Skeletons in multigraphs

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Abstract. Under a multigraph it is meant in this paper a general incidence structure with finitely many points and blocks such that there are at least two blocks through any point and also at least two points on any block. Using submultigraphs with saturated points there are defined generating point sets, point bases and point skeletons. The main result is that the complement to any basis (skeleton) is a skeleton (basis).

Keywords: multigraph, submultigraph with saturated vertices, generating vertex set, vertex basis, skeleton

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1. Definitions.

An *incidence structure in wide sense* is defined as a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are sets with $\mathcal{A} \cap \mathcal{B} = \emptyset$, $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{B}$. Elements of $\mathcal{C}$ are called *flags*. The fact that $(a, b) \in \mathcal{C}$ or, equivalently, $a\mathcal{C}b$, can be expressed by words as "a lies on b", "b goes through a".

Note that $\mathcal{C}$ will be often replaced by its "symmetric prolongation" $\widetilde{\mathcal{C}} \subseteq (\mathcal{A} \cup \mathcal{B}) \times (\mathcal{A} \cup \mathcal{B})$ such that $x\widetilde{\mathcal{C}}y$ means either $x\mathcal{C}y$ or $y\mathcal{C}x$. The relation $\mathcal{C}$ (or $\widetilde{\mathcal{C}}$) is called the *incidence*. The *dual structure* $(\overline{\mathcal{A}}, \overline{\mathcal{B}}, \overline{\mathcal{C}})$ of the incidence structure $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is defined by $\mathcal{A} = \mathcal{B}$, $\mathcal{B} = \mathcal{A}$, $\overline{\mathcal{C}} = \mathcal{C}^{-1}$.

If $\mathfrak{I}_1 = (\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1)$, $\mathfrak{I}_2 = (\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2)$ are incidence structures with $\mathcal{A}_1 \subseteq \mathcal{A}_2$, $\mathcal{B}_1 \subseteq \mathcal{B}_2$, $\mathcal{C}_1 \subseteq \mathcal{C}_2$ then $\mathfrak{I}_1$ is said to be a *substructure* of $\mathfrak{I}_2$.

If $\mathfrak{I} = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ is an incidence structure then we put $[a]_{\mathfrak{I}} = \{b \mid a\mathcal{C}b,\ b \in \mathcal{B}\}$ for all $a \in \mathcal{A}$ and $[b_{\mathfrak{I}} = \{a \mid a\mathcal{C}b, a \in \mathcal{A}\}$ for all $b \in \mathcal{B}$. The index $\mathfrak{I}$ may be omitted if there is no danger of confusion.

Now we introduce several special kinds of general incidence structures $\mathfrak{I} = (\mathcal{A}, \mathcal{B}, \mathcal{C})$.

1. If $\mathcal{A}, \mathcal{B} \neq \emptyset$ and $\#[x] \geq 2$ for all $x \in \mathcal{A} \cup \mathcal{B}$ then $\mathfrak{I}$ is called here a *multigraph*. Elements of $\mathcal{A}$ are called *vertices*, those of $\mathcal{B}$ *edges* and $\mathcal{C}$ is called the *adjacency relation*. If $\mathfrak{I}$ is a multigraph then $\overline{\mathfrak{I}}$ is a multigraph too.

2. If $\mathcal{A} \neq \emptyset$ and $\#[b] = 2$ for all $b \in \mathcal{B}$ then $\mathfrak{I}$ is called a *graph*.

3. If for any two distinct $x, y \in \mathcal{A}$ there is just one $z \in \mathcal{B}$ such that $x, y \in [z]$ then $\mathfrak{I}$ is called a *configuration*. If $\mathfrak{I}$ is a configuration then $\overline{\mathfrak{I}}$ is a configuration too. The terminology "vertices — edges — adjacency' is for graphs and configurations convenient too, mainly for finite ones. Another terminology is mentioned below.

4. A configuration $(\mathcal{A},\mathcal{B},\mathcal{C})$ which is simultaneously a multigraph with $\#[a]=k$ (a constant) for all $a\in\mathcal{A}$ is called a *k-configuration*. A k-configuration $(\mathcal{A},\mathcal{B},\mathcal{C})$ satisfying in addition $\#[b]=\ell$ (a constant) for all $b\in\mathcal{B}$ is said to be a *(k,ℓ)-configuration*.

5. A configuration $(\mathcal{A},\mathcal{B},\mathcal{C})$ is also called an *incidence structure in narrower sense* or a *partial plane*. Elements of $\mathcal{A}$ are then often called *points*, those of $\mathcal{B}$ *lines* and $\mathcal{C}$ is called the *incidence relation*.

The definition of substructures (submultigraphs, subgraphs, subconfigurations, sub-k-configurations, sub-(k,ℓ)-configurations) is obvious and can be omitted.

An incidence structure $\mathfrak{I}=(\mathcal{A},\mathcal{B},\mathcal{C})$ is said to be *connected* if for any two distinct elements $a,a'\in\mathcal{A}$ there exists a finite sequence $(a_0,b_1,a_1,b_2,a_3,\dots,b_n,a_n)$ such that $a_0,a_1,\dots,a_n\in\mathcal{A}$; $b_1,\dots,b_n\in\mathcal{B}$; $a_0=a$; $a_n=b$ and that every element b_i passes through both neighboring elements a_{i-1},a_i $(i\in\{1,\dots,\ell\})$. One can also say that the connectness is, as a binary relation on $\mathcal{A}$, the transitive closure of the binary relation "to lie on the same element of $\mathcal{B}$" on $\mathcal{A}$.

2. Closed sets and generating sets in multigraphs.

In the sequel we shall restrict ourselves to finite multigraphs (i.e. with finite sets of vertices and edges). We shall use the denotation $\mathfrak{G}=(\mathcal{V},\mathcal{E},\mathrm{I})$ or a similar one, for a multigraph.

A submultigraph $\mathfrak{G}'=(\mathcal{V}',\mathcal{E}',\mathrm{I}')$ of a multigraph $\mathfrak{G}=(\mathcal{V},\mathcal{E},\mathrm{I})$ is called to be *with saturated vertices*, respectively *with saturated edges*, if $\#[x]_{\mathfrak{G}}=\#[x]_{\mathfrak{G}'}$ for all $x\in\mathcal{V}'$, respectively for all $x\in\mathcal{E}'$.

Let $\mathfrak{G}=(\mathcal{V},\mathcal{E},\mathrm{I})$ be a multigraph. A set $\mathcal{W}\subseteq\mathcal{V}$ is called *closed* in $\mathfrak{G}$ if for every edge $x\in\mathcal{E}$ the set $[x]\cap(\mathcal{V}\setminus\mathcal{W})$ is either empty or contains at least two elements.

Proposition 1. *Let $\mathfrak{G}=(\mathcal{V},\mathcal{E},\mathrm{I})$ be a multigraph and $\mathcal{W}$ a proper subset of $\mathcal{V}$ closed in $\mathfrak{G}$. Putting $\mathcal{V}'=\mathcal{V}\setminus\mathcal{W}$, $\mathcal{E}'=\mathcal{E}\setminus\{y\mid y\in\mathcal{E},\ [y]\subseteq\mathcal{W}\}$, $\mathrm{I}'=\mathrm{I}\cap(\mathcal{V}'\times\mathcal{E}')$ we get a submultigraph $\mathfrak{G}_{\mathcal{W}}=(\mathcal{V}',\mathcal{E}',\mathrm{I}')$ of $\mathfrak{G}$ with saturated vertices.*

Proof: As $\mathcal{W}\neq\mathcal{V}$, the set $\mathcal{V}'$ is not empty. For the same reason also $\mathcal{E}'$ must be nonempty. The assumption of closeness of $\mathcal{W}$ guarantees that every edge in $\mathcal{E}'$ passes through at least two vertices from $\mathcal{V}'$. Further, the definition of $\mathcal{V}'$ and $\mathcal{E}'$ guarantees that all edges going through a vertex in $\mathcal{V}'$ lie already in $\mathcal{E}$ so that all vertices in $\mathcal{V}'$ are saturated. □

Corollaries. (i) *If moreover $\mathfrak{G}$ is a 3-configuration then $\mathfrak{G}_{\mathcal{W}}$ is a 3-configuration too.* (ii) *If moreover $\mathfrak{G}$ is a connected graph then there is no proper subgraph $\mathfrak{G}_{\mathcal{W}}$ obtainable.*

Proof: As $\mathfrak{G}_{\mathcal{W}}$ is with saturated vertices then, under assumption $\mathfrak{G}$ being a 3-configuration, $\mathfrak{G}_{\mathcal{W}}$ must be a 3-configuration too. However, if $\mathfrak{G}$ is a multigraph with $\#[x]=2$ for all $x\in\mathcal{E}$ then there holds for all $x\in\mathcal{E}$ either $[x]\subseteq\mathcal{W}$ or $[x]\cap\mathcal{W}=\emptyset$. Hence from the connectivity of $\mathfrak{G}$ it follows either $\mathcal{W}=\mathcal{V}$ (which is excluded) or $\mathcal{W}=\emptyset$. □

Remark. As no connected (k,ℓ)-configuration $\mathfrak{C}$ has proper sub-(k,ℓ)-configurations, there are only two closed vertex subsets in $\mathfrak{C}$: the empty set and the full vertex set.

Proposition 2. *Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a multigraph and $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathrm{I}')$ its submultigraph with saturated vertices. Then $\mathcal{W} = \mathcal{V} \backslash \mathcal{V}'$ is a closed set*/ and $\mathfrak{G}' = \mathfrak{G}_{\mathcal{W}}$.*

PROOF: The case $\mathfrak{G}' = \mathfrak{G}$, $\mathcal{W} = \emptyset$ can be left aside as a trivial one. Thus assume that $\mathfrak{G}' \neq \mathfrak{G}$, $\mathcal{V}' \neq \mathcal{V}$. If there exists an edge e such that among its adjacent vertices there is just one from $\mathcal{V}'$ then $e \in \mathcal{E}'$. However, as $\mathfrak{G}'$ is a multigraph, it must be $\#[e]_{\mathfrak{G}'} \geq 2$. We have obtained a contradiction. □

Corollary. *If $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ is a 3-configuration and $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathrm{I}')$ its sub-3-configuration then $\mathcal{W} = \mathcal{V} \setminus \mathcal{V}'$ is a closed set and $\mathfrak{G}' = \mathfrak{G}_{\mathcal{W}}$.*

Proposition 3. (i) *If $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ is a multigraph and $(\mathcal{W}_\iota)_{\iota \in \mathcal{I}}$ a family (with an arbitrary index set $\mathcal{I}$) of closed sets then $\bigcup_{\iota \in \mathcal{I}} \mathcal{W}_\iota$, $\bigcap_{\iota \in \mathcal{I}} \mathcal{W}_\iota$ are also closed sets.*

(ii) *If $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ is a multigraph and $\mathcal{W}$ a subset of $\mathcal{V}$ then there is just one minimal closed set $\mathcal{W}_{\mathfrak{G}}$ such that $\mathcal{W} \subseteq \mathcal{W}_{\mathfrak{G}}$.**/*

(iii) *If $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ is a multigraph and $\mathcal{W}, \mathcal{W}'$ vertex subsets with $\mathcal{W} \subseteq \mathcal{W}'$ then $\mathcal{W}_{\mathfrak{G}} \subseteq \mathcal{W}'_{\mathfrak{G}}$.*

(iv) *Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a multigraph, $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathrm{I}')$ its submultigraph with saturated vertices and $\mathcal{S}$ a subset of $\mathcal{V}'$. Then $\mathcal{S}_{\mathfrak{G}} \cap \mathcal{V}' \subseteq \mathcal{S}_{\mathfrak{G}'}$.*

PROOF: For parts (i)–(ii)–(iii): straightforward.
Let the assumption in the part (iv) be satisfied. The set $\mathcal{S}_{\mathfrak{G}'} \cup (\mathcal{V} \setminus \mathcal{V}')$ as a union of two closed sets, is a closed set too. By (iii), $\mathcal{S} \subseteq \mathcal{S}_{\mathfrak{G}'}$ implies $\mathcal{S}_{\mathfrak{G}} \subseteq \mathcal{S}_{\mathfrak{G}'} \cup (\mathcal{V} \setminus \mathcal{V}')$, so that consequently $\mathcal{S}_{\mathfrak{G}} \cap \mathcal{V}' \subseteq (\mathcal{S}_{\mathfrak{G}'} \cup (\mathcal{V} \setminus \mathcal{V}')) \cap \mathcal{V}' = (\mathcal{S}_{\mathfrak{G}'} \cap \mathcal{V}') \cup ((\mathcal{V} \setminus \mathcal{V}') \cap \mathcal{V}') = \mathcal{S}_{\mathfrak{G}'} \cap \mathcal{V}' \subseteq \mathcal{S}_{\mathfrak{G}'}$. □

Corollary of (iv). *If especially $\mathcal{V}' \subseteq \mathcal{S}_{\mathfrak{G}}$ then $\mathcal{S}_{\mathfrak{G}'} = \mathcal{V}'$.*

PROOF: Let $\mathcal{V}' \subseteq \mathcal{S}_{\mathfrak{G}}$. Then the conclusion $\mathcal{S}_{\mathfrak{G}} \cap \mathcal{V}' \subseteq \mathcal{S}_{\mathfrak{G}'}$ of (iv) can be written as $\mathcal{V}' \subseteq \mathcal{S}_{\mathfrak{G}'}$ so that $\mathcal{V}' = \mathcal{S}_{\mathfrak{G}'}$. □

If $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ is a multigraph then every set $\mathcal{W} \subseteq \mathcal{V}$ with $\mathcal{W}_{\mathfrak{G}} = \mathcal{V}$ (so that $\mathcal{W} \neq \emptyset$) is called the *generating set* of $\mathfrak{G}$.

If, moreover, every set $\mathcal{W} \setminus \{w\}$, $w \in \mathcal{W}$, is not more generating, $\mathcal{W}$ is said to be a *basis* of $\mathfrak{G}$.

A generating set of $\overline{\mathfrak{G}}$ can be also designed as a *dually generating* set of $\mathfrak{G}$. Similarly, a basis of $\overline{\mathfrak{G}}$ can be called a *dual basis* of $\mathfrak{G}$.

Another terminology: *vertex generating set* and *edge generating set* of $\mathfrak{G}$, *vertex basis* and *edge basis* of $\mathfrak{G}$, respectively. Observe that for a $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ with a generating set $\mathcal{W} \neq \mathcal{V}$ there exists a dually generating set $\overline{\mathcal{W}}$ such that the fundamental equality $\#\mathcal{V} + \#\overline{\mathcal{W}} = \#\mathcal{E} + \#\mathcal{W}$ holds.

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*/ Here and hereinafter the closed vertex subsets are meant as "closed in $\mathfrak{G}$".

**/ Here "minimal" means that for every closed vertex set $\mathcal{W}'$ containing $\mathcal{W}$ it is $\mathcal{W}' \supseteq \mathcal{W}_{\mathfrak{G}}$. It holds $\emptyset_{\mathfrak{G}} = \emptyset$. If $\mathcal{W}$ is a one-vertex set ($= \{w\}$) we shall write $w_{\mathfrak{G}}$ instead of $\{w\}_{\mathfrak{G}}$.

We will give a purely algebraic proof of it in a separate article together with some specializations of results of this paper to 3-configurations and to $(3,2)$-configurations.

3. Bases and skeletons.

A multigraph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$, every one-vertex set of which is generating, is said to be *primitive*. A multigraph $\mathfrak{G}$ with $\overline{\mathfrak{G}}$ primitive is called *dually primitive*.

We have already taken notice of the fact that multigraphs which are simultaneously graphs are primitive.

Proposition 4. *A* $\begin{cases} \textit{multigraph} \\ \textit{3-configuration} \end{cases}$ *is primitive if and only if it admits no proper* $\begin{cases} \textit{submultigraph with saturated vertices.} \\ \textit{sub-3-configuration.} \end{cases}$

A $\begin{cases} \textit{multigraph} \\ \textit{3-configuration} \end{cases}$ *is dually primitive if and only if it admits no proper* $\begin{cases} \textit{submultigraph with saturated edges.} \\ \textit{submultigraph which is a graph.} \end{cases}$

Proof: Obvious. □

Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a multigraph. A set $\mathcal{W} \subseteq \mathcal{V}$ is called a *skeleton* if

(i) $\mathcal{V}' \setminus \mathcal{W} \neq \emptyset$ for all submultigraphs $(\mathcal{V}', \mathcal{E}', \mathrm{I}')$ with saturated vertices and

(ii) for every set $\mathcal{W}'$, $\mathcal{W} \subset \mathcal{W}' \subseteq \mathcal{V}$, there is a primitive submultigraph $(\mathcal{V}', \mathcal{E}', \mathrm{I}')$ with saturated vertices such that $\mathcal{V}' \subseteq \mathcal{W}'$. A *dual skeleton* of $\mathfrak{G}$ is defined as a skeleton of $\overline{\mathfrak{G}}$.

Theorem. *Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a multigraph. A set $\mathcal{W} \subseteq \mathcal{V}$ is a basis of $\mathfrak{G}$ if and only if $\mathcal{V} \setminus \mathcal{W}$ is a skeleton of $\mathfrak{G}$.*

Proof: Necessity. Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a multigraph with a basis $\mathcal{W}$ and let $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathrm{I}')$ be its primitive submultigraph with saturated vertices.

(i) We assert that $\mathcal{V}' \setminus (\mathcal{V} \setminus \mathcal{W}) \neq \emptyset$: Assume $\mathcal{V}' \setminus (\mathcal{V} \setminus \mathcal{W}) = \emptyset$, i.e. $\mathcal{V}' \subseteq \mathcal{V} \setminus \mathcal{W}$. Since $\mathcal{V} \setminus \mathcal{V}'$ is a closed set and $\mathcal{W} \subseteq \mathcal{V} \setminus \mathcal{V}'$ it follows $\mathcal{W}_{\mathfrak{G}} \subseteq \mathcal{V} \setminus \mathcal{V}'$ which contradicts the fact that $\mathcal{W}$ is a basis.

(ii) Let $\mathcal{W}'$ be a set such that $\mathcal{V} \setminus \mathcal{W} \subset \mathcal{W}' \subseteq \mathcal{V}$. Thus there is a vertex $w \in \mathcal{W} \cap \mathcal{W}'$ and we can form the set $\mathcal{S} = (\mathcal{W} \setminus \{w\})_{\mathfrak{G}}$. This set $\mathcal{S}$ is closed and does not contain w because for $w \in \mathcal{S}$, $\mathcal{W}$ could not be a basis.

The submultigraph $\mathfrak{G}_{\mathcal{S}}$ has a generating set $\{w\}$ since $w_{\mathfrak{G}_{\mathcal{S}}} = \mathcal{V} \setminus \mathcal{S}$ (this last equality can be verified by contradiction: if $w_{\mathfrak{G}_{\mathcal{S}}} \neq \mathcal{V} \setminus \mathcal{S}$ then $\mathcal{W}_{\mathfrak{G}} \neq \mathcal{V}$). As $\mathcal{V} \setminus \mathcal{W} \subset \mathcal{W}'$ we have also $\mathcal{V} \setminus \mathcal{S} \subset \mathcal{W}'$.

We still need to prove the lemma: *For a multigraph $\mathfrak{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathrm{I}_1)$ with a one-vertex basis $\{v\}$ there exists its primitive submultigraph $\mathfrak{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \mathrm{I}_0)$ with saturated vertices and with $v \in \mathcal{V}_0$.*

In fact, if $\mathfrak{G}_1$ is primitive, we are ready. So let $\mathfrak{G}_1$ be non primitive so that there must exist a vertex w with $\mathcal{U} = w_{\mathfrak{G}_1} \neq \mathcal{V}_1$. Thus $(\mathfrak{G}_1)_{\mathcal{U}} \neq \mathcal{V}_1$ and $v \in \mathcal{V}_1 \setminus \mathcal{U}$. We see that v is a vertex of $(\mathfrak{G}_1)_{\mathcal{U}}$ but that the full vertex set of $(\mathfrak{G}_1)_{\mathcal{U}}$ is less than $\mathcal{V}_1$.

By finitely many repeatings of this step we get a primitive submultigraph $\mathfrak{G}_0$ in $\mathfrak{G}_1$ with saturated vertices and with v as its vertex.

Sufficiency. Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a multigraph and $\mathcal{W}$ a vertex subset such that $\mathcal{V} \setminus \mathcal{W}$ is a skeleton. We have first to verify that $\mathcal{W}$ is a generating set and secondly that it is, as a generating set, minimal.

(1) Suppose, on the contrary, that $\mathcal{T} = \mathcal{W}_{\mathfrak{G}} \neq \mathcal{V}$. Then $\mathcal{V} \setminus \mathcal{T}$ is the full vertex set of the submultigraph $\mathfrak{G}_{\mathcal{T}}$ with saturated vertices. Since $\mathcal{W} \subseteq \mathcal{T}$, it is also $\mathcal{V} \setminus \mathcal{T} \subseteq \mathcal{V} \setminus \mathcal{W}$. By the above lemma $\mathfrak{G}_{\mathcal{T}}$ contains a primitive submultigraph $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathrm{I}')$ so that $\mathcal{V}' \subseteq \mathcal{V} \setminus \mathcal{T} \subseteq \mathcal{V} \setminus \mathcal{W}$ in contradiction to the assumption that $\mathcal{V} \setminus \mathcal{W}$ is a skeleton.

(2) Let us investigate a set $\mathcal{W}' \subset \mathcal{W}$. Then $\mathcal{V} \setminus \mathcal{W} \subset \mathcal{V} \setminus \mathcal{W}'$. As $\mathcal{V} \setminus \mathcal{W}$ is a skeleton, there exists a primitive submultigraph $\mathfrak{G}'' = (\mathcal{V}'', \mathcal{E}'', \mathrm{I}'')$ such that $\mathcal{V}'' \subseteq \mathcal{V} \setminus \mathcal{W}'$ and consequently $\mathcal{W}' \subseteq \mathcal{V} \setminus \mathcal{V}''$. Because $\mathcal{V} \setminus \mathcal{V}''$ is a closed set in $\mathfrak{G}$ we have finally $\mathfrak{G}_{\mathcal{W}'} \subseteq \mathcal{V} \setminus \mathcal{V}'' \neq \mathcal{V}$. □

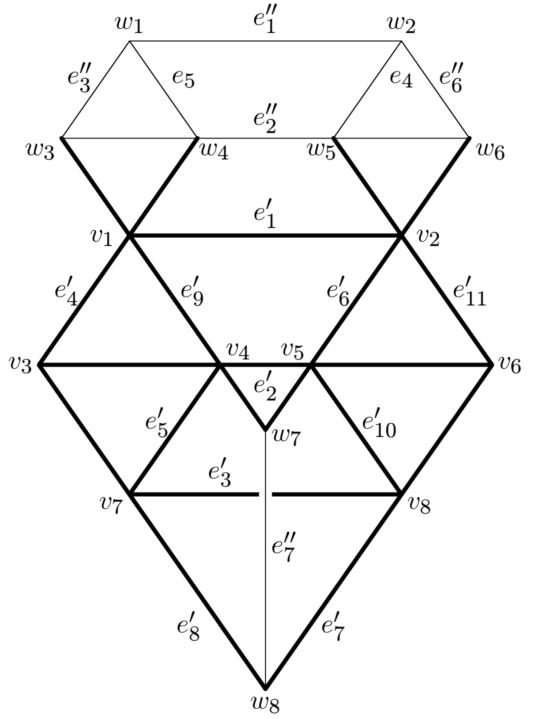
Corollaries. (i) *Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a multigraph. A set $\mathcal{F} \subseteq \mathcal{E}$ is a dual basis of $\mathfrak{G}$ if and only if $\mathcal{E} \setminus \mathcal{F}$ is a dual skeleton of $\mathfrak{G}$.* (ii) *If a multigraph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ is simultaneously a graph then every skeleton has cardinality $\#\mathcal{V} - 1$ (as $\mathfrak{G}$ is primitive and by the fundamental equality).*

Example (of a 3-configuration $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$, $\mathcal{V} = \{v_1, \dots, v_{10}\}$, $\mathcal{E} = \{e_1, \dots, e_{12}\}$, I as on Figure 2a, with one- and two-element bases). There are primitive sub-3-configurations $\mathfrak{G}_1, \mathfrak{G}_2$ (cf. Figures 2b–c) and bases $\{v_7, v_{10}\}$, $\{v_1\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$.

Further examples (of bases and dual bases):

Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$, $\mathcal{V} = \{v_1, \dots, v_8\}$, $\mathcal{E} = \{e_1, \dots, e_7\}$ be a multigraph, which is a configuration, described on Figure 3a. The set $\{v_2, v_3, v_4, v_7\}$ is closed and the corresponding submultigraph is $(\{v_1, v_5, v_6, v_8\}, \mathcal{E} \setminus \{e_4\}, \mathrm{I})$. Every vertex subset $\mathcal{W} \supset \{v_1, v_5, v_6, v_8\}$ is generating and every set $\{v_1, v_5, v_6, v_8, v_i\}$, $i \in \{1, 5, 6, 8\}$ is a basis. The set $\{e_2, e_4, e_5\}$ is closed in $\overline{\mathfrak{G}}$ and the corresponding submultigraph in $\overline{\mathfrak{G}}$ is $(\{e_1, e_3, e_6, e_7\}, \{v_1, v_2, v_4, v_5, v_5, v_8\}, \mathrm{I}^{-1})$ (cf. Figure 3b). Every vertex subset $\mathcal{W} \supset \{e_1, e_4, e_5\}$ in $\overline{\mathfrak{G}}$ is generating in $\overline{\mathfrak{G}}$ and every set $\{e_2, e_4, e_5, e_j\}$, $j \in \{1, 3, 6, 7\}$ is a basis of $\overline{\mathfrak{G}}$, i.e. a dual basis of $\mathfrak{G}$. Thus for every 5-element basis $\mathcal{B}$ mentioned above there exists a 4-element dual basis $\overline{\mathcal{B}}$ so that it holds $\#\mathcal{V} + \#\overline{\mathcal{B}} = \#\mathcal{E} + \#\mathcal{B}$, in agreement with the fundamental equality.

$\mathfrak{G} = (\{v_1, v_2\}, \{e_1, e_2, e_3\}, \{v_1, v_2\} \times \{e_1, e_2, e_3\})$ is a multigraph which is not a configuration (cf. Figure 4). Bases are the sets $\{v_1\}$, $\{v_2\}$, dual bases are the sets $\{e_1, e_2\}$, $\{e_1, e_3\}$, $\{e_2, e_3\}$. If $\mathcal{B}$ is a basis and $\overline{\mathcal{B}}$ a dual basis then $\#\{v_1, v_2\} + \#\overline{\mathcal{B}} = \#\{e_1, e_2, e_3\} + \#\mathcal{B}$ (as the fundamental equation says). Skeletons of $\mathfrak{G}$ are set complements of bases in the full vertex set of $\mathfrak{G}$.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w_2
e''_3
e_5
e_4
e''_6
e''_2
w_3
w_4
w_5
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e'_1
v_1
v_2
e'_4
e'_9
e'_6
e'_{11}
v_3
v_4
v_5
v_6
e'_2
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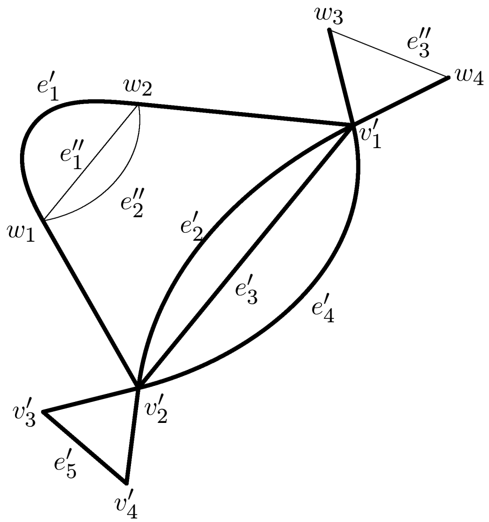
w_8

Fig. 1a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v'_4

Fig. 1b

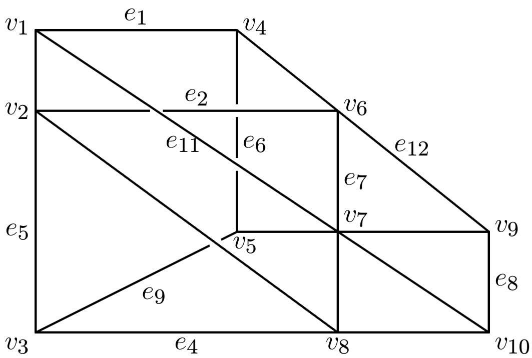

Fig. 2a

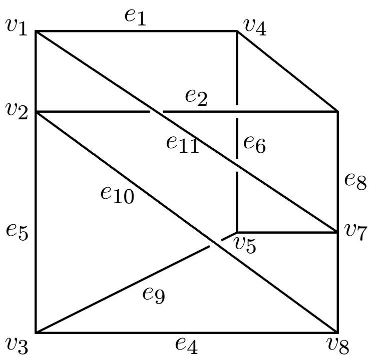

Fig. 2b

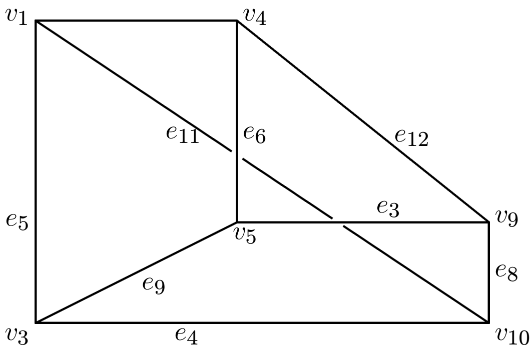

Fig. 2c

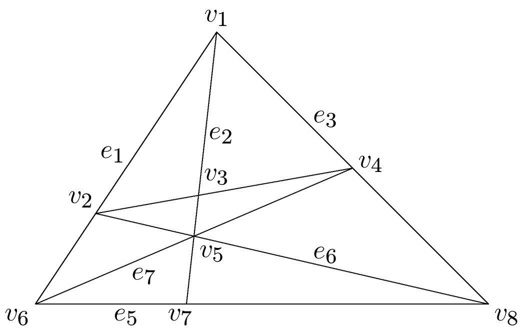

Fig. 3a

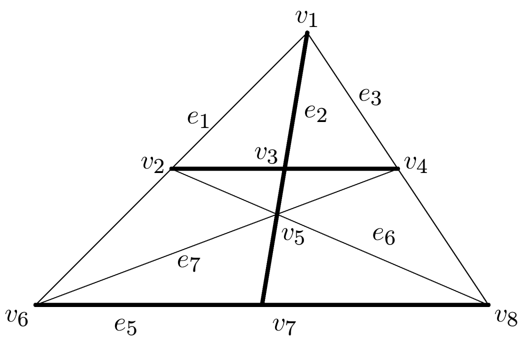

Fig. 3b

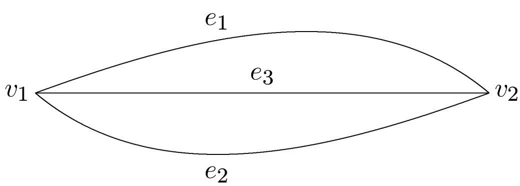

Fig. 4

References

[1] Dembowski P., *Finite Geometries*, Heidelberg, New York, 1968.

[2] Krapež A., Taylor M.A., *Bases of web configurations*, Publ de l'Inst. Math., nouvelle série **38** (52) (1985), 21–30.

[3] Belousov V.D., *Configurations in Algebraic Nets* (in Russian), Kishinev, 1979.

[4] Havel V., *Regulated bildups of* 3*-configurations*, Archivum Mathematicum, 1993, to appear.

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