n+1}\}. \end{aligned}$$

If x is not a vertex point of x we first define what s is in the vertex points of the simplex σ_i that contains x and extend it over the rest of the vertex set.

$$\begin{aligned} s_x(v) = \min\{|r_i^{-1}(x) - r_i^{-1}(v_1)| + \sum_{k=1}^{m} d(v_k, v_{k+1}) : \\ &\text{where } v_1 \text{ is a vertex point of } \sigma_i \text{ and} \\ &v_1, v_2, \dots, v_{k+1} = v \text{ is a path between} \\ &v_1 \text{ and } v \text{ in the graph } \Gamma_{n+1}\}. \end{aligned}$$

Note that the map s_x measures in some sense how far a vertex point is away from x. Observe that the map $x \to s_x$ is continuous. To obtain the desired homotopy R_n, we need to extend s_x to a map on all of Γ_{n+1}. To this end, let

$$s_x(y) = |r_i^{-1}(x) - r_i^{-1}(y)|,$$

if x and y are both contained in the simplex σ_i, and

$$s_x(y) = \min\{s_x(v_j) + |r_j^{-1}(v_j) - r_j^{-1}(y)|, s_x(w_j) + |r_j^{-1}(w_j) - r_j^{-1}(y)|\}$$

if y is an element of the simplex σ_j and $x \notin \sigma_j$. An easy check shows that the map s_x is continuous and that moreover the map $S : \Gamma_n \times \Gamma_{n+1} \to [0, \infty)$ is continuous. Now we have what we need to define our homotopy R_n. Let

$$R_n(x,t) = \{y \in \Gamma_{n+1} : S(x,y) \le t\}.$$

It is clear that $R_n(x,t)$ is continuous and $R_n(x,0)=\{x\}$ for all $x\in\Gamma_n$. We claim that $d_H(R_n(x,t),\{x\})\leq t+\frac{2}{7n}$. Let $y\in R_n(x,t)$. If x and y are in the same simplex, obviously $d(x,y)\leq\frac{1}{7n}\leq t+\frac{2}{7n}$. If x and y are in different simplices, then there is a path $v_1,\dots,v_k$ of vertex points in Γ_n such that v_1 and x are contained in the simplex σ_i and v_k and y are contained in some other simplex σ_j. By the definition of s_x it follows that if $y\in R_n(x,t)$, we have that

$$|r_i^{-1}(x)-r_i^{-1}(v_1)|+\sum_{n=1}^{k-1}d(v_n,v_{n+1})+|r_j^{-1}(v_k)-r_j^{-1}(y)|\leq t.$$

Hence

$$|r_i^{-1}(x)-r_i^{-1}(v_1)|+d(v_1,v_k)+|r_j^{-1}(v_k)-r_j^{-1}(y)|\leq t.$$

We find that

$$d(x,y)\leq d(x,v_1)+d(v_1,v_k)+d(v_k,y)\leq\frac{1}{7n}+t+\frac{1}{7n}=t+\frac{2}{7n}.$$

It also is clear that if $t>0$ then the set $R_n(x,t)$ consists of more than one point and is connected. Now we have to prove that for points $x,y\in\Gamma_n$ with $d(x,y)=t$, the set $R_n(x,t+\frac{3}{7n})$ is connected and contains $\{x,y\}$. That $x\in R_n(x,t+\frac{3}{7n})$ and $R_n(x,t+\frac{3}{7n})$ is connected is clear. It remains to prove that this set contains y. Note that there are simplexes σ and τ containing x and y respectively, with $\operatorname{diam}(\sigma)\leq\frac{1}{7n}$ and $\operatorname{diam}(\tau)\leq\frac{1}{7n}$, hence there are vertices v of σ and w of τ such that $d(v,w)\leq d(x,y)+\frac{2}{7n}$. This implies that we added a path α_j^* between v and w to Γ_n in the procedure to obtain Γ_{n+1}. The way we defined R_n now implies that

$$R_n(x,\frac{1}{7n})\supseteq\sigma,$$
$$R_n(x,t+\frac{3}{7n})\supseteq\sigma\cup\alpha_j^*$$

and

$$R_n(x,t+\frac{4}{7n})\supseteq\sigma\cup\alpha_j^*\cup\tau.$$

The last equation implies that $R_n(x,t+\frac{4}{7n})$ contains y, as desired. □

Lemma 3.5. *Let $(\Gamma_i)_{i=1}^\infty$ be a tower of graphs as in Theorem 3.3 for some countable infinite product of Peano continua X. Then for every $n\in\mathbb{N}$ there exists a homotopy $G:2^{\Gamma_n}\times I\to 2^{\Gamma_{n+1}}$ such that for every $A\in 2^{\Gamma_n}$ that is t-close to a connected subset of X, and every $s\geq 2t$ we have $G(A,s)\in C(\Gamma_{n+1})$ and $\hat{d}(G_t,1_{2^{\Gamma_n}})\leq t+\frac{1}{n}$.*

PROOF: Fix $n\in\mathbb{N}$ for the rest of the proof. Let $R_n:\Gamma_n\times I\to C(\Gamma_{n+1})$ denote the "growth homotopy" for Γ_n in Γ_{n+1} as in Corollary 3.4. Define

$$G(A,t)=\bigcup\{R_n(a,t+\frac{4}{7n}):a\in A\}.$$

We claim that this is the map as required. Note that $G(A,t)$ is compact for every t and A. Assume that A is t-close to a connected set of X. We shall prove that for $s \geq 2t$ the set $G(A,s)$ is a connected subset of Γ_{n+1}. If A itself is connected, this is clear. We prove that $G(A,2t)$ is connected. The result for $s \geq 2t$ then follows easily. Assume that $G(A,2t) = R_n(A, 2t + \frac{4}{7n})$ is not connected. Then we can write $A = B \cup C$ where B and C are non-empty closed subsets of A and

$$R_n(B, 2t + \frac{4}{7n}) \cap R_n(C, 2t + \frac{4}{7n}) = \emptyset.$$

Since A is t-close to a connected subset of X, we have that

$$\{x \in X : d(x,B) \leq t\} \cap \{x \in X : d(x,C) \leq t\} \neq \emptyset.$$

We can find points $b \in B$ and $c \in C$ such that $d(b,c) \leq 2t$. Now the assertion follows from (3) in Lemma 3.4.

Choose $A \in 2^{\Gamma_n}$ and $t > 0$. For every $a \in A$ we have

$$d_H(\{a\}, R_n(a, t + \frac{4}{7n})) \leq t + \frac{4}{7n} + \frac{2}{7n} \leq t + \frac{1}{n}\,.$$

Thus $d_H(A, G(A,t)) \leq t + \frac{1}{n}$. □

We now come to an important result.

Proposition 3.6. *Let X be a Peano continuum that contains no free arcs. Then $\dim_{\geq n+1}(C(X))$ is a σZ-set in $C(X)$ for all $n \in \mathbb{N}$.*

PROOF: It is clear that for all $n \in \mathbb{N}$ the set $\dim_{\geq n+1}(C(X))$ is σ-compact, thus the only thing to prove is the approximation property. We shall show that $\dim_{\geq 2}(C(X))$ is σZ-set. It follows that for any $n \geq 2$, the set $\dim_{\geq n}(C(X))$ is a σ-compact subset of a σZ-set and hence a σZ-set itself.

Find a tower $(\Gamma_i)_{i=1}^{\infty}$ of graphs in X such that $(\mathcal{F}_i(\Gamma_i))_{i=1}^{\infty}$ has the mapping approximation property as in Theorem 3.3. Choose $\varepsilon > 0$ and a map $f : Q \to C(X)$. Let $H : C(X) \times I \to 2^X$ be a map such that $H_0 = 1_{C(X)}$ and $H[C(X) \times [\frac{1}{n}, 1]] \subseteq \mathcal{F}_{j(n)}(\Gamma_{j(n)})$ for a strictly increasing sequence $j(1), j(2), \ldots \subseteq \mathbb{N}$. Note that we may assume that $\hat{d}(1_{2^X}, H_t) \leq t$ for all t. Find $K \in \mathbb{N}$ such that $\frac{1}{K} \leq \frac{\varepsilon}{8}$. Define $h : Q \to 2^X$ by $h(x) = H(f(x), \frac{1}{K})$. Then $\hat{d}(f,h) \leq \frac{1}{K}$ and $h[Q] \subseteq \mathcal{F}_{j(m)}(\Gamma_{j(m)})$ for all $m \geq K$. We adjust the map h to get the approximation of f that we need. Observe that for any $x \in Q$, $h(x)$ is $\frac{1}{K}$-close to a connected subset of X. Let $G : \Gamma_{j(K)} \times I \to 2^{\Gamma_{j(K)+1}}$ be the homotopy of Lemma 3.5. Then $g(x) = G(h(x), \frac{2}{K})$ is the desired approximation of f. Because $h(x)$ is $\frac{1}{K}$-close to a connected subset of X, $G(h(x), \frac{2}{K})$ is connected. We find $\hat{d}(f,g) \leq \hat{d}(f,h) + \hat{d}(h,g) \leq \frac{1}{K} + \frac{2}{K} + \frac{1}{j(K)} \leq \frac{4}{K} < \varepsilon$. Observe that for every $x \in Q$, $g(x) \in C(\Gamma_{j(K)+1})$, thus $\dim(g(x)) \leq 1$ for all $x \in Q$. This implies that $\dim_{\geq 2}(C(X)) \in \mathcal{Z}_\sigma(C(X))$. □

We now derive the main result of this section.

Theorem 3.7. *Let X be a countable infinite product of non-degenerate Peano continua and $\mathcal{D}$ a true dimension function. Then $\mathcal{D}_{\geq n+1}(C(X))$ is a σZ-set in $C(X)$ for all $n \geq 1$.*

PROOF: We may assume that X is a closed subset of Q. By [9, Corollary 3.3] we have that $\mathcal{D}_{\geq n}(2^X)$ is an F_σ subset of 2^Q. Thus $\mathcal{D}_{\geq n}(2^X)$ is an F_σ subset of 2^X and hence $\mathcal{D}_{\geq n}(C(X))$ is an F_σ subset of $C(X)$.

To see that $\mathcal{D}_{\geq n+1}(C(X))$ is locally homotopy negligible, we only have to show that this is the case for $n = 2$. As in [9, Lemma 5.3] observe that $\mathcal{D}_{\geq 2}(C(X)) \subseteq \dim_{\geq 2}(C(X))$ ([9, Theorem 2.9]) and use Proposition 3.6 to conclude that $\mathcal{D}_{\geq 2}(C(X))$ is locally homotopy negligible. $\square$

4. Auxiliary homotopies in $C(X)$.

In this section we construct some useful homotopies. The homotopy in the following lemma is essential in the proof of the main result in this paper.

Lemma 4.1. *Let $X = \prod_{k=1}^\infty X_k$ be a countable infinite product of Peano continua. Then for every point $(c_1, c_2, \dots) \in X$, we can find a sequence $i_1 < i_2 < i_3 < \dots \in \mathbb{N}$ and homotopies $\Lambda : C(X) \times I \to C(X)$ and $\text{Ю} : C(X) \times I \to 2^X$ such that*

(1) $\text{Ю}_0 = 1 \mid C(X)$,
(2) $\text{Ю}[C(X) \times (0, 1]] \subseteq \mathcal{F}(X)$,
(3) $\hat{d}(\text{Ю}_t, 1) \leq t$,
(4) $\Lambda_0 = 1 \mid C(X)$,
(5) $\text{Ю}(A, t) \subseteq \Lambda(A, t)$ *for all A and t,*
(6) $\hat{d}(\Lambda_t, 1) \leq 13t$,
(7) *if $t \in [2^{-n}, 2^{-(n-1)}]$ then for all $m \geq i_n + 1$ we have $\pi_m[\text{Ю}(A, t)] = \pi_m[\Lambda(A, t)] = \{c_m\}$,*
(8) *if $t \in [2^{-n}, 2^{-(n-1)}]$ then for all $m \geq i_n$ we have that the projection $\Pi_m[\Lambda(A, t)]$ of $\Lambda(A, t)$ on $X_1 \times \dots \times X_m$ is at most one-dimensional for every true dimension function.*

PROOF: Choose a point $(c_1, c_2, \dots) \in X$. Find a tower $(\Gamma_i)_{i=1}^\infty$ of graphs as in Theorem 3.3 for the space X and the point $(c_1, c_2, \dots)$. Let $\text{Ю} : C(X) \times I \to 2^X$ be a homotopy such that

(i) $\hat{d}(\text{Ю}_t, 1) \leq t$,
(ii) $\text{Ю}[C(X) \times [\frac{1}{n}, 1]] \subseteq \mathcal{F}_{j_n}(\Gamma_{j_n})$, for all $n \in \mathbb{N}$ and some strictly increasing sequence $j_1, j_2, \dots$.

Note that this map Ю and the point $(c_1, c_2, \dots)$ are as required.

We now construct the homotopy Λ. Let $i_n = j_{n+1} + 1$. Note that the sequence $i_1, i_2, \dots$ is strictly increasing. Let $R_n : \Gamma_n \times I \to C(\Gamma_{n+1})$ be the "growth homotopy" of Corollary 3.4. Define maps $g_n : [0, 1] \to [0, 2]$ such that $g_{j_n} \mid [\frac{1}{n+1}, \frac{1}{n}] = \frac{2}{n}$ and $g_{j_n} \mid [0, \frac{1}{n+2}] \cup [\frac{1}{n-1}, 1] = 0$.

Define $\Lambda : C(X) \times I \to C(X)$ by the formula

$$\Lambda(A, t) = \bigcup_{n=1}^\infty \{R_{j_n}[\text{Ю}(A, t) \times \{g_{j_n}(t)\}]\}.$$

We claim that Λ is as required. That Λ is continuous follows from the continuity of all used maps. Note that

$$
\begin{aligned}
\Lambda(A,0) &= \bigcup_{n=1}^{\infty} \{R_{j(n)}[\text{Ю}(A,0) \times \{g_{j_n}(0)\}]\} \\
&= \bigcup_{n=1}^{\infty} \{R_{j(n)}[A \times \{0\}]\} \\
&= A.
\end{aligned}
$$

It is clear that Λ_0 has all the required properties. Now take $t > 0$. Assume $t \in [\frac{1}{m+1}, \frac{1}{m}]$. Then

$$
\begin{aligned}
\Lambda(A,t) &= \bigcup_{n=1}^{\infty} \{R_{j_n}[\text{Ю}(A,t) \times \{g_{j_n}(t)\}]\} \\
&= \bigcup_{n=m-1}^{m+1} \{R_{j_n}[\text{Ю}(A,t) \times \{g_{j_n}(t)\}]\}.
\end{aligned}
$$

Note that $\Lambda(A,t)$ is compact, because it is a finite union of continuous images of $\text{Ю}(A,t)$. Observe that $\text{Ю}(A,t)$ is finite, say $\{a_1,\dots,a_k\}$. For every $1 \le i \le k$, the set $R_s(a_i, g_s(t))$ is connected and contains a_i, thus in particular $\Lambda(A,t) \supseteq \text{Ю}(A,t)$. We also know that the set $R_{j_m}[\text{Ю}(A,t) \times \{g_{j_m}(t)\}] = R_{j_m}[\text{Ю}(A,t) \times \{\frac{2}{m}\}]$ is connected, because $\text{Ю}(A,t)$ is $t \le \frac{1}{m}$-close to the connected set A. This implies that $\Lambda(A,t)$ is connected.

Observe that the sets $R_{j_{m-1}}[\text{Ю}(A,t) \times \{g_{j_{m-1}}(t)\}]$, $R_{j_m}[\text{Ю}(A,t) \times \{g_{j_m}(t)\}]$ and $R_{j_{m+1}}[\text{Ю}(A,t) \times \{g_{j_{m+1}}(t)\}]$ are subsets of finite graphs. Since for compact polyhedra P we know that $\dim(P) = 1$ implies $\mathcal{D}(P) = 1$ [9], we find that $\Lambda(A,t)$ is at most one-dimensional for every true dimension function.

Note that for $k \ge i_m = j_{m+1} + 1$ we have

$$
\begin{aligned}
\pi_k[\Lambda(A,t)] &= \pi_k[\bigcup_{n=m-1}^{m+1} \{R_{j_n}[\text{Ю}(A,t) \times \{g_{j_n}(t)\}]\}] \\
&= \bigcup_{n=m-1}^{m+1} \{\pi_k[R_{j_n}[\text{Ю}(A,t) \times \{g_{j_n}(t)\}]]\} \\
&= \{c_k\}.
\end{aligned}
$$

This implies also that $\Lambda(A,t)$ is of the form $\tilde{A} \times \{c_{i_m}\} \times \{c_{i_m+2}\} \times \cdots$, where $\tilde{A}$ has the same dimension as $\Lambda(A,t)$. The conclusion is that Λ satisfies (8).

We also have

$$\begin{aligned}
d_H(\Lambda(A,t),A) &\le d_H(A,\text{Ю}(A,t)) + d_H(\text{Ю}(A,t),\Lambda(A,t))\\
&\le t + d_H(R_{j_{m-1}}[\text{Ю}(A,t)\times\{g_{j_{m-1}}(t)\}],\text{Ю}(A,t))\\
&\quad + d_H(R_{j_m}[\text{Ю}(A,t)\times\{g_{j_m}(t)\}],\text{Ю}(A,t))\\
&\quad + d_H(R_{j_{m+1}}[\text{Ю}(A,t)\times\{g_{j_{m+1}}(t)\}],\text{Ю}(A,t))\\
&\le t + g_{j_{m-1}}(t) + \frac{1}{j_{m-1}} + g_{j_m}(t) + \frac{1}{j_m} + g_{j_{m+1}}(t) + \frac{1}{j_{m+1}}\\
&\le t + \frac{2}{m-1} + \frac{1}{m-1} + \frac{2}{m} + \frac{1}{m} + \frac{2}{m+1} + \frac{1}{m+1}\\
&\le t + \frac{3}{m-1} + \frac{6}{m}\\
&\le t + 6t + 6t\\
&\le 13t.
\end{aligned}$$

We conclude that Λ is as required. $\square$

The next lemma that we need is similar to [7, Lemma 3.2]. The reader easily verifies that the map defined there has the additional property (4) that we need in the following lemma.

Lemma 4.2. *Let $\overline{Q} = \prod_{n=1}^\infty I^n$ and $0_m = (0,\dots,0) \in I^m$. There is a homotopy $\mathcal{W} : 2^Q \times I \to 2^{\overline{Q}}$ such that*

1. *for $t > 0$, $\mathcal{W}_t : 2^Q \to 2^{\overline{Q}}$ is an embedding,*
2. *if $t \le 2^{-n}$, $\pi_m[\mathcal{W}(A,t)] = \{0_m\}$ for every $A \in 2^Q$ and $m \le n$,*
3. *if $t \ge 2^{-n}$, $\pi_m[\mathcal{W}(A,t)] = \{\pi_m(A)\} \times \{0_{m-1}\}$ for every $A \in 2^Q$ and $m \ge n+3$,*
4. *$\mathcal{W}_t[C(Q)] \subseteq C(\overline{Q})$ for all t,*
5. *for $t > 0$, $\mathcal{W}_t(A)$ is homeomorphic to A for every $A \in 2^Q$.*

Note that (2) and the continuity of $\mathcal{W}$ imply that $\pi_i[\mathcal{W}(A,0)] = 0_i$ for every $A \in 2^Q$ and $i \in \mathbb{N}$.

$\square$

5. An F_σ-absorbing sequence in $C(X)$.

The main result of this section is that if X is a countable product of non-degenerate Peano continua and $\mathcal{D}$ a true dimension function, then the sequence $(\mathcal{D}_{\ge n}(C(X)))_{n=2}^\infty$ is strongly F_σ-universal in $C(X)$.

Proposition 5.1. *For every n let X_n be a non-degenerate Peano continuum. Let $X = \prod_{n=1}^\infty X_n$ and $\mathcal{D}$ a true dimension function. Then the sequence $\big(\mathcal{D}_{\ge n}(C(X))\big)_{n=2}^\infty$ is strongly F_σ-universal in $C(X)$.*

Proof: Let $\varepsilon > 0$. Choose a decreasing sequence of σ-compact subsets $(A_n)_{n=1}^\infty$ in Q and a map $f : Q \to C(X)$ that restricts to a Z-embedding on some compact subset K of Q. Without loss of generality we may assume that f is a Z-embedding because $C(X)$ is a Hilbert cube.

Choose $c = (c_1, c_2, \dots) \in X$ and maps $R_n : (X \times (0, \dots, 0)) \cup (\{c_n\} \times I^n) \to X_n$ such that $R_n(x, 0_n) = x$ for all x and $R_n \mid (\{c_n\} \times I^n)$ is an embedding. (We can do this because we may assume that the space X_n contains an n-cube. This is where we use that the spaces X_n are non-degenerate.)

For the space X and the point $(c_1, c_2, \dots)$, find a sequence $i_1 < i_2 < i_3 < \dots \in \mathbb{N}$ and homotopies $\Lambda : C(X) \times I \to C(X)$ and $\text{Ю} : C(X) \times I \to 2^X$ with the properties as stated in Lemma 4.1. Define $\delta : Q \to I$ by

$$\delta(x) = \frac{\varepsilon}{16} d(f(x), f(K)).$$

Let $\varrho : I \to I$ be a map such that $\varrho^{-1}(0) = 0$ and $\varrho(2^{-n}) = 2^{-i_n}$ for all $n \in \mathbb{N}$. Let $v : Q \to Q$ be a map such that

$$v(x_1, x_2, x_3, \dots) = (\underbrace{x_1}, \underbrace{x_1, x_2}, \underbrace{x_1, x_2, x_3}, \underbrace{x_1, \dots}, \dots).$$

Observe that if for all, but finitely many, m we have $\pi_m(v(x)) = \pi_m(v(y))$, then $x = y$.

By [9, Remark 5.15 and Proposition 5.10] it is possible to find an embedding $j : Q \times I \to C(Q)$ such that

(1) $\pi_1[j^{-1}[\mathcal{D}_{\geq k+1}(C(Q))]] = A_k$,
(2) $(0, 0, \dots) \in j(x, t)$ for every x and t,
(3) $\pi_{4k+1}[j(x, t)] = [0, \pi_k(v(x))]$,
(4) $\pi_{4k+3}[j(x, t)] = [0, t]$.

Let $\mathcal{W} : C(Q) \times I \to C(\overline{Q})$ as in Lemma 4.2.

The map that approximates f is given by

$$\begin{aligned} h(x) = \{ &\prod_{n=1}^{\infty} R_n(y_n, w_n) : (y_1, y_2, \dots) \in \text{Ю}(f(x), \delta(x)), \\ &(w_1, w_2, \dots) \in \mathcal{W}(j(x, \varrho(\delta(x))), 2^{-5} \varrho(\delta(x)))\} \\ &\cup \Lambda(f(x), \delta(x)). \end{aligned}$$

We claim that this is the approximation of f as in the definition of strong $F_{\sigma\delta}$-universality.

The proofs that h is well-defined, $h \mid K = f \mid K$ and $\hat{d}(f, h) < \varepsilon$ are similar to the proofs of corresponding results in [6, Theorem 3.4] and [7, Proposition 4.1].

We will prove that h is a Z-embedding and that h satisfies the desired embedding property.

Claim 1. h is a Z- embedding.

First observe that $h[K] \cap h[Q \setminus K] = \emptyset$. One easily verifies that for every $x \in Q \setminus K$ we have $d_H(h(x), f(x)) < d_H(f(x), h[K])$. Since $f \mid K = h \mid K$

and $f \mid K$ is an embedding, it follows that $h \mid K$ is an embedding. The only thing left to prove therefore is that if $x, y \in Q \setminus K$ and $h(x) = h(y)$ implies $x = y$. To this end, choose $x, y \in Q \setminus K$ such that $h(x) = h(y)$ and let $\varrho(\delta(x)) = t$ and $\varrho(\delta(y)) = r$. First we shall prove that in this situation $r = t$. Observe that by the above $t, r > 0$. There consequently exists $m \in \mathbb{N}$ such that $2^{-m} \leq \min\{2^{-5}t, 2^{-5}r\}$. Let $\Pi : C(X) \to C(\prod_{n=m+3}^{\infty} X_n)$ denote the projection. Then we have,

$$\Pi\Big[\text{Ю}(f(x), r)\Big] = \{(c_{m+3}, c_{m+4}, \dots)\} = \Pi\Big[y\text{Ю}(f(y), t)\Big]$$

and

$$\Pi\big[\Lambda(f(x), r)\big] = \{(c_{m+3}, c_{m+4}, \dots)\} = \Pi\big[\Lambda(f(y), t)\big].$$

Consequently,

$$\Pi\big[h(x)\big] = \{ \prod_{n=m+3}^{\infty} \{R_k(c_n, w_n)\} : w \in \mathcal{W}(j(x, r), 2^{-5}r)\}$$

and

$$\Pi\big[h(y)\big] = \{ \prod_{n=m+3}^{\infty} \{R_n(c_n, w_n)\} : w \in \mathcal{W}(j(y, t), 2^{-5}t)\}.$$

Choose $k \geq m + 3$ such that k has the form $4p + 3$. Then

$$\begin{aligned} R_k\big[\{c_k\} \times ([0, t] \times \{0\}^{k-1})\big] &= \pi_k[h(x)] \\ &= \pi_k[h(y)] \\ &= R_k\big[\{c_k\} \times ([0, r] \times \{0\}^{k-1})\big]. \end{aligned}$$

Because $R_k \mid \{c_k\} \times I^k$ is an embedding, we find $t = r$. To see that $x = y$, note that for any $k \geq m + 3$ that is of the form $k = 4p + 1$ we have

$$\begin{aligned} R_k\big[\{c_k\} \times ([0, \pi_p(v(x))] \times \{0\}^{k-1})\big] &= \pi_k[h(x)] \\ &= \pi_k[h(y)] \\ &= R_k\big[\{c_k\} \times ([0, \pi_p(v(y))] \times \{0\}^{k-1})\big]. \end{aligned}$$

Using again that $R_k \mid \{c_k\} \times I^k$ is an embedding we find that for any p with $4p + 1 \geq m + 3$ we have

$$\pi_p(v(x)) = \pi_p(v(y)).$$

Because of the way we constructed the coordinate shifting map v this implies that $x = y$.

Since for all $x \in Q \setminus K$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $\pi_n[h(x)] \neq X_n$ and $h[K] \in \mathcal{Z}(Q)$, h is a Z-embedding.

Claim 2. For all $k \in \mathbb{N}$, $h^{-1}\big[\mathcal{D}_{\geq k+1}(C(X))\big] \setminus K = A_k \setminus K$.

Note that for $x \in Q \setminus K$ we have that $h(x)$ is the union of an at most one-dimensional set $\Lambda(f(x), \delta(x))$ and a finite union of copies of

$$\mathcal{W}\Big(j(x, \varrho(\delta(x))), 2^{-5}\varrho(\delta(x))\Big).$$

For $x \in Q \setminus K$ we have

$$\mathcal{D}\Big[\mathcal{W}\Big(j(x, \varrho(\delta(x))), 2^{-5}\varrho(\delta(x))\Big)\Big] = \mathcal{D}[j(x, \varrho(\delta(x)))].$$

As a consequence, for $x \in Q \setminus K$:

$$\mathcal{D}[h(x)] = \max\{1, \mathcal{D}[j(x, \varrho(\delta(x)))]\}.$$

By the construction of the map j we have for all $k \in \mathbb{N}$,

$$h^{-1}[\mathcal{D}_{\geq k+1}(C(X))] \setminus K = j^{-1}[\mathcal{D}_{\geq k+1}(C(Q))] \setminus K = A_k \setminus K.$$

□

Theorem 5.2. *If X is a countable infinite product of non-degenerate Peano continua and $\mathcal{D}$ a true dimension function, then the sequence $\big(\mathcal{D}_{\geq n}(C(X))\big)_{n=2}^{\infty}$ is F_σ-absorbing in $C(X)$.*

Proof: The sequence $\big(\mathcal{D}_{\geq n}(C(X))\big)$ is strongly F_σ-universal in $C(X)$ by Proposition 5.1. By Theorem 3.7 we have that for every $n \geq 2$, $\mathcal{D}_{\geq n}(C(X))$ is a σZ-subset of $C(X)$. □

Corollary 5.3. *Let X be a countable product of non-degenerated Peano continua and $\mathcal{D}$ a true dimension function. Then there is a homeomorphism $f : C(X) \to Q^\infty$ such that for all $n \in \mathbb{N}$,*

$$f[\mathcal{D}_{\geq n+1}(C(X))] = \underbrace{B \times \ldots B}_{n \text{ times}} \times Q \times Q \times \ldots.$$

Proof: Since as was observed in § 2 the sequence $(B^n \times Q \times Q \times \ldots)_{n=1}^{\infty}$ is F_σ-absorbing in Q^∞, this follows from Theorem 5.2 and Theorem 2.1.

□

Corollary 5.4. *Let X be a countable infinite product of Peano continua. Then $\mathcal{D}_{\geq n}(C(X))$ is a capset in $C(X)$ for every $n \geq 2$.*

Proof: By Corollary 5.3, for every $n \geq 2$ the pair $\big(C(X), \mathcal{D}_{\geq n}(C(X))\big)$ is homeomorphic to the pair $(Q^\infty, B^{n-1} \times Q \times Q \times \ldots)$. $B^{n-1} \times Q \times Q \times \ldots$ is a capset in Q^∞, thus $\mathcal{D}_{\geq n}(C(X))$ is a capset in $C(X)$. □

Remark. Note that the results in this section hold in particular for the covering dimension dim and the cohomological dimension $\dim_G$ for any Abelian group G.

Remark. The techniques used in this paper combined with the result in [7], can be used to prove that for X a countable infinite product of non- degenerate Peano continua and $\mathcal{D}$ an arbitrarily true dimension function, the sequence $(\mathcal{D}_{\geq n}(2^X))_{n=1}^\infty$ is F_σ-absorbing in 2^X.

It is easy to see that for every n the set $\mathcal{D}_{\geq n}(2^X)$ is a σZ-set in 2^X.

To prove the strong F_σ-universality of the sequence, repeat the proof that the sequence $(\dim_{\geq n}(2^X))_{n=1}^\infty$ is strongly F_σ-universal in [7]. Use in that proof a map $\tilde{j} : Q \times I \to 2^Q$, that can be found in the same way as the map $j : Q \times I \to C(Q)$ that we used in the proof of Proposition 5.1 in this paper.

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