The index ${}_2F_1$-transform of generalized functions

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Abstract. In this paper the index transformation

$$F(\tau) = \int_0^\infty f(t)_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t)t^\alpha \, dt$$

${}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t)$ being the Gauss hypergeometric function, is defined on certain space of generalized functions and its inversion formula established for distributions of compact support on $\mathbf{I} = (0, \infty)$.

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1. Introduction.

The index ${}_2F_1$-transform (see [6]) of a real valued function f is defined by:

$$F(\tau) = \int_0^\infty \mathbf{F}(\mu, \alpha, \tau, t) f(t) \, dt \tag{1.1}$$

where

$$\mathbf{F}(\mu, \alpha, \tau, t) = {}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t)t^\alpha \tag{1.2}$$

and ${}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t)$ is the Gauss hypergeometric function, α and μ are complex parameters and τ real.

In this paper, according to Zemanian [14], we introduce the testing function space $U_{a,\mu,\alpha}$ containing the kernel of the transform. As usual, $U'_{a,\mu,\alpha}$ denotes the dual space of $U_{a,\mu,\alpha}$. The generalized index ${}_2F_1$-transformation of $f \in U'_{a,\mu,\alpha}$ is defined by:

$${}_2\mathcal{F}_1(f) = F(\tau) = \langle f(t), \mathbf{F}(\mu, \alpha, \tau, t) \rangle, \qquad \tau \in \mathbb{R}_+ .$$

An inversion formula on the space $\mathcal{E}'(\mathbf{I})$ is proved.

The notation and terminology used here is that of Zemanian [14]. In the following $\mathbf{I}$ denotes the open interval $(0, \infty)$ and $\mathbb{R}_+$ the set of the positive real numbers. The spaces $\mathcal{D}(\mathbf{I})$, $\mathcal{D}'(\mathbf{I})$, $\mathcal{E}(\mathbf{I})$ and $\mathcal{E}'(\mathbf{I})$ have their usual meaning [11]. The parameter a is always in $[0, \frac{1}{2})$.

2. The testing function space and its dual.

Let $U_{a,\mu,\alpha}$ be the linear space of $\mathcal{C}^\infty$-functions on $\mathbf{I}$ according to:

$$U_{a,\mu,\alpha} = \{\phi \in \mathcal{C}^\infty : \gamma_{k,a,\mu,\alpha}(\phi) < \infty, \quad \text{for} \quad k \in \mathbb{N} \cup \{0\}\}$$

where

$$\gamma_{k,a,\mu,\alpha}(\phi) = \sup_{0<t<\infty} \left|(2t+1)^a t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}} A_t^k \phi(t)\right| \tag{2.1}$$

A_t being the differential operator:

$$t^{\alpha-\mu}(t+1)^{-\mu} D_t t^{\mu+1}(t+1)^{\mu+1} D_t t^{-\alpha} \tag{2.2}$$

$U_{a,\mu,\alpha}$ equipped with the topology arising from the family $\{\gamma_{k,a,\mu,\alpha}\}$ of seminorms of which $\gamma_{0,a,\mu,\alpha}$ is a norm, is a countably multinormed, locally convex, Hausdorff space. By using a technique of Zemanian [14] it follows immediately that $U_{a,\mu,\alpha}$ is sequentially complete, i.e. a Fréchet space.

From the relation:

$$A_t \mathbf{F}(\mu, \alpha, \tau, t) = -\left[\left(\mu + \frac{1}{2}\right)^2 + \tau^2\right] \mathbf{F}(\mu, \alpha, \tau, t) \tag{2.3}$$

and by the asymptotic behavior of the hypergeometric function it follows that $\mathbf{F}(\mu, \alpha, \tau, t) \in U_{a,\mu,\alpha}$.

The dual space $U'_{a,\mu,\alpha}$ of $U_{a,\mu,\alpha}$ is a space of generalized functions. Equipped with the usual weak topology it is a separated multinormed space which is sequentially complete.

The assertions of the following proposition can be proved by using standard techniques (cf. [14]):

Proposition 2.1.

(i) *$\mathcal{D}(\mathbf{I})$ is a subspace of $U_{a,\mu,\alpha}$ and the topology of $\mathcal{D}(\mathbf{I})$ is stronger than that induced on it by $U_{a,\mu,\alpha}$. Consequently, the restriction of any $f \in U'_{a,\mu,\alpha}$ to $\mathcal{D}(\mathbf{I})$ is in $\mathcal{D}'(\mathbf{I})$. $\mathcal{D}(\mathbf{I})$ is not dense in $U_{a,\mu,\alpha}$.*

(ii) *$U_{a,\mu,\alpha}$ is a dense subspace of $\mathcal{E}(\mathbf{I})$. Hence $\mathcal{E}'(\mathbf{I})$ is a subspace of $U'_{a,\mu,\alpha}$.*

(iii) *For $f \in U'_{a,\mu,\alpha}$ there exists $C > 0$ and $r \in \mathbb{N} \cup \{0\}$ such that*

$$|\langle f, \phi \rangle| \le C \max_{0 \le k \le r} \gamma_{k,a,\mu,\alpha}(\phi)$$

for all $\phi \in U_{a,\mu,\alpha}$.

(iv) *The differential operator A_t is a continuous linear mapping from $U_{a,\mu,\alpha}$ into $U_{a,\mu,\alpha}$. Its adjoint operator A'_t maps $U'_{a,\mu,\alpha}$ continuously into $U'_{a,\mu,\alpha}$.*

(v) *A locally integrable function f on $\mathbf{I}$ such that*

$$(2t+1)^{-a} t^{\alpha-\frac{\mu}{2}}(t+1)^{-\frac{\mu}{2}} f(t)$$

is absolutely integrable on $\mathbf{I}$, gives rise to a regular generalized function on $U'_{a,\mu,\alpha}$ with

$$\langle f, \phi \rangle = \int_0^\infty f(t)\phi(t)\, dt, \qquad \phi \in U_{a,\mu,\alpha}$$

(vi) *If $R_e(2\alpha - \frac{\mu}{2}) > -1$ and $a + R_e(\mu - \alpha) < -\frac{1}{2}$, $U_{a,\mu,\alpha}$ is contained in $U'_{a,\mu,\alpha}$.*

Lemma 2.1. *For each compact subset $\mathbb{K}$ contained in $\mathbf{I}$ and $k \in \mathbb{N} \cup \{0\}$ let the seminorm $\gamma_{k,K}$ be defined by*

$$\gamma_{k,K}(\phi) = \sup_{t \in K} \left| A_t^k \phi(t) \right|, \qquad \phi \in \mathcal{E}(\mathbf{I})$$

where A_t is defined by (2.2). Then, $\{\gamma_{k,K}\}$ gives rise to a topology in $\mathcal{E}(\mathbf{I})$ which coincides with its usual topology.

Proof: From an inductive argument it can be proved that:

$$A_t^k \phi(t) = \sum_{j=0}^{2k} t^{j-k} p_{j,k}(t) D_t^j \phi(t)$$

with

$$p_{2k,k}(t) = (t+1)^k \ \text{ and } \ p_{2k-1,k}(t) = k(t+1)^{k-1}[\mu - 2\alpha + k + 2t(\mu - \alpha + k)]$$

$p_{j,k}(t)$ being polynomials of degree k, $0 \le j \le 2k$. Therefore, if a sequence $\{\phi_n(t)\}_{n \in N} \subset \mathcal{E}(\mathbf{I})$ converges to zero in the usual topology on $\mathcal{E}(\mathbf{I})$, then ϕ_n converges to zero in the topology generated from $\gamma_{k,K}$.

Conversely, let $\{\phi_n(t)\}_{n \in N}$ be a sequence on $\mathcal{E}(\mathbf{I})$ converging to zero in the topology generated from $\gamma_{k,K}$. Obviously, $\phi_n(t)$ and $A_t \phi_n(t)$ tend to zero uniformly on every compact $\mathbb{K} \subset \mathbf{I}$.

Moreover,

$$\begin{aligned} A_t \phi_n(t) &= t(t+1) D_t^2 \phi_n(t) + [\mu - 2\alpha + 1 + 2t(\mu - \alpha + 1)] D_t \phi_n(t) + \\ &\quad + \left[\alpha(\alpha - 2\mu - 1) + \frac{\alpha(\alpha - \mu)}{t} \right] \phi_n(t). \end{aligned} \tag{2.4}$$

Thus,

$$\begin{aligned} A_t \phi_n(t) - &\left[\alpha(\alpha - 2\mu - 1) + \frac{\alpha(\alpha - \mu)}{t} \right] \phi_n(t) = \\ &= t(t+1) D_t^2 \phi_n(t) + [\mu - 2\alpha + 1 + 2t(\mu - \alpha + 1)] D_t \phi_n(t) \end{aligned} \tag{2.5}$$

tends uniformly to zero on $\mathbb{K}$. Now, taking into account that (2.5) can be written as:

$$t^{2\alpha - \mu}(t+1)^{-\mu} D_t \left[t^{\mu - 2\alpha + 1}(t+1)^{\mu+1} D_t \phi_n(t) \right] \tag{2.6}$$

by an integration it follows that $D_t \phi_n(t)$ and also $D_t^2 \phi_n(t)$ tends to zero uniformly in $\mathbb{K}$. By a similar argument it is proved for every non negative integer k, that $D_t^k \phi_n(t)$ converges uniformly to zero in $\mathbb{K}$.

Finally, since $\mathcal{E}(\mathbf{I})$ is a metrizable space, the conclusion follows. □

3. The generalized transform.

For $f \in U'_{a,\mu,\alpha}$ the generalized index ${}_2F_1$-transform is defined by

$$
{}_2\mathcal{F}_1(f) = F(\tau) = \langle f(t), \mathbf{F}(\mu,\alpha,\tau,t)\rangle, \qquad \tau \in \mathbb{R}^+ . \tag{3.1}
$$

For regular generalized functions this formula coincides with (1.1).

Proposition 3.1. *For all $f \in U'_{a,\mu,\alpha}$, and $k \in \mathbb{N}\cup\{0\}$, one has:*

$$
{}_2\mathcal{F}_1(A_t'^{k} f) = (-1)^k \left[\left(\mu+\frac{1}{2}\right)^2 + \tau^2\right]^k {}_2\mathcal{F}_1(f)
$$

A'_t being the adjoint operator of A_t.

Proof: By making use of the relation (2.3) the conclusion follows. □

Now, the analyticity of the index ${}_2F_1$-transform will be established. For it, the next two lemmas are required.

Lemma 3.1. *For each non negative integer m and $R_e\mu > -\frac{1}{2}$, one has:*

$$
\begin{aligned}
(3.2)\quad & |D_\tau^m \mathbf{F}(\mu,\alpha,\tau,t)| \le \\
&\le M t^{R_e\alpha}\left[\log\left(2t+1+2\sqrt{t(t+1)}\right)\right]^m [t(t+1)]^{-R_e\frac{\mu}{2}} P_{-\frac{1}{2}}^{-R_e\mu}(2t+1)
\end{aligned}
$$

$P_{-\frac{1}{2}}^{-R_e\mu}$ being the well-known associated Legendre function.

Proof: The integral representation ([1, p. 155]),

$$
\begin{aligned}
(3.3)\quad & \mathbf{F}(\mu,\alpha,\tau,t) = \\
&= \frac{\Gamma(\mu+1)t^\alpha}{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})}\int_0^\pi \left(2t+1+2\sqrt{t(t+1)}\cos\,\xi\right)^{-\mu-\frac{1}{2}-i\tau}(\sin\,\xi)^{2\mu}\,d\xi
\end{aligned}
$$

is valid for $R_e\mu > -\frac{1}{2}$. Now, differentiating with respect to the parameter τ, (3.2) holds. □

Lemma 3.2. *Let μ be a complex parameter with $R_e\mu > -\frac{1}{2}$ and k, m non negative integers. Then there exists $C > 0$ such that:*

$$
\gamma_{k,a,\mu,\alpha}(D_\tau^m \mathbf{F}(\mu,\alpha,\tau,t)) \le C\left|\left(\mu+\frac{1}{2}\right)^2 + \tau^2\right|^k . \tag{3.4}
$$

Proof: For $k = 0$, making use of the asymptotic behavior:

$$
P_{-\frac{1}{2}}^{-R_e\mu}(2t+1) \sim \frac{1}{\Gamma(\mu+\frac{1}{2})}\left(\frac{2}{\pi(2t+1)}\right)^{\frac{1}{2}}\log(2t+1), \qquad t\to\infty
$$

(cf. [9, p. 173 (12.20)]), it follows from Lemma 3.1:

$$\gamma_{0,a,\mu,\alpha}(D_\tau^m \mathbf{F}(\mu,\alpha,\tau,t)) \le$$
$$\le M_1 \sup_{0<t<\infty} \left|(2t+1)^a \left[\log\left(2t+1+2\sqrt{t(t+1)}\right)\right]^m P_{-\frac{1}{2}}^{-R_e\mu}(2t+1)\right| \le M_2$$

with $M_1, M_2 > 0$.

For $k > 0$, by using the commutativity of A_t^k and D_τ^m, (2.3) and Lemma 3.1, one has:

$$\gamma_{k,a,\mu,\alpha}(D_\tau^m \mathbf{F}(\mu,\alpha,\tau,t)) \le$$
$$\le \sum_{j=0}^{m} \binom{m}{j} H_j \left| D_\tau^j \left[\left(\mu+\frac{1}{2}\right)^2 + \tau^2\right]^k \right| \le C \left| \left(\mu+\frac{1}{2}\right)^2 + \tau^2 \right|^k$$

H_j, $j = 1, 2, \ldots m$ and C being suitable constants. □

Theorem 3.1. *For $f \in U'_{a,\mu,\alpha}$, $R_e\mu > -\frac{1}{2}$, the generalized transform $F(\tau)$ defined by (3.1) is an analytic function and*

$$D_\tau^m F(\tau) = \langle f(t), D_\tau^m \mathbf{F}(\mu,\alpha,\tau,t)\rangle . \tag{3.5}$$

Proof: By Lemmas 3.1 and 3.2 it follows that (3.5) has a sense. Moreover, set

$$\frac{F(\tau+\Delta\tau) - F(\tau)}{\Delta\tau} - \langle f(t), D_\tau \mathbf{F}(\mu,\alpha,\tau,t)\rangle = \langle f(t), \Upsilon_{\Delta\tau}(t)\rangle$$

where

$$\begin{aligned}\Upsilon_{\Delta\tau}(t) &= \frac{1}{\Delta\tau}[\mathbf{F}(\mu,\alpha,\tau+\Delta\tau,t) - \mathbf{F}(\mu,\alpha,\tau,t)] - D_\tau \mathbf{F}(\mu,\alpha,\tau,t) = \\ &= \frac{1}{\Delta\tau}\int_\tau^{\tau+\Delta\tau} dx \int_\tau^x D_y^2 \mathbf{F}(\mu,\alpha,y,t)\, dy.\end{aligned} \tag{3.6}$$

Thus, from (2.3), for any k non negative integer,

$$\left|(2t+1)^a t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}} A_t^k \Upsilon_{\Delta\tau}(t)\right| \le$$
$$\le \frac{|\Delta\tau|}{2} \left|(2t+1)^a t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}}\right| \sup_{y\in\Lambda} \left| D_y^2 \left[\left(\mu+\frac{1}{2}\right)^2 + y^2\right]^k \mathbf{F}(\mu,\alpha,y,t)\right|$$

Λ being the interval $\tau - |\Delta\tau| < y < \tau + |\Delta\tau|$.

Now, by the boundedness on $0 < t < \infty$ of

$$\left|(2t+1)^a t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}}\right| \sup_{y\in\Lambda} \left| D_y^2 \left[\left(\mu+\frac{1}{2}\right)^2 + y^2\right]^k \mathbf{F}(\mu,\alpha,y,t)\right|$$

for $|\Delta\tau| < 1$, it follows that $\Upsilon_{\Delta\tau}(t) \to 0$ in $U_{a,\mu,\alpha}$ as $\Delta\tau \to 0$. With this the proof is finished. □

Theorem 3.2. *Let $F(\tau)$ be the generalized ${}_2F_1$-transform of f given by (3.1). Then:*

(3.7)
$$\begin{cases} \text{(i)} \;\; \textit{For } \tau \to 0, \;\; \textit{one has } \; F^{(m)}(\tau) = O(1), \;\; \textit{for all } \; m \in \mathbb{N} \cup \{0\}. \\ \text{(ii)} \;\; \textit{There exists } \; a \; p \in \mathbb{N} \cup \{0\} \; \textit{ such that } \; F(\tau) = O\left(\tau^{2p - R_e\mu - \frac{1}{2}}\right), \; \tau \to \infty. \end{cases}$$

Proof: It follows immediately from (2.3), Proposition 2.1 (iii), and taking into account that:

$$|\mathbf{F}(\mu, \alpha, \tau, t)| \leq M t^{-\frac{1}{2} - R_e(\alpha+\mu)} (t+1)^{-\frac{1}{2} - R_e \frac{\mu}{2}} \tau^{-\frac{1}{2} - R_e\mu}, \qquad \tau \to \infty$$

(cf. [10, (24), p. 231]).

4. Generalized inversion formula.

In this paragraph we state the main result of this work. For it we recall the definition of the $\mathcal{M}_{c,\gamma}^{-1}(L)$ spaces introduced in [13].

Let c and γ be real numbers such that $2 \;\text{sgn}\; c + \text{sgn}\; \gamma \geq 0$. The space of functions $f(x)$ which can be represented in the form of:

$$f(x) = \frac{1}{2\pi i} \int_\sigma \rho(s) x^{-s}\, ds, \qquad x \in (0, \infty), \qquad \sigma = \{s \in \mathbb{C} : R_e s = \frac{1}{2}\}$$

where

$$\rho(s) = s^{-\gamma} e^{-c\pi |Ims|} F(s) \quad \text{ with } \quad \int_\sigma |F(s)|\, ds < \infty,$$

is denoted by $\mathcal{M}_{c,\gamma}^{-1}(L)$. Before giving the inversion theorem we need to prove the following lemmas:

Lemma 4.1. *If* $2 \;\text{sgn}\; (c+1) + \text{sgn}\; (\gamma - R_e\mu) > 0$, *there exists the integral*

$$F(\tau) = \frac{1}{2\pi i} \frac{\Gamma(\mu+1)}{\Gamma(\mu + \frac{1}{2} + i\tau)\Gamma(\mu + \frac{1}{2} - i\tau)} \cdot$$
$$\int_\sigma \frac{\Gamma(\mu + \frac{1}{2} - \alpha + i\tau - s)\Gamma(\mu + \frac{1}{2} - \alpha - i\tau - s)\Gamma(\alpha + s)}{\Gamma(1 + \mu - \alpha - s)} f^*(1-s) ds$$

f^* being the Mellin transform of $f \in \mathcal{M}_{c,\gamma}^{-1}(L)$, $\alpha, \mu \in \mathbb{C}$, $\tau \in \mathbb{R}_+$, $\sigma = \{s \in \mathbb{C} : R_e s = \frac{1}{2}\}$.

Moreover, if $R_e\alpha > -\frac{1}{2}$ and $R_e(\mu - \alpha) > 0$, then:

$$F(\tau) = \int_0^\infty f(t)\, \mathbf{F}(\mu, \alpha, \tau, t)\, dt. \tag{4.2}$$

Proof: From the asymptotic behavior of the Gamma function (see [1, p. 47]) and since $f \in \mathcal{M}_{c,\gamma}^{-1}(L)$ it follows the existence of the first integral.

On the other hand, if $R_e\alpha > -\frac{1}{2}$ and $R_e(\mu-\alpha) > 0$,

$$\int_0^\infty \mathbf{F}(\mu,\alpha,\tau,t)t^{s-1}\,dt$$

converges absolutely $\forall s \in \sigma$.

Moreover, $f^* \in L(\sigma)$ and consequently:

$$\int_0^\infty f(t)\mathbf{F}(\mu,\alpha,\tau,t)\,dt = \frac{1}{2\pi i}\int_0^\infty \mathbf{F}(\mu,\alpha,\tau,t)\,dt \int_\sigma f^*(1-s)t^{s-1}\,ds.$$

Now the absolute convergence of this integral allows us to interchange the order of integration to obtain:

$$\frac{1}{2\pi i}\int_\sigma f^*(1-s)\,ds \int_0^\infty \mathbf{F}(\mu,\alpha,\tau,t)t^{s-1}\,dt$$

and the conclusion follows. □

Lemma 4.2. *Let α, μ and s be complex parameters with $R_e\alpha > 0$, $R_e\mu > 0$, $R_e s = \frac{1}{2}$, $\frac{1}{8} < R_e(\mu-\alpha) < \frac{1}{4}$, $R_e(\mu-2\alpha) < -1$. Then one has the following integral representation:*

(4.3)

$$\frac{1}{2\Gamma(\mu+1)}\, sh\,\pi\tau\Gamma(\mu+\frac{1}{2}-\alpha+i\tau-s)\Gamma(\mu+\frac{1}{2}-\alpha-i\tau-s)t^{\alpha-\mu}\mathbf{G}(\mu,\alpha,\tau,t) =$$
$$= \int_0^\infty z^{\mu-\alpha-s}C_\mu(tz)\,dz \int_{-\infty}^\infty e^{2\theta(\mu-\alpha+\frac{1}{2}-s)}\,d\theta \int_{|\theta|}^\infty C_0(ze^\theta\Psi)\,\sin\,2\tau u\,du$$

where $\Psi = 2\ ch\ u - 2\ ch\ \theta$ and

$$\mathbf{G}(\mu,\alpha,\tau,t) = x^{\mu-\alpha}{}_2F_1\big(\frac{1}{2}+i\tau,\frac{1}{2}-i\tau;\mu+1;-t\big).$$

Remark 4.1. C_μ denotes the Bessel-Clifford function of the first kind and order μ. This function is related with the Bessel function J_μ through $C_\mu(z) = z^{-\frac{\mu}{2}}J_\mu(2\sqrt{z})$ (see [4]).

Proof: Let us consider the integral representation (cf. [7]):

$$\frac{2}{\pi}K_{2\tau i}(2\sqrt{z})K_{2\tau i}(2\sqrt{y})\ sh\ 2\pi\tau = \tag{4.4}$$
$$= \int_{|\frac{1}{2}\log\frac{y}{z}|}^\infty C_0(2\sqrt{zy}chu - z - y)\ \sin\ \ 2\tau u\ du$$

and also that (cf. [12, p. 248] and [2, 10.2 (2)] resp.)

$$\int_0^\infty K_{2\tau i}(2\sqrt{z})z^{-\frac{1}{2}}C_\mu(tz)\ dz = \frac{\Gamma(\frac{1}{2}+i\tau)\Gamma(\frac{1}{2}-i\tau)}{2\Gamma(\mu+1)}t^{\alpha-\mu}\mathbf{G}(\mu,\alpha,\tau,t)$$

$$\int_0^\infty y^{\mu-\alpha-\frac{1}{2}-s}K_{2\tau i}(2\sqrt{y})dy = \frac{1}{2}\Gamma(\mu+\frac{1}{2}-\alpha+i\tau-s)\Gamma(\mu+\frac{1}{2}-\alpha-i\tau-s).$$

Now, by means of the change $\frac{1}{2}\log\frac{y}{z} = \theta$, one has (4.3). The existence of the integral (4.3) follows from the asymptotic behavior of the Bessel-Clifford functions $C_\nu(x)$ (cf. [4]) and the hypotheses. □

Lemma 4.3. *Let $F(\tau) = {}_2\mathcal{F}_1(f)$, $\phi \in \mathcal{D}(\mathbf{I})$ be and set*

$$\varphi(\tau) = S(\mu,\tau)\int_0^\infty \phi(t)\mathbf{G}(\mu,\alpha,\tau,t)\,dt \tag{4.5}$$

then

$$\int_0^N \varphi(\tau)\,\langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle\,d\tau = \left\langle f(x), \int_0^N \varphi(\tau)\,\mathbf{F}(\mu,\alpha,\tau,x)\,d\tau\right\rangle \tag{4.6}$$

where

$$S(\mu,\tau) = \frac{2}{\pi\Gamma(\mu+1)^2}\tau\ sh\ \pi\tau\ \Gamma(\mu+\frac{1}{2}+i\tau)\Gamma(\mu+\frac{1}{2}-i\tau).$$

Proof: By the asymptotic behavior of the hypergeometric function, one has that

$$\Theta_N(x) = \int_0^N \varphi(\tau)\ \mathbf{F}(\mu,\alpha,\tau,x)\,d\tau \tag{4.7}$$

belongs to $U_{a,\mu,\alpha}$.

Moreover, if we put

$$Q(x,n) = \frac{N}{n}\sum_{p=1}^{n}\varphi\left(\frac{pN}{n}\right)\mathbf{F}\left(\mu,\alpha,\frac{pN}{n},x\right)$$

it follows

$$\langle f(x), Q(x,n)\rangle = \frac{N}{n}\sum_{p=1}^{n}\varphi\left(\frac{pN}{n}\right)\left\langle f(x), \mathbf{F}\left(\mu,\alpha,\frac{pN}{n},x\right)\right\rangle \tag{4.8}$$

and it can be easily proved that (4.8) tends to

$$\int_0^N \varphi(\tau)\langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle\,d\tau \qquad \text{for} \qquad n\to\infty.$$

Now, by (2.3) and the asymptotic behavior of $\mathbf{F}(\mu,\alpha,\tau,x)$ it follows the existence of an $X>0$ and $n_0\in\mathbb{N}$ such that

$$\left|(2x+1)^a x^{\frac{\mu}{2}-\alpha}(x+1)^{\frac{\mu}{2}}A_x^k[\Theta_N(x)-Q(x,n)]\right| < \varepsilon$$

for $x>X$ and $n>n_0$.

Furthermore, by the uniform continuity of $\mathbf{F}(\mu,\alpha,\tau,x)$ ($R_e\alpha>0$) on the domain $E=\{(x,\tau):\ \ 0\le x\le X,\ \ 0\le\tau\le N\}$, there exists $n_1\in\mathbb{N}$ such that

$$\left|(2x+1)^a x^{\frac{\mu}{2}-\alpha}(x+1)^{\frac{\mu}{2}}A_x^k[\Theta_N(x)-Q(x,n)]\right| < \varepsilon$$

for $0\le x\le X$ and $n>n_1$. This fact implies that $Q(x,n)\to\Theta_N(x)$ in $U_{a,\mu,\alpha}$ as $n\to\infty$ and therefore (4.6) holds. □

Lemma 4.4. *Assume that $\phi \in \mathcal{D}(\mathbf{I})$ and let $\Theta_N(x)$ be given as in Lemma 4.5. If α and μ are complex parameters such that $R_e\alpha > 0$, $R_e\mu > 0$, $\frac{1}{8} < R_e(\mu - \alpha) < \frac{1}{4}$ and $R_e(\frac{\mu}{2} - \alpha) < -\frac{1}{2}$, then $\Theta_N(x)$ converges in $\mathcal{E}(\mathbf{I})$ to $\phi(x)$ as $N \to \infty$.*

PROOF: Let ϕ be in $\mathcal{D}(\mathbf{I})$. If the support of ϕ is contained in the closed interval $[c,d]$, $0 < c < d < \infty$, one has:

$$\Theta_N(x) = \int_0^N S(\mu,\tau)\mathbf{F}(\mu,\alpha,\tau,x)\,d\tau \int_c^d \phi(t)\mathbf{G}(\mu,\alpha,\tau,t)\,dt.$$

By virtue of the smoothness of the functions and the finiteness of the limits of integration we may repeatedly differentiate under the integral sign. By using the identity (2.3) we get:

$$A_x^k\Theta_N(x) =$$
$$= \int_0^N S(\mu,\tau)(-1)^k\left[\left(\mu+\frac{1}{2}\right)^2+\tau^2\right]^k \mathbf{F}(\mu,\alpha,\tau,x)\,d\tau\,\cdot$$
$$\int_c^d \phi(t)\mathbf{G}(\mu,\alpha,\tau,t)\,dt =$$
$$= \int_0^N S(\mu,\tau)\;\mathbf{F}(\mu,\alpha,\tau,x)\,d\tau\;\int_c^d \phi(t)(-1)^k\left[\left(\mu+\frac{1}{2}\right)^2+\tau^2\right]^k \mathbf{G}(\mu,\alpha,\tau,t)\,dt.$$

Integrating by parts and using the identity

$$A_t'\mathbf{G}(\mu,\alpha,\tau,t) = -\left[\left(\mu+\frac{1}{2}\right)^2+\tau^2\right]\mathbf{G}(\mu,\alpha,\tau,t) \tag{4.9}$$

A_t' being the adjoint operator of A_t. It follows by applying some properties of the hypergeometric function that (see [1, p. 105]):

$$A_x^k\Theta_N(x) = \tag{4.10}$$
$$= \int_0^N S(\mu,\tau)x^\alpha(x+1)^{-\mu}\mathbf{G}(\mu,\alpha,\tau,x)\,d\tau \int_c^d t^{\mu-2\alpha}(t+1)^\mu A_t^k\phi(t)\mathbf{F}(\mu,\alpha,\tau,t)\,dt.$$

By virtue of our assumptions, $\mathcal{D}(\mathbf{I}) \subset \mathcal{M}_{0,n}^{-1}(L)$ $(\forall n \in \mathbb{N})$ and by Lemma 4.1, (4.10) can be rewritten as follows:

$$2x^\alpha(x+1)^{-\mu}\int_0^N S(\mu,\tau)\mathbf{G}(\mu,\alpha,\tau,x)\,d\tau\frac{1}{2\pi i}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\frac{1}{2}+i\tau)\Gamma(\mu+\frac{1}{2}-i\tau)}\,\cdot$$
$$\int_\sigma \frac{\Gamma(\mu+\frac{1}{2}-\alpha+i\tau-s)\Gamma(\mu+\frac{1}{2}-\alpha-i\tau-s)\Gamma(\alpha+s)}{\Gamma(1+\mu-\alpha-s)}\,\cdot \tag{4.11}$$
$$\left[t^{\mu-2\alpha}(t+1)^\mu A_t^k\phi(t)\right]^*(1-s)\,ds$$

with $\sigma = \{s \in \mathbb{C} : R_e s = \frac{1}{2}\}$, and where

$$\left[t^{\mu-2\alpha}(t+1)^{\mu} A_t^k \phi(t)\right]^*(1-s)$$

is the Mellin transform of the function within the square brackets calculated at the point $1-s$.

Taking into account that

$$\int_0^N \tau \ \sin\ 2\tau u \, d\tau = -\frac{\partial}{\partial u}\frac{\sin\ 2Nu}{u}$$

by reversing the order of integration and using Lemma 4.2 we obtain:

$$\frac{x^{\alpha}(x+1)^{-\mu}}{2\pi i}\int_{\sigma}\frac{\Gamma(\alpha+s)}{\Gamma(1+\mu-\alpha-s)}\left[t^{\mu-2\alpha}(t+1)^{\mu}A_t^k\phi(t)\right]^*(1-s)\,ds$$
$$\frac{1}{\pi}\int_0^{\infty} z^{\mu-\alpha-s}C_{\mu}(xz)\,dz\int_{-\infty}^{\infty}e^{2\theta(\mu-\alpha+\frac{1}{2}-s)}\,d\theta\,\cdot$$
$$\int_{|\theta|}^{\infty}C_0(ze^{\theta}\Psi)\left(-\frac{\partial}{\partial u}\frac{\sin\ 2Nu}{u}\right)\,du.$$

The absolute convergence allows the interchanging of the order of integration and it follows

$$\begin{aligned} A_x^k\Theta_N(x) = \frac{x^{\alpha}(x+1)^{-\mu}}{\pi}&\int_0^{\infty}\left(-\frac{\partial}{\partial u}\frac{\sin\ 2Nu}{u}\right)\,du\int_{-u}^{u}e^{2\theta(\mu-\alpha+\frac{1}{2}-s)}\,d\theta\,\cdot \\ &\int_0^{\infty}z^{\mu-\alpha}C_{\mu}(xz)C_0\left(ze^{\theta}\Psi\right)\,dz\cdot \\ \frac{1}{2\pi i}&\int_{\sigma}\left(t^{\mu-2\alpha}(t+1)^{\mu}A_t^k\phi(t)\right)^*(1-s)\left(ze^{2\theta}\right)^{-s}\,ds. \end{aligned} \tag{4.12}$$

Observe that

$$\frac{1}{2\pi i}\int_{\sigma}\left(t^{\mu-2\alpha}(t+1)^{\mu}A_t^k\phi(t)\right)^*(1-s)\left(ze^{2\theta}\right)^{-s}\,ds \tag{4.13}$$

represents the G_{02}^{10}-transform of

$$t^{\mu-2\alpha}(t+1)^{\mu}A_t^k\phi(t)$$

evaluated at the point $ze^{2\theta}$ (see [13]). This transform exists since it can be proved that $\mathcal{D}(\mathbf{I}) \subset \mathcal{M}_{0,n}^{-1}(L)$, $\forall n \in \mathbb{N}$. We denote (4.13) by $G(\phi_k)(ze^{2\theta})$.

Now, by making the change of variable $ze^{2\theta} = y$, (4.12) can be written as

$$\begin{aligned} &\frac{x^{\alpha}(x+1)^{-\mu}}{\pi}\int_0^{\infty}\left(-\frac{\partial}{\partial u}\frac{\sin\ 2Nu}{u}\right)\,du\int_{-u}^{u}e^{-\theta}\,d\theta\,\cdot \\ &\int_0^{\infty}G(\phi_k)(y)y^{\mu-\alpha}C_{\mu}\left(xye^{-2\theta}\right)C_0\left(ye^{-2\theta}\Psi\right)\,dy. \end{aligned} \tag{4.14}$$

A partial integration leads to:

(4.15)
$$\begin{aligned}
&A_x^k \Theta_N(x) = \\
&= -\frac{\sin\, 2Nu}{u}\, \frac{x^\alpha (x+1)^{-\mu}}{\pi} \int_{-u}^{u} e^{-\theta}\, d\theta \cdot \\
&\int_0^\infty G(\phi_k)(y) y^{\mu-\alpha} C_\mu \left(xye^{-2\theta} \right) C_0 \left(ye^{-2\theta}\Psi \right) \, dy \bigg|_0^\infty \\
&+ \frac{x^\alpha (x+1)^{-\mu}}{\pi} \int_0^\infty \Phi(x,u) \frac{\sin\, 2Nu}{u}\, du
\end{aligned}$$

where

(4.16)
$$\begin{aligned}
\Phi(x,u) &= e^{-u} \int_0^\infty G(\phi_k)(y) y^{\mu-\alpha} C_\mu \left(xye^{2u} \right) \, dy + \\
&+ e^{u} \int_0^\infty G(\phi_k)(y) y^{\mu-\alpha} C_\mu \left(xye^{-2u} \right) \, dy + \\
&+ \int_{-u}^{u} e^{-\theta}\, d\theta \frac{\partial}{\partial u} \int_0^\infty G(\phi_k)(y) y^{\mu-\alpha} C_\mu \left(xye^{-2\theta} \right) C_0 \left(ye^{\theta}\Psi \right) \, dy.
\end{aligned}$$

It can be shown that the first term of (4.15) tends uniformly to zero for $u \to 0$ and $u \to \infty$ if $\frac{1}{8} < R_e(\mu - \alpha) < \frac{1}{4}$ when x belongs to any compact $\mathbb{K} \subset \mathbf{I}$.

Next, by the absolute convergence, one can differentiate under the integral sign in the last term of (4.16). By using the identity

$$\frac{\partial}{\partial u} C_0 \left(ye^{-\theta}\Psi \right) = 2ye^{-2\theta}(e^{\theta-u} - 1) C_1 \left(ye^{-\theta}\Psi \right) - \frac{\partial}{\partial \theta} C_0 \left(ye^{-\theta}\Psi \right)$$

we obtain

(4.17)
$$\begin{aligned}
&\Phi(x,u) = \\
&= 2e^{u} \int_0^\infty G(\phi_k)(y) y^{\mu-\alpha} C_\mu \left(xye^{-2u} \right) \, dy + F_1(x,u) - F_2(x,u) - F_3(x,u)
\end{aligned}$$

where

$$\begin{aligned}
F_1(x,u) &= 2e^{-u} \int_{-u}^{u} e^{-2\theta}\, d\theta \int_0^\infty G(\phi_k)(y) y^{\mu-\alpha+1} C_\mu (xye^{-2\theta}) C_1 (ye^{-\theta}\Psi) \, dy, \\
F_2(x,u) &= 2 \int_{-u}^{u} e^{-3\theta}\, d\theta \int_0^\infty G(\phi_k)(y) y^{\mu-\alpha+1} C_\mu (xye^{-2\theta}) C_1 (ye^{-\theta}\Psi) \, dy, \\
F_3(x,u) &= \int_{-u}^{u} e^{-\theta}\, d\theta \cdot \\
&\int_0^\infty G(\phi_k)(y) y^{\mu-\alpha} \left[e^{-\theta} C_\mu (xye^{-2\theta}) + 2xye^{-3\theta} C_{\mu+1}(xye^{-2\theta}) \right] \, dy.
\end{aligned}$$

Now, observe that (see [13])

$$G(\phi_k)(y) = \int_0^\infty t^{\mu-2\alpha}(t+1)^\mu A_t^k \phi(t)\, t^\mu C_\mu(ty)\, dt.$$

According to the inversion formula of the Hankel-Clifford transform (see [5] and [8]) we get:

$$\begin{aligned}(4.18)\qquad \Phi(x,u) &= \\ &= 2x^{\alpha-\mu}(xe^{2u}+1)^\mu e^{-2u(\mu-\alpha-\frac{1}{2})} A_x^k \phi(xe^{2u}) + F_1(x,u) - F_2(x,u) - F_3(x,u).\end{aligned}$$

Thus

$$\begin{aligned}A_x^k \Theta_N(x) = \frac{1}{\pi}\int_0^\infty 2e^{-2u(\mu-\alpha-\frac{1}{2})}\left(\frac{xe^{2u}+1}{x+1}\right)^\mu A_x^k\phi(xe^{2u})\frac{\sin\ 2Nu}{u}\, du + \\ + x^\alpha(x+1)^{-\mu}\int_0^\infty (F_1(x,u) - F_2(x,u) - F_3(x,u))\,\frac{\sin\ 2Nu}{u}\, du.\end{aligned}$$

Let us consider now

$$\begin{aligned}&A_x^k(\Theta_N(x) - \phi(x)) = \\ (4.19)\qquad &\frac{2}{\pi}\int_0^\infty \left[e^{-2u(\mu-\alpha-\frac{1}{2})}\left(\frac{xe^{2u}+1}{x+1}\right)^\mu A_x^k\phi(xe^{2u}) - A_x^k\phi(x)\right]\frac{\sin\ 2Nu}{u}\, du + \\ &+\frac{1}{\pi}\int_0^\infty (F_1(x,u) - F_2(x,u) - F_3(x,u))\,\frac{\sin\ 2Nu}{u}\, du.\end{aligned}$$

For x in a compact $\mathbb{K} \subset \mathbf{I}$,

$$\begin{aligned}(4.20)\quad A_x^k(\Theta_N(x) - \phi(x)) = &\left(\int_0^\delta + \int_\delta^\infty\right) v(x,u)\ \sin\ 2Nu\, du + \\ &+\frac{1}{\pi}\int_0^\infty (F_1(x,u) - F_2(x,u) - F_3(x,u))\,\frac{\sin\ 2Nu}{u}\, du\end{aligned}$$

where

$$v(x,u) = \frac{2}{\pi u}\left[e^{-2u(\mu-\alpha-\frac{1}{2})}\left(\frac{xe^{2u}+1}{x+1}\right)^\mu A_x^k\phi(xe^{2u}) - A_x^k\phi(x)\right]$$

with $\delta > 0$.

From the boundedness of $v(x,u)$ on $E = \{(x,u) : x \in \mathbb{K},\ 0 \le u \le 1\}$, for a given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that for each δ in the interval $(0, \delta_1]$, $x \in \mathbb{K}$ and $N > 0$, we have

$$\left|\int_0^\delta v(x,u)\sin\ 2Nu\, du\right| < \frac{\varepsilon}{2}\,.$$

In order to study $\int_\delta^\infty$, set

$$\lambda(x,u)=\frac{1}{u}e^{-2u(\mu-\alpha-\frac{1}{2})}\left(\frac{xe^{2u}+1}{x+1}\right)^\mu A_x^k\phi(xe^{2u}).$$

Since $\phi\in\mathcal{D}(\mathbf{I})$ there exists a constant $m>0$ such that the support of $\lambda(x,u)$ with respect to u is upperly bounded by m whatever $x\in\mathbb{K}$ may be. An integration by parts yields

$$\int_\delta^\infty \sin\ 2Nu\ \lambda(x,u)\,du=$$
$$=\frac{1}{2N}\left[(\cos\ 2N\delta\ )\lambda(x,\delta)\right]+\int_\delta^h(\cos\ 2Nu\ )\frac{\partial}{\partial u}\lambda(x,u)\,du.$$

But $\lambda(x,u)$ is a bounded function of x and $\frac{\partial}{\partial u}\lambda(x,u)$ is a bounded function of (x,u) for all $x\in\mathbb{K}$ and $u\in[\delta,m]$. Moreover,

$$\int_{2N\delta}^\infty\frac{\sin\ u}{u}\,du\to 0,\qquad\text{as}\qquad N\to\infty.$$

These facts imply that there exists an N_1 such that, for every $N>N_1$, and every $x\in\mathbb{K}$

$$\left|\int_\delta^\infty v(x,u)\ \sin\ 2Nu\,du\right|<\frac{\varepsilon}{2}\,.$$

Finally, by the boundedness of the functions $x^{\frac{1}{4}}C_0(x)$ and $x^{\frac{1}{4}}C_1(x)$ [4], the imposed conditions and the estimation

$$|G(\psi)(w)|<Cw^{-\frac{1}{2}}$$

C being a suitable constant, it follows after some manipulations that

$$x^\alpha(1+x)^{-\mu}\frac{F_i(x,u)}{u}\in L(0,\infty),\qquad i=1,2,3.$$

Furthermore, from the Riemann lemma, we can conclude that

$$x^\alpha(x+1)^{-\mu}\int_0^\infty F_i(x,u)\frac{\sin\ 2Nu}{u}\,du\to 0$$

uniformly in $\mathbb{K}$, as $N\to\infty$, $i=1,2,3$. Therefore, by Lemma 2.1, $\Theta_N(x)\to\phi(x)$ in $\mathcal{E}(\mathbf{I})$, and the lemma is proved. □

Now, we establish an inversion formula on the subspace $\mathcal{E}'(\mathbf{I})$ of the distributions of compact support which is a subspace of $U'_{a,\mu,\alpha}$.

Theorem 4.1. *Let $f \in \mathcal{E}'(\mathbf{I})$ be and set*

$$F(\tau) = \langle f(t), \mathbf{F}(\mu, \alpha, \tau, t)\rangle.$$

Then, for every $\phi \in \mathcal{D}(\mathbf{I})$

$$\langle f, \phi\rangle = \lim_{N\to\infty} \left\langle \int_0^N S(\mu,\tau)\mathbf{G}(\mu,\alpha,\tau,t)F(\tau)\,d\tau, \phi(t)\right\rangle \tag{4.21}$$

with $R_e\alpha > 0$, $R_e\mu > 0$, $\frac{1}{8} < R_e(\mu-\alpha) < \frac{1}{4}$ and $R_e(\frac{\mu}{2}-\alpha) < -\frac{1}{2}$.

Proof: Let $\phi \in \mathcal{D}(\mathbf{I})$ be. We shall show that

$$\left\langle \int_0^N S(\mu,\tau)\mathbf{G}(\mu,\alpha,\tau,t)F(\tau)\,d\tau, \phi(t)\right\rangle \tag{4.22}$$

tends to $\langle f, \phi\rangle$ as $N \to \infty$. From the analyticity of $F(\tau)$ and the fact that the support of $\phi(t)$ is a compact subset of $\mathbf{I}$, it follows that (4.22) is really a repeated integral in (t,τ) having a continuous integrand on a closed bounded domain of integration. Thus, we may change the order of integration to obtain from (4.22):

$$\int_0^N \langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle\,d\tau \int_0^\infty \phi(t)S(\mu,\tau)\mathbf{G}(\mu,\alpha,\tau,t)\,dt.$$

By Lemma 4.3, this is equal to

$$\left\langle f(x), \int_0^N \mathbf{F}(\mu,\alpha,\tau,x)\,d\tau \int_0^\infty \phi(t)S(\mu,\tau)\mathbf{G}(\mu,\alpha,\tau,t)\,dt\right\rangle. \tag{4.23}$$

Then, $f \in \mathcal{E}'(\mathbf{I})$, and according to Lemma 4.4, the testing function inside (4.23) converges in $\mathcal{E}(\mathbf{I})$ to $\phi(x)$ as $N \to \infty$, and this completes the proof. □

An immediate consequence of the above inversion theorem is the following uniqueness theorem:

Theorem 4.2. *Let $F(\tau) = {}_2\mathcal{F}_1(f)$ and $G(\tau) = {}_2\mathcal{F}_1(g)$ with $f, g \in \mathcal{E}'(\mathbf{I})$ and assume that $F(\tau) = G(\tau)$ for all $\tau > 0$. Then $f = g$.*

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