Entropies of self-mappings of topological spaces with richer structures

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Abstract. For mappings $f : S \to S$, where S is a merotopic space equipped with a diameter function, we introduce and examine an entropy, called the δ-entropy. The topological entropy and the entropy of self-mappings of metric spaces are shown to be special cases of the δ-entropy. Some connections with other characteristics of self-mappings are considered. We also introduce and examine an entropy for subsets of S^N, which is closely connected with the δ-entropy of $f : S \to S$.

Keywords: entropy, merotopic space, self-mapping, diameter function

Classification: 54C70, 54E17

In this article, we introduce an entropy, which we call the δ-entropy, for self-mappings of certain enriched topological spaces, namely for those equipped with a merotopy μ and a diametric function d, as defined, in a more special setting, by Z. Frolík in 1962. If d is the unit diameter function, i.e. $d(X) = 1$ whenever card $X > 1$, then the δ-entropy is shown to coincide with the usual topological entropy if the underlying space is compact and with the entropy introduced by R. Bowen in 1973, if the space is metrizable.

We examine some properties of the δ-entropy for self-mappings and its relations to an analogous entropy for subsets of S^N, where S is a space equipped with a diameter function and a merotopy. We also present several examples and some simple propositions on self-mappings of R^n showing certain connections between the δ-entropy and other characteristics of self-mappings.

The article is organized as follows.

Diametric spaces (D-spaces) are introduced in Section 1. For these spaces we define the δ-entropy; in the case of diametric functions induced by a semimetric, this entropy coincides with the δ-entropy considered in [K90] and [K92a]. Some properties of the δ-entropy for diametric spaces are examined and the concept of a relative δ-entropy is considered.

In Section 2, we recall some basic facts connected with the well-known concepts of the topological entropy and of the entropy (which we call the Bowen entropy) for uniformly continuous self-mappings of metric spaces.

After stating some definitions and facts concerning merotopic spaces, we introduce, in Section 3, diametric spaces equipped with a merotopy; we call them merotopic diametric spaces or MD-spaces. For these spaces, we introduce an entropy (again called the δ-entropy), a special case of which is the δ-entropy for D-spaces.

In Section 4, we introduce and examine the central concept of the article, namely that of the δ-entropy $\delta\langle f, S\rangle$ of a self-mapping $f : S \to S$, also denoted by $\langle f, S\rangle$, of an MD-space S. Among other things, it is shown that both the usual topological entropy and the entropy of uniformly continuous self-mappings of metric spaces are special cases of $\delta\langle f, S\rangle$.

Objects of the form $\langle P, S\rangle$, where S is an MD-space and P is a non-void subset of S^N, are considered in Section 5; we call them "polydromic processes" in view of a certain analogy with stochastic processes. For objects of this kind, an entropy is introduced. It turns out that with every self-mapping $\langle f, S\rangle$, where S is an MD-space, there is associated a polydromic process $\langle P, S\rangle$ and that, under certain conditions, the entropies of $\langle f, S\rangle$ and $\langle P, S\rangle$ coincide.

Section 6 contains a number of examples and some propositions concerning relations between $\delta\langle f, S\rangle$ and some other characteristics of $f : S \to S$, in particular for the case $S = R^n$.

1.

1.1. Notation. (A) The letters $N, R, R_+, \overline{R}_+$ have their usual meaning. — (B) If X is a set, then $\exp X = \{Y : Y \subset X\}$. — (C) If f is a self-mapping, i.e. a mapping of the form $f : X \to X$, then f^k, $k \in N$, is defined by $f^{k+1} = f \circ f^k$, where f^0 is the identity mapping $\mathrm{id} : X \to X$. — (D) If $x, y \in R$, we put $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$. — (E) Parentheses are used to denote indexed collections, e.g. $(X_a : a \in A)$. In particular, $(x_k : k < n)$ and $(x_k : k \in N)$ denote sequences. If $x = (x_k : k < n)$, we put $|x| = n$. If $x = (x_k : k < n)$ or $x = (x_k : k \in N)$ is a sequence and $m \in N$, then $x \restriction n$ denotes $(x_k : k < m \wedge n)$ or $(x_k : k < m)$, respectively. If x and y are sequences, and $x = y \restriction n$ for some n, we write $x \leq y$. — (F) The concatenation of finite sequences x and y is denoted by $x \cdot y$. — (G) If $x \in R$, we put $\exp x = 2^x$. If $x \in R$, $x > 0$, we put $\log x = \log_2 x$, $\mathrm{Log}\ x = m$, where m is the least integer satisfying $\log x \leq m$. We put $\log \infty = \mathrm{Log}\ \infty = \infty$.

1.2. Notation and conventions. (A) Parentheses are often omitted provided there is no danger of confusion. E.g. we write, if f is a mapping, fx instead of $f(x)$, $f^{-1}Y$ instead of $f^{-1}(Y)$. — (B) If $f : X \to X$ is a mapping and $\mathscr{Y} \subset \exp Y$, then $f^{-1}\mathscr{Y}$ denotes $\{f^{-1}Z : Z \in \mathscr{Y}\}$, and similarly for other analogous cases. — (C) Brackets of the form $\langle\ \rangle$ are used to denote spaces and similar objects. Instead of e.g. $\langle f, \langle Q, d, \mu\rangle\rangle$ we often write $\langle f; Q, d, \mu\rangle$; details concerning the use of this notation are given at pertinent places. — (D) The same symbol is sometimes used to denote a space and its underlying set.

1.3. Definitions. Let Q be a set. If $\varrho : Q \times Q \to R$ satisfies, for all $x, y \in Q$, the conditions (i) $\varrho(x, y) = \varrho(y, x)$, (ii) $\varrho(x, x) = 0$, then ϱ is called a <u>semimetric</u> on Q and $\langle Q, \varrho\rangle$ is called a <u>semimetric space</u>. If $d : \exp Q \to \overline{R}_+$ satisfies (1) $d(X) \leq d(Y)$ whenever$X \subset Y \subset Q$, (2) $d(X) = 0$ if $X \subset Q$, card $X \leq 1$, then d is called a <u>diametric function</u> or a <u>diameter</u> on Q and $\langle Q, d\rangle$ is called a <u>diametric space</u> or a <u>D-space</u>. — If $\langle Q, d\rangle$ is a D-space and $\mathscr{X} \subset \exp Q$, we put $d(\mathscr{X}) = \sup\{d(X) : X \in \mathscr{X}\}$. If Q is a set and $b \in R_+$, then b will also denote the diameter d on Q such that $d(X) = b$ if card $X > 1$; the diameter 1 will be called the unit diameter.

For diametric spaces, we introduce two kinds of subspaces. If $\langle Q, d\rangle$ is a D-space and $X \subset Q$, then the diameter $d \restriction \exp X$ is denoted by $d \restriction X$ and $\langle X, d \restriction X\rangle$, often denoted simply by $\langle X, d\rangle$, is called a subspace of the first kind, abbreviated subspace (I), of $\langle Q, d\rangle$. Subspaces of the first kind will also be called simply "subspaces" provided there is no danger of confusion. Another kind of subspaces, which we are going to define, will not occur very often. If $S_1 = \langle Q, d_1\rangle$ and $S_2 = \langle Q, d_2\rangle$ are D-spaces and $d_1 \le d_2$, we will say that S_1 is a subspace of the second kind, abbreviated subspace (II), of S_2. If S_1 is a subspace (I) of S, we write $S_1 \subset S$; if S_1 is a subspace (II) of S, we write $S_1 \le S$.

1.4. Remark. It seems that diameters, defined as non-negative functions on $\exp Q$, appear for the first time in [F62], where it is assumed that Q is endowed with the topology and that, besides (1) and (2) from 1.3, the following condition holds: $d(X) = \inf\{d(U) : X \subset U \subset Q,\ U \text{ open}\}$.

1.5. If ϱ is a semimetric on Q, then the function $d : \exp Q \to \overline{R}_+$, defined by $d(X) = \sup\{\varrho(x,y) : x, y \in X\}$, is a diameter on Q, which will be denoted by $d[\varrho]$. We will say that $d[\varrho]$ is induced by ϱ.

Clearly, $\langle Q, \varrho\rangle \mapsto \langle Q, d[\varrho]\rangle$ is an embedding of the class of semimetric spaces into that of D-spaces.

With every W-space (see e.g. [K90]), there is associated a D-space. Namely, if $\langle Q, \varrho, \mu\rangle$ is a W-space, i.e. μ is a bounded measure and ϱ is a measurable semimetric on Q, let, for every $X \subset Q$, $d(X)$ be the least $t \in \overline{R}_+$ such that $\{(x,y) \in X \times X : \varrho(x,y) > t\}$ is of measure zero. Then $\langle Q, d\rangle$ is a D-space.

In [K90] and [K92a], we have examined the functional δ defined for semimetric spaces. In what follows, we introduce (1.9) an analogous functional, also denoted by δ, for D-spaces. It is easy to show (by considering dyadic expansions) that we always have $\delta\langle Q, d[\varrho]\rangle = \delta\langle Q, \varrho\rangle$. Hence we can limit ourselves to the functional δ defined for D-spaces.

It will turn out that many, though not all, results about δ for semimetric spaces remain valid for D-spaces. However, some assertions concerning $\langle Q, d\rangle$ are true only if d is induced by a semimetric or, as the case may be, by a metric.

1.6. Dyadic expansions (abbreviated d.e.) of D-spaces are defined in a way completely analogous to the case of semimetric spaces. Nevertheless, we state the pertinent definitions in full.

(A) $\mathscr{D}$ will denote the collection of all finite non-void $A \subset \bigcup(\{0,1\}^n : n \in N)$ such that (1) $x \in A$, $y \le x$ implies $y \in A$, (2) if $x \in A$, then either $\{x0, x1\} \subset A$ or $\{x0, x1\} \cap A = \emptyset$. — If $A \in \mathscr{D}$, we put $A' = \{x \in A : \{x0, x1\} \subset A\}$, $A'' = A \setminus A'$.

(B) A <u>dyadic expansion</u> of a set Q is an indexed collection $(Q_x : x \in A)$ such that $A \in \mathscr{D}$, $Q_\emptyset = Q$, and $Q_{x0} \cup Q_{x1} = Q_x$, $Q_{x0} \cap Q_{x1} = \emptyset$ for $x \in A'$. A <u>dyadic expansion</u> of a D-space S is an indexed collection $\mathscr{S} = (S_x : x \in A)$ of subspaces of S such that $(|S_x| : x \in A)$ is a d.e. of $|S|$. — If $\mathscr{S} = (S_x : x \in A)$ is a d.e., we put $\mathscr{S}'' = \{S_x : x \in A''\}$. Observe that this notation is different from the notation in [K90] and [K92a], where $\mathscr{S}''$ is the indexed collection $(S_x : x \in A'')$.

1.7. Notation. If $\mathscr{S} = (S_x : x \in A)$ is a dyadic expansion of a D-space $S = \langle Q, d\rangle$, then we put $\delta(\mathscr{S}) = \max\{\sum(d(S_x) : x < z) : z \in A''\}$.

1.8. Notation and definition. (A) A cover of a set Q or of a space $S = \langle Q, \dots \rangle$ is a collection $\mathscr{X}$ of subsets of Q such that $\bigcup X = Q$. The set of all covers of Q will be denoted by $\operatorname{Cov}(Q)$. — (B) If $\mathscr{X} \in \operatorname{Cov}(Q)$, $A \subset Q$, then $\mathscr{X} \restriction A$ denotes the collection $\{X \cap A : X \in \mathscr{X}\}$. — (C) If $\mathscr{X}$, $\mathscr{Y}$ are covers of Q and, for any $X \in \mathscr{X}$, there is $Y \in \mathscr{Y}$ with $X \subset Y$, then we say that $\mathscr{X}$ refines $\mathscr{Y}$ ($\mathscr{X}$ is finer than $\mathscr{Y}$) or that $\mathscr{Y}$ is coarser than $\mathscr{X}$, and we write $\mathscr{X} \geq \mathscr{Y}$. If $\mathscr{X} \geq \mathscr{Y}$ and $\mathscr{Y} \geq \mathscr{X}$, we write $\mathscr{X} \cong \mathscr{Y}$ and say that $\mathscr{X}$ and $\mathscr{Y}$ are equivalent. — (D) If $\mathscr{X}$, $\mathscr{Y}$ are covers of Q, then $\mathscr{X} \vee \mathscr{Y}$ denotes the cover $\{X \cap Y : X \in \mathscr{X}, Y \in \mathscr{Y}\}$. — (E) If $\langle Q, d \rangle$ is a D-space and $\varepsilon \geq 0$, then the cover $\{X : X \subset Q, d(X) \leq \varepsilon\}$ is denoted by $\mathscr{M}(d, \varepsilon)$, abbreviated $\mathscr{M}(\varepsilon)$ or $\mathscr{M}_\varepsilon$.

1.9. Definition. Let $S = \langle Q, d \rangle$ be a D-space. If $\mathscr{U} \in \operatorname{Cov}(Q)$, then $\delta\langle S \mid \mathscr{U} \rangle$ or $\delta\langle Q, d \mid \mathscr{U} \rangle$ denotes the infimum of all $\delta(\mathscr{S})$, where $\mathscr{S} = (S_x : x \in A)$ is a dyadic expansion of S such that $\mathscr{S}'' \geq \mathscr{U}$ (i.e. for every $x \in A''$ there is $U \in \mathscr{U}$ with $S_x \subset U$). Instead of $\delta\langle S \mid \mathscr{M}(d, \varepsilon) \rangle$ we often write $\delta\langle S \mid \varepsilon \rangle$. We put $\delta S = \sup\{\delta\langle S \mid \varepsilon \rangle : \varepsilon > 0\}$. — We will call $\delta\langle S \mid \mathscr{U} \rangle$ the δ-entropy of S with respect to $\mathscr{U}$; δS will be called the δ-<u>entropy</u> (<u>entropic content</u>) of the D-space $S = \langle Q, d \rangle$.

1.10. Definition. A D-space $S = \langle Q, d \rangle$ is called <u>bounded</u>, if $d(Q) < \infty$, <u>totally bounded</u> (abbreviated t.b.), if $d(Q) < \infty$ and, for every $\varepsilon > 0$, there is a finite cover $\mathscr{U}$ with $d(\mathscr{U}) \leq \varepsilon$.

1.11. The following statement, which is almost evident, will be used in some proofs.

Let $\mathscr{S} = (S_x : x \in A)$ be a d.e. of a D-space $S = \langle Q, d \rangle$. For every $a \in A''$ let $\mathscr{Y}_a = (Y_{a,b} : b \in B_a)$ be a d.e. of S_a. Let $\widehat{A}$ consists of all $x \in A'$ and all concatenations $a \cdot b$, where $a \in A''$, $b \in B_a$. For $a \in A'$ put $T_a = S_a$; for $a \in A''$, $b \in B_a$ put $T_{a \cdot b} = Y_{a,b}$. Then $\mathscr{T} = (T_c : c \in \widehat{A})$ is a d.e. of S.

1.12. Proposition. *Let $S = \langle Q, d \rangle$ be a D-space. Then (1) $\delta\langle S \mid \mathscr{U} \rangle \leq \delta S$ for every finite cover $\mathscr{U}$ of S, (2) if S is totally bounded or d is induced by a metric, then δS is equal to the supremum of all $\delta\langle S \mid \mathscr{U} \rangle$, where $\mathscr{U}$ is a finite cover of S.*

Proof: I. Let $\mathscr{V} = (V_x : x \in A)$ be a d.e. of S such that $\mathscr{V}''$ refines $\mathscr{U}$; let $n = \max\{|x| : x \in A\}$; let $\varepsilon > 0$ be arbitrary. Clearly, there exists a d.e. $\mathscr{S} = (S_y : y \in B)$ of S such that $d(\mathscr{S}'') \leq \varepsilon$ and $\delta(\mathscr{S}) \leq \delta\langle S \mid \varepsilon \rangle + \varepsilon$. Let $\mathscr{T} = (T_z : z \in C)$ be the d.e. of S constructed from $\mathscr{S}$ and $\mathscr{V}_y = (V_x \cap S_y : x \in A)$, $y \in B''$, in the way described in 1.11. Since $\mathscr{T}''$ refines $\mathscr{U}$, we get $\delta\langle S \mid \mathscr{U} \rangle \leq \delta(\mathscr{T}) \leq \delta(\mathscr{S}) + n\varepsilon \leq \delta\langle S \mid \varepsilon \rangle + (n+1)\varepsilon \leq \delta S + (n+1)\varepsilon$. This proves the first assertion. — II. If S is t.b. then every $\mathscr{M}_\varepsilon$ is refined by a finite cover, which implies $\delta S = \sup\{\delta\langle S \mid \mathscr{U} \rangle : \mathscr{U} \in \operatorname{Cov}(Q), \mathscr{U} \text{ finite}\}$. If $d = d[\varrho]$, ϱ a metric, and S is not t.b., then there is an $\varepsilon > 0$ and an infinite $M \subset Q$ such that $\varrho(x, y) \geq \varepsilon$ for $x, y \in M$, $x \neq y$. Clearly, $\delta\langle M \mid \mathscr{U} \rangle \geq m\varepsilon$ whenever $\mathscr{U}$ is a finite cover, $\operatorname{card} \mathscr{U} \geq \exp m$. $\square$

1.13. Example. Let $\mathscr{F}$ be a free filter on N. For $X \subset N$ put $d(X) = 1$ if $X \in \mathscr{F}$, $d(X) = 0$ if $X \notin \mathscr{F}$. If $\mathscr{F}$ is not an ultrafilter, then $\langle N, d \rangle$ is easily seen to be totally bounded, $\delta\langle N, d \rangle = 1$. If $\mathscr{F}$ is an ultrafilter, then $\langle N, d \rangle$ is not totally bounded, $\delta\langle N, d \rangle = \infty$, but $\delta\langle N \mid \mathscr{U} \rangle = 1$ for every finite cover $\mathscr{U}$.

1.14. Lemma. *Let $\mathscr{S} = \langle Q, d\rangle$ be a* D*-space. Assume that there exists a finite cover $\mathscr{U}$ of S such that $d(\mathscr{U}) = 0$. Then $\delta S = \inf\{\delta(\mathscr{S}) : d(\mathscr{S}'') = 0\}$ and $\delta S = \delta\langle S \mid \mathscr{V}\rangle$ for every finite cover $\mathscr{V}$ satisfying $d(\mathscr{V}) = 0$.*

Proof: By definition, for every $\varepsilon > 0$ we have $\delta\langle S \mid \varepsilon\rangle = \inf\{\delta(\mathscr{S}) : d(\mathscr{S}'') \le \varepsilon\}$ and therefore $\delta\langle \mathscr{S} \mid \varepsilon\rangle \le \inf\{\delta(\mathscr{S}) : d(\mathscr{S}'') = 0\}$. This implies $\delta S \le \inf\{\delta(\mathscr{S}) : d(\mathscr{S}'') = 0\}$. If $\mathscr{V}$ is a finite cover and $d(\mathscr{V}) = 0$, then, by 1.12, $\delta S \ge \delta\langle S \mid \mathscr{V}\rangle = \inf\{\delta(\mathscr{S}) : \mathscr{S}''$ refines $\mathscr{V}\} \ge \inf\{\delta(\mathscr{S}) : d(\mathscr{S}'') = 0\}$, which proves $\delta S = \inf\{\delta(\mathscr{S}) : d(\mathscr{S}'') = 0\}$. This equality implies $\delta S \le \inf\{\delta(\mathscr{S}) : \mathscr{S}''$ refines $\mathscr{V}\} = \delta\langle S \mid \mathscr{V}\rangle$. □

1.15. The product of finitely many semimetric spaces is defined in the usual way, see e.g. [K90, 1.4]. To introduce the product of finitely many D-spaces, it is sufficient to define $S_1 \times S_2$, where $S_i = \langle Q_i, d_i\rangle$ are D-spaces. For every $X \subset Q_1 \times Q_2$, let X_i, $i = 1, 2$, denote the projection of X into Q_i; put $d(X) = d_1(X_1) \vee d_2(X_2)$. Clearly, d is a diameter on $Q_1 \times Q_2$, which will be denoted by $d_1 \times d_2$. The space $S_1 \times S_2 = \langle Q_1 \times Q_2, d_1 \times d_2\rangle$ will be called the product of S_1 and S_2.

1.16. Lemma. *Let $S = \langle Q_1, d_1\rangle$ and $T = \langle Q_2, d_2\rangle$ be* D*-spaces. Let $\mathscr{S} = (S_a \mid a \in A)$ and $\mathscr{T} = (T_b : b \in B)$ be dyadic expansion of S and T, respectively. Let $\varepsilon \ge 0$. Assume that $d_1(\mathscr{S}'') \le \varepsilon$, $d_2(\mathscr{T}'') \le \varepsilon$, $d_1(S_a) > \varepsilon$ for $a \in A'$ and $d_2(T_b) > \varepsilon$ for $b \in B'$. Then there exists a dyadic expansion $\mathscr{U} = (U_c : c \in C)$ of $S \times T = \langle Q_1 \times Q_2, d\rangle$ such that* (1) *$d(\mathscr{U}'') \le \varepsilon$,* (2) *$\delta(\mathscr{U}) \le \delta(\mathscr{S}) + \delta(\mathscr{T})$,* (3) *every U_c, $c \in C$, is of the form $S_a \times T_b$, $a \in A$, $b \in B$.*

Proof: It is easy to see that, starting from the trivial d.e. $(U_c : c \in \{\emptyset\})$, $U_\emptyset = S \times T$, we can construct, step by step, a dyadic expansion $\mathscr{U} = (U_c : c \in C)$ with the following properties: $\mathscr{U}$ satisfies (3); if $c \in C$, $d(U_c) \le \varepsilon$, then $c \in C''$; if $c \in C$, $d(U_c) > \varepsilon$, $U_c = S_a \times T_b$, then, for $i = 0, 1$, $U_{ci} = S_{ai} \times T_b$ if $d_1(S_a) \ge d_2(T_b)$, whereas $U_{ci} = S_a \times T_{bi}$ if $d_1(S_a) < d_2(T_b)$. Clearly, $\mathscr{U}$ satisfies (1) and (3). To prove that (2) is satisfied, it is sufficient to show that, for $c \in C$, $U_c = S_a \times T_b$, we have

$$\sum(d(U_z) : z < c) = \sum(d_1(S_x) : x < a) + \sum(d_2(T_y) : y < b).$$

This equality is easily proved by induction on the length $|c|$ of c. □

1.17. Proposition. *For any* D*-spaces and T, $\delta(S \times T) \le \delta S + \delta T$.*

Proof: Let $\varepsilon > 0$. Then, for any $\vartheta > 0$, there are d.e. $\mathscr{S} = (S_a : a \in A)$ of $S = \langle Q_1, d_1\rangle$ and $\mathscr{T} = (T_b : b \in B)$ of $T = \langle Q_1, d_2\rangle$ such that $d_1(\mathscr{S}'') \le \varepsilon$, $d_2(\mathscr{T}'') \le \varepsilon$, $\delta(\mathscr{S}) < \delta\langle S \mid \varepsilon\rangle + \vartheta/2$, and $\delta(\mathscr{T}) < \delta\langle T \mid \varepsilon\rangle + \vartheta/2$. Clearly, we can assume that $d_1(S_a) > \varepsilon$ for $a \in A'$, and $d_2(T_b) > \varepsilon$ for $b \in B'$. By 1.16, there exists a d.e. $\mathscr{U} = (U_c : c \in C)$ of $S \times T$ such that $\delta(\mathscr{U}) \le \delta(\mathscr{S}) + \delta(\mathscr{T})$ and $d(\mathscr{U}'') \le \varepsilon$. Hence $\delta\langle S\times T \mid \varepsilon\rangle \le \delta\langle S \mid \varepsilon\rangle + \delta\langle T \mid \varepsilon\rangle + \vartheta$. Therefore $\delta\langle S\times T \mid \varepsilon\rangle \le \delta\langle S \mid \varepsilon\rangle + \delta\langle T \mid \varepsilon\rangle$ for every $\varepsilon > 0$. By 1.9, this implies $\delta(S \times T) \le \delta S + \delta T$. □

1.18. Notation. Let $S = \langle Q, d\rangle$ be a D-space and let $\mathscr{U}$ be a cover of S. Then (1) $\mathscr{U} \odot d$ denotes the diameter defined as follows: (i) $(\mathscr{U} \odot d)(X) = 0$ if $X \subset U$ for some $U \in \mathscr{U}$, (ii) $(\mathscr{U} \odot d)(X) = d(X)$ if $X \subset U$ for no $U \in \mathscr{U}$, (2) $\mathscr{U} \odot S$ denotes the D-space $\langle Q, \mathscr{U} \odot d\rangle$. — If $\varepsilon \geq 0$, we put $\varepsilon \odot d = \mathscr{M}(d, \varepsilon) \odot d$, $\varepsilon \odot S = \langle Q, \varepsilon \odot d\rangle$; thus, $(\varepsilon \odot d)(X) = 0$ if $d(X) \leq \varepsilon$, and $(\varepsilon \odot d)(X) = d(X)$ if $d(X) > \varepsilon$.

The following assertions are obvious. Let $S = \langle Q, d\rangle$ be a D-space and let $\mathscr{U}$ and $\mathscr{V}$ be covers of S. Then (1) $(\mathscr{U} \vee \mathscr{V}) \odot d = (\mathscr{U} \odot d) \vee (\mathscr{V} \odot d)$, (2) *if* $\mathscr{V}$refines $\mathscr{U}$, then $\mathscr{V} \odot d \geq \mathscr{U} \odot d$.

1.19. Proposition. *Let $S = \langle Q, d\rangle$ be a D-space. Then* (1) $\delta(\mathscr{U} \odot S) \leq \delta(S \mid \mathscr{U})$ *for every cover $\mathscr{U}$ of Q,* (2) $\delta(\mathscr{U} \odot S) = \delta\langle S \mid \mathscr{U}\rangle$ *if $\mathscr{U}$ is finite.*

PROOF: Write d^* instead of $\mathscr{U} \odot d$, S^* instead of $\mathscr{U} \odot S$. To prove (1), we can assume that $\delta\langle S \mid \mathscr{U}\rangle < \infty$. Let $\varepsilon > 0$. Then there is a d.e. $\mathscr{S} = (S_a : a \in A)$ of S such that $\delta(\mathscr{S}) < \delta\langle S \mid \mathscr{U}\rangle + \varepsilon$ and $\mathscr{S}''$ refines $\mathscr{U}$. Put $T_a = \mathscr{U} \odot S_a$, $\mathscr{T} = (T_a : a \in A)$; thus $\mathscr{T}''$ is a d.e. of $\mathscr{S}^*$. Clearly, $\mathscr{T}''$ refines $\mathscr{M}(d^*, 0)$ and therefore $\delta S^* \leq \delta(\mathscr{T})$. Since $\delta(\mathscr{T}) \leq \delta(\mathscr{S})$, we get $\delta S^* \leq \delta\langle S \mid \mathscr{U}\rangle + \varepsilon$, which implies, $\varepsilon > 0$ being arbitrary, the inequality (1). — To prove (2), we can assume that $\delta(\mathscr{U} \odot S) < \infty$. Let $\mathscr{U}$ be finite. We have $d^*(\mathscr{U}) = 0$, hence, by 1.14, $\delta S^* = \delta\langle S^* \mid \mathscr{U}\rangle$. Let $\varepsilon > 0$. There exists a d.e. $\mathscr{T} = (T_a : a \in A) = (\mathscr{U} \odot S_a : a \in A)$ of S^* such that $\delta(\mathscr{T}) < \delta S^* + \varepsilon$ and $\mathscr{T}''$ refines $\mathscr{U}$. Clearly, we can assume that if $a \in A'$, then $T_a \subset \mathscr{U}$ for $U \notin \mathscr{U}$, and therefore $d^*(S_a) = d(S_a)$ for $a \in A'$. Put $\mathscr{S} = (S_a : a \in A)$. Then $\mathscr{S}$ is a d.e. of S, $\delta(\mathscr{S}) = \delta(\mathscr{T}) \leq \delta S^* + \varepsilon$ and $\mathscr{S}''$ refines $\mathscr{U}$. This implies $\delta\langle S \mid \mathscr{U}\rangle < \delta S^* + \varepsilon$, which proves, $\varepsilon > 0$ being arbitrary, the inequality $\delta\langle S \mid \mathscr{U}\rangle \leq \delta S^*$. ☐

1.20. We are going to consider some questions connected with the notion of relative entropy including certain conceptual and intuitive aspects. An examination of these aspects can throw light on some ideas investigated later (see e.g. 5.5), though the relative entropy itself will occur only seldom in the subsequent sections. From several approaches to the relative δ-entropy, we choose the one which is fairly simple and fits well into the general framework introduced in [K92b].

1.21. Recall that in [K92b, 1.3] a piece of information is defined as a pair (S, S'), where S and S' are V-fields (in the case considered here, D-spaces); it is required that S' should be a subspace of S. In [K92b, 1.25] it is stated that the concept of a piece of information (S, S') can be extended by dropping this requirement, and a (non-mathematical) example is given, where S is a subfield of S'. In [K92b], it is also stated (1.21) that subfields can be introduced in various ways and that the concept of a piece of information depends on how subfields are defined.

In accordance with this approach and with the fact that two kinds of subspaces have been introduced for D-spaces, we proceed as follows. For the case of D-spaces, two different kinds of pieces of information will be considered. A piece of information of the first kind, abbreviated a piece of information (I), is a pair (S, S') such that $S \supset S'$, i.e. S' is a subspace (I) of S.; a piece of information of the second kind, abbreviated a piece of information (II), is a pair (S, S'), where $S \leq S'$, i.e. S is a subspace (II) of S'. A piece of information of the first kind could also be called a localizing piece of information, and that of the second kind, an enriching piece of information.

1.22. Definition. Let (S, S'), where S and S' are D-spaces, be a piece of information of the first or the second kind. The evaluation or "measure" (cf. [K92b, 1.6]) of (S, S') is defined to be the value $\delta S - \delta S'$ or $\delta S' - \delta S$ according to the whether $S \supset S'$ or $S \leq S'$ provided, of course, that these expressions are meaningful (i.e. $\delta S' < \infty$ or $\delta S < \infty$, respectively). The value $\delta S - \delta S'$ or $\delta S' - \delta S$ will be called the <u>relative</u> δ<u>-entropy</u> of S with respect to S' (if $S \supset S'$) or of S' with respect to S (if $S \leq S'$).

If $\mathscr{X}$ and $\mathscr{Y}$ are covers of a D-space S and $\delta(\mathscr{X} \odot S) < \infty$, then $\delta((\mathscr{X} \vee \mathscr{Y}) \odot S) - \delta(\mathscr{X} \odot S)$ will be called the <u>relative</u> δ<u>-entropy</u> of $\mathscr{Y}$ with respect to $\mathscr{X}$ on S.

1.23. The intuitive meaning of the transition from a cover $\mathscr{U}$ to a finer one $\mathscr{V}$ can be outlined as follows (for the sake of simplicity we consider finite partitions only).

It is not possible, in general, to localize (to identify) elements of a given D-space $S = \langle Q, d\rangle$ with a full precision in finitely many steps.

In fact, we rather proceed as follows. A finite partition $\mathscr{U} = \{U_i : 1 \leq i \leq m\}$ is taken and it is determined to which U_i does the element in question belong. The overall information gain, with respect to all possible $x \in Q$, can be evaluated as $\delta\langle S \mid \mathscr{U}\rangle$, i.e., by 1.19, as $\delta(\mathscr{U} \odot S)$.

In the next step, a finer partition $\mathscr{V} = \{V_k : 1 \leq k \leq n\}$ is taken; thus, we pass from a view characterized by $\mathscr{U}$ to a sharper and more informative view corresponding to $\mathscr{V}$. The evaluation of the overall information gain is now $\delta\langle S \mid \mathscr{V}\rangle = \delta(\mathscr{V} \odot S)$. Clearly, $\mathscr{U} \odot S \leq \mathscr{V} \odot S$; in other words, $\mathscr{U} \odot S$ is a subspace (II) of $\mathscr{V} \odot S$.

In this way, we can proceed indefinitely, passing over from a given "visualization" to a sharper one. A more general procedure consists in transitions from $\langle Q, d_1\rangle$ to $\langle Q, d_2\rangle \geq \langle Q, d_1\rangle$. Since d_2 discerns points of Q more effectively than d_1, transitions of this kind also mean taking a sharper view. They will play an important role in Section 5.

2.

In this section, we recall the definitions of the topological entropy of continuous self-mappings of compact spaces, introduced in [AKM65], and of the entropy of uniformly continuous self-mappings of metric spaces, introduced in [B71]. We also recall some auxiliary concepts and known facts.

2.1. Notation. (A) If $\mathscr{X} \in \operatorname{Cov}(Q)$, then $\nu(\mathscr{X})$ denotes the infimum of card $\mathscr{Y}$, where $\mathscr{Y} \subset \mathscr{X}$ is a finite cover of Q; thus, if there is no finite cover $\mathscr{Y} \subset \mathscr{X}$, then $\nu(\mathscr{X}) = \infty$. We put $h(\mathscr{X}) = \log \nu(\mathscr{X})$. — (B) If $\mathscr{X} \in \operatorname{Cov}(Q)$, then $\nu^*(\mathscr{X})$ denotes the supremum of card M for finite subsets M of Q such that $\operatorname{card}(M \cap X) \leq 1$ for all $X \in \mathscr{X}$. We put $h^*(\mathscr{X}) = \log \nu^*(\mathscr{X})$. — (C) If $S = \langle Q, d\rangle$ is a D-space and $\varepsilon > 0$, we put $h(S, \varepsilon) = h(\mathscr{M}(d, \varepsilon))$, $h^*(S, \varepsilon) = h^*(\mathscr{M}(d, \varepsilon))$. For a semimetric space $S = \langle Q, \varrho\rangle$, we put $h(S, \varepsilon) = h(\langle Q, d[\varrho]\rangle, \varepsilon)$, $h^*(S, \varepsilon) = h^*(\langle Q, d[\varrho]\rangle, \varepsilon)$. If S is fixed, we write $h(\varepsilon)$ and $h^*(\varepsilon)$ instead of $h(S, \varepsilon)$ and $h^*(S, \varepsilon)$, respectively. Instead of $h(\langle X, d\rangle, \varepsilon)$ and $h^*(\langle X, d\rangle, \varepsilon)$, we sometimes write $h(X, \varepsilon)$ and $h^*(X, \varepsilon)$, respectively, provided there is no danger of confusion.

2.2. We list some well-known simple facts. (A) If $\mathscr{U} \in \operatorname{Cov}(Q)$, then $h(\mathscr{U}) \geq h^*(\mathscr{U})$. — (B) If $\mathscr{X}, \mathscr{Y} \in \operatorname{Cov}(Q)$ and $\mathscr{X} \geq \mathscr{Y}$, then $h(\mathscr{X}) \geq h(\mathscr{Y})$, $h^*(\mathscr{X}) \geq h^*(\mathscr{Y})$. — (C) If $\mathscr{X}, \mathscr{Y} \in \operatorname{Cov}(Q)$, then $h(\mathscr{X} \vee \mathscr{Y}) \leq h(\mathscr{X}) + h(\mathscr{Y})$. — (D) If ϱ is a metric on Q, $S = \langle Q, d\rangle$, then $h(S, \varepsilon) \leq h^*(S, \varepsilon/2)$ for every $\varepsilon > 0$. — Observe that this last assertion may be false if ϱ is merely a semimetric.

2.3. Recall that a mapping of the form $f : X \to X$ is called a <u>self-mapping</u> (of X). A self-mapping $f : X \to X$ will often be denoted by $\langle f, X\rangle$, in particular if X is a space.

2.4. If $\langle f, S\rangle$ is a self-mapping, $\mathscr{U}$ is a cover of S, $m, n \in N$, and $n > 0$, we put $\mathscr{U}^{(n)} = \mathscr{U}_f^{(n)} = \bigvee(f^{-k}\mathscr{U} : 0 \leq k < n)$, $\mathscr{U}^{(m,n)} = \mathscr{U}_f^{(m,n)} = \bigvee(f^{-k}\mathscr{U} : m \leq k < m+n)$.

The following assertions are evident. If $\mathscr{U} \in \operatorname{Cov}(Q)$, $\langle f, Q\rangle$ is a self-mapping, $m, n \in N$, and $n > 0$, then (A) $h(f^{-1}\mathscr{U}) = h(\mathscr{U})$, (B) $\mathscr{U}^{(m+n)} = \mathscr{U}^{(m)} \vee \mathscr{U}^{(m,n)}$, (C) $h(\mathscr{U}^{(m+n)}) \leq h(\mathscr{U}^{(m)}) + h(\mathscr{U}^{(n)})$.

2.5. Fact. If $\mathscr{U} \in \operatorname{Cov}(Q)$, then either $h(\mathscr{U}) = \delta\langle Q, 1 \mid \mathscr{U}\rangle = \infty$ or $\delta\langle Q, 1 \mid \mathscr{U}\rangle - 1 \leq h(\mathscr{U}) \leq \delta\langle Q, 1 \mid \mathscr{U}\rangle$, where 1 denotes the unit diameter.

The proof is easy and can be omitted.

2.6. Lemma. *Let $x_n, n \in N$ be non-negative reals. If $x_{m+n} \leq x_m + x_n$ for all $m, n \in N$, then the sequence $(x_n/n : n \in N)$ converges in R.*

This is well known.

2.7. We recall the definition of the topological entropy of self-mappings in its usual form (with a partly changed notation). Let $\langle f, X\rangle$ be a continuous self-mapping of a compact Hausdorff topological space, in particular, of a compact metric space $X = \langle Q, \varrho\rangle$. For every finite open cover $\mathscr{U}$ of X, put $h\langle f, X \mid \mathscr{U}\rangle = \lim_{n\to\infty}(h(\mathscr{U}^{(n)})/n)$ (this limit does exist by 2.4 (C) and 2.6); we say that $h\langle f, X \mid \mathscr{U}\rangle$ is the topological entropy of $\langle f, X\rangle$ with respect to $\mathscr{U}$. The supremum of all $h\langle f, X \mid \mathscr{U}\rangle$, where $\mathscr{U}$ is a finite open cover, is denoted by $h\langle f, X\rangle$ and it is called the <u>topological entropy</u> of the self-mapping f of X.

Remark. The definition remains meaningful, if (1) instead of finite open covers, any family of covers is considered, (2) the assumption of continuity of f is dropped. We will return to these questions in Section 4.

2.8. Notation. Let $f : Q \to Q$ be a mapping. Let d be a diameter on Q. If $n \in N$, $n > 0$, then $d^{(n)} = d_f^{(n)} = d[f, n]$ denotes the diameter defined by $d^{(n)}(X) = \max\{d(f^k X) : 0 \leq k < n\}$. If ϱ is a semimetric on Q, then $\varrho^{(n)} = \varrho_f^{(n)} = \varrho[f, n]$ is defined in a similar way: $\varrho^{(n)}(x, y) = \max\{\varrho(f^k x, f^k y) : 0 \leq k < n\}$.

2.9. We are going to recall the definition of the entropy of a uniformly continuous self-mapping of a metric space. It will be stated in a form which is equivalent to but formally different from that given in [B71].

Let $S = \langle Q, \varrho\rangle$ be a metric space and let $\langle f, S\rangle$ be a uniformly continuous self-mapping. If $K \subset S$ is compact, then we put (1) for every $\varepsilon > 0$, $h^*\langle f, S; K \mid$

$\varepsilon\rangle = \overline{\lim}_{n\to\infty}(h^*(\langle K,\varrho^{(n)}\rangle,\varepsilon)/n)$, (2) $h^*\langle f,S;K\rangle = \sup\{h^*\langle f,S;K \mid \varepsilon\rangle : \varepsilon > 0\}$. We put $h^*\langle f,S\rangle = \sup\{h^*\langle f,S;K\rangle : K \subset S \text{ compact}\}$. We will call $h^*\langle f,S\rangle$ the <u>Bowen entropy</u> of $\langle f,S\rangle$; $h^*\langle f,S;K\rangle$ will be called the Bowen entropy of $\langle f,S\rangle$ with respect to K.

2.10. The following important well-known result can be easily proved using the inequality $h^*(S,\varepsilon) \le h(S,\varepsilon) \le h^*(S,\varepsilon/2)$, see 2.2 (A) and 2.2 (D).

Proposition. *Let $\langle f,S\rangle$ be a self-mapping of a compact metric space. Then $h^*\langle f,S\rangle = h\langle f,S\rangle$.*

2.11. The definition of the uniform equivalence of metrics can be extended top semimetrics and diameters as follows. Let ϱ_1 and ϱ_2 be semimetrics and let d_1 and d_2 be diameters on Q. We will say that ϱ_1 and ϱ_2 or, respectively, d_1 and d_2 are uniformly equivalent, if there are positive realvalued functions u and v on $R_+ \setminus \{0\}$ such that $\varrho_1(x,y) < \eta$ implies $\varrho_2(x,y) < u(\eta)$ and $\varrho_2(x,y) < \varepsilon$ implies $\varrho_1(x,y) < v(\varepsilon)$ (or, respectively, $d_1(X) < \eta$ implies $d_2(X) < u(\eta)$ and $d_2(X) < \varepsilon$ implies $d_1(X) < v(\varepsilon)$).

2.12. The following result is also known. Nevertheless, we give a short proof.

Proposition. *Let ϱ and σ be uniformly equivalent metrics on Q. Let $f : Q \to Q$ be uniformly continuous with respect to ϱ or σ. Then $h^*\langle f,\langle Q,\varrho\rangle\rangle = h^*\langle f,\langle Q,\sigma\rangle\rangle$.*

PROOF: Let u and v possess, with respect to ϱ and σ, the properties stated in 2.11. It is easy to verify that they also possess these properties with respect to $\varrho^{(n)}$ and $\sigma^{(n)}$ for all $n \in N$. This implies that, for every compact $K \subset S$, every $n \in N$, $n > 0$, and every $\varepsilon > 0$, we have $h^*(\langle K,\varrho^{(n)}\rangle,\varepsilon) \le h^*(\langle K,\sigma^{(n)}\rangle,u(\varepsilon))$, $h^*(\langle K,\sigma^{(n)}\rangle,\varepsilon) \le h^*(\langle K,\varrho^{(n)}\rangle,v(\varepsilon))$. Hence, we get $h^*\langle f,S_1;K \mid \varepsilon\rangle \le h^*\langle f,S_2;K \mid u(\varepsilon)\rangle$, $h^*\langle f,S_2,K \mid \varepsilon\rangle \le h^*\langle f,S_1,K \mid v(\varepsilon)\rangle$, where $S_1 = \langle Q,\varrho\rangle$, $S_2 = \langle Q,\sigma\rangle$. From this we obtain, by definition 2.9, $h^*\langle f,S_1;K\rangle = h^*\langle f,S_2;K\rangle$ for every compact (with respect to ϱ or, equivalently, to σ) subset $K \subset Q$. Hence $h^*\langle f,S_1\rangle = h^*\langle f,S_2\rangle$. □

Remark. The proposition just proved shows that, in fact, h^* is an entropy of a uniformly continuous self-mappings of a metrizable uniform space; it does not depend on the choice of metric inducing the uniformity.

2.13. Proposition. *Let $S = \langle Q,d\rangle$ be a compact metrizable space. If $f : S \to S$ is a continuous mapping, then $h\langle f,S\rangle$ is equal to the supremum of all $\overline{\lim}(\delta\langle Q,1 \mid \mathscr{U}^{(n)}\rangle/n)$, where $\mathscr{U}$ is an open cover of $\mathscr{S}$.*

PROOF: Since S is compact, $h(\mathscr{U}) < \infty$ for every open cover $\mathscr{U}$. By 2.5, we have $|h(\mathscr{U}) - \delta\langle Q,1 \mid \mathscr{U}\rangle| \le 1$. □

3.

In Section 2, we mentioned the fact that the definition 2.7 remains meaningful if, given a non-void $\xi \subset \operatorname{Cov}(Q)$, we define $h(f,Q,\xi)$ as the supremum of all $h\langle f,Q \mid \mathscr{U}\rangle$ for $\mathscr{U} \in \xi$ (observe that for a fixed $\mathscr{U}$, $h\langle f,X \mid \mathscr{U}\rangle$, as defined in 2.7, does not depend on the topology of the space in question). Before proceeding to develop this observation, let us note that, in connection with entropies, there do occur systems ξ of covers of a topological space $\langle Q, \tau \rangle$ which differ from the system of all $\mathscr{U} \in \mathrm{Cov}\,(Q)$ refinable by some open cover; cf. e.g. the example in Section 1 of [B73].

Since it seems superfluous to examine quite arbitrary systems of covers, we restrict ourselves to merotopies.

3.1. From various equivalent definitions of a merotopy (see e.g. [K75]), we choose the following one.

Definition. Let Q be a set and let $\emptyset \neq \mu \subset \mathrm{Cov}\,(Q)$. Assume that (1) if $\mathscr{X}, \mathscr{Y} \in \mathrm{Cov}\,(Q)$, $\mathscr{X} \leq \mathscr{Y}$, $\mathscr{Y} \in \mu$, then $\mathscr{X} \in \mu$, (2) $\mathscr{X}, \mathscr{Y} \in \mu$ implies $\mathscr{X} \vee \mathscr{Y} \in \mu$. Then μ is called a <u>merotopy</u> on Q and $\langle Q, \mu \rangle$ is called a <u>merotopic space</u>. — A cover $\mathscr{X} \in \mu$ will sometimes be called a μ-cover.

Observe that every merotopy induces, in a natural way, a topology and that merotopic spaces can be considered as topological spaces equipped, in addition, with a merotopy inducing the given topology.

3.2. We are now going to recall some simple basic concepts and facts concerning merotopic spaces. For further information see e.g. [H74], [H82] or [K76].

(A) If μ is a merotopy on Q and $X \subset Q$, then $\mu \restriction X$ denotes the collection of all $\{U \cap X : U \in \mathscr{U}\}$, where $\mathscr{U} \in \mu$. It is easy to see that $\mu \restriction X$ is a merotopy. The space $\langle X, \mu \restriction X \rangle$, often denoted simply by $\langle X, \mu \rangle$, will be called a subspace of $\langle Q, \mu \rangle$. — (B) Let μ be a merotopy on Q. A collection $\beta \subset \mu$ will be called a <u>base</u> of μ (or of $\langle Q, \mu \rangle$), if for every $\mathscr{X} \in \mu$ there is a $\mathscr{Y} \in \beta$ refining $\mathscr{X}$. — (C) If μ_1 and μ_2 are merotopies on Q, $\mu_1 \supset \mu_2$, we say that μ_1 is finer than μ_2 or that μ_2 is coarser than μ_1. — (D) A merotopic space $\langle Q, \mu \rangle$ is called <u>totally bounded</u> (abbreviated t.b.) if, for every $\mathscr{U} \in \mu$, there is a finite cover $\mathscr{V} \subset \mathscr{U}$ (it is not required that $\mathscr{V}$ should be a μ-cover). A set $X \subset Q$ is called <u>totally bounded</u> if so is $\langle X, \mu \rangle$. — (E) If $S_1 = \langle Q_i, \mu_i \rangle$, $i = 1, 2$, are merotopic spaces, then a mapping $f : S_1 \to S_2$ is called continuous, if $f^{-1}(\mu_2) \subset \mu_1$. — (F) If $S_i = \langle Q_i, \mu_i \rangle$, $i = 1, 2$, are merotopic spaces, then $\mu_1 \times \mu_2$ denotes the merotopy on $Q_1 \times Q_2$ with a base consisting of all $\{X_1 \times X_2 : X_i \in \mathscr{X}_i\}$, where $\mathscr{X}_i \in \mu_i$. The space $\langle Q_1 \times Q_2, \mu_1 \times \mu_2 \rangle$, denoted by $S_1 \times S_2$, is called the product of S_1 and S_2.

3.3. We list some important classes of merotopies. (A) Let $S = \langle Q, \tau \rangle$ be a topological space, and let $\mu[\tau]$ be the merotopy with a base consisting of all open covers; we say that $\mu[\tau]$ is induced by τ (or by S). — (B) If $S = \langle Q, \tau \rangle$ is a topological space and $X \subset Q$, we say that the merotopy $\mu[\tau] \restriction X$ on X is induced by the topology of S. — (C) If $S = \langle Q, \xi \rangle$ is a uniform space, then $\mu[\xi]$ denotes the merotopy consisting of all uniform covers (thus, if uniformities are defined as systems of covers, we have $\mu[\xi] = \xi$). (D) If d is a diameter on Q, then $\mu[d]$ denotes the merotopy with a base consisting of all $\mathscr{M}(d, \varepsilon)$, $\varepsilon > 0$. — (E) If $\mathscr{U} \in \mathrm{Cov}\,(Q)$, then $[\mathscr{U}]$ will denote the merotopy of which $\{\mathscr{U}\}$ is a base.

We shall see in what follows that the approach based on arbitrary merotopies enables the uniform treatment of quite different objects (uniformities, diameters, individual covers) in the same way.

3.4. We are going to introduce spaces whose structure consists of a diameter d and a merotopy μ; in general, no relationship between d and μ is assumed. Self-mappings of these spaces, entropies of self-mappings, etc. will form the main object of investigations in the subsequent sections.

3.5. Definition and convention. Let Q be a set, d a diameter and μ a merotopy on Q. We will call $\langle Q, d, \mu\rangle$ a merotopized diametric space or simply an MD-space. If $S = \langle Q, d\rangle$ is a D-space and no merotopy on Q is explicitly given, we will consider S as an MD-space $\langle Q, d, \mu[d]\rangle$.

3.6. Definition. If $S = \langle Q, d, \mu\rangle$ is an MD-space, then δS will denote the supremum of all $\delta\langle Q, d \mid \mathscr{U}\rangle$, where $\mathscr{U} \in \mu$. We will call δS the δ-entropy of the MD-space S.

3.7. Fact. If $\mathscr{U}$ is a cover of a D-space $\langle Q, d\rangle$, then $\delta\langle Q, d, [\mathscr{U}]\rangle = \delta\langle Q, d \mid \mathscr{U}\rangle$.
This is an immediate consequence of the definition.

3.8. The following equality justifies the convention in 3.5 and also the use of the same letter δ to denote the entropy of D-spaces as well as of MD-spaces.

Fact. If $\langle Q, d\rangle$ is a D-space, then $\delta\langle Q, d, \mu[d]\rangle = \delta\langle Q, d\rangle$.

Proof: By definition, $\delta\langle Q, d\rangle$ is equal to the supremum of all $\delta\langle Q, d \mid \mathscr{M}(d, \varepsilon)\rangle$, where $\varepsilon > 0$. Since (see 3.3 (D)) $\mathscr{M}(d, \varepsilon)$, $\varepsilon > 0$, form a base for $\mu[d]$, this proves the assertion. $\square$

3.9. We are going to introduce some terminology concerning MD-spaces. Though it is often self-explanatory, we prefer to state the exact definitions. — (A) Let $S = \langle Q, d, \mu\rangle$ be an MD-space. If $X \subset Q$, then $\langle X, d \restriction X, \mu \restriction X\rangle$, often denoted simply by $\langle X, d, \mu\rangle$ (or also by $S \restriction X$), will be called a subspace of S. — (B) If $S_i\langle Q_i, d_i, \mu_i\rangle$, $i = 1, 2$, then $\langle Q_1 \times Q_2, d_1 \times d_2, \mu_1 \times \mu_2\rangle$ is called the product of S_1 and S_2 and it is denoted by $S_1 \times S_2$. — (C) If $S_i = \langle Q_i, d_i \mu_i\rangle$, $i = 1, 2$, then a mapping $f : S_1 \to S_2$ is called (1) continuous, if $f : \langle Q_1, \mu_1\rangle \to \langle Q_2, \mu_2\rangle$ is continuous, i.e. $f^{-1}(\mu_2) \subset \mu_1$, (2) bounded, if $d_2(fX) < \infty$ whenever $d_1(X) < \infty$.

3.10. Definition. An MD-space $\langle Q, d, \mu\rangle$ is called totally bounded, abbreviated t.b., if $d(Q) < \infty$ and $\langle Q, \mu\rangle$ is totally bounded.

3.11. We list some simple facts, omitting their proofs. Observe that the assertion (D) is an easy consequence of 1.12.

(A) If S' is a subspace of an MD-space S, then $\delta S' \leq \delta S$. — (B) If $S_i = \langle Q, d_i, \mu_i\rangle$, $i = 1, 2$, are MD-spaces and $d_1 \leq d_2$, $\mu_1 \subset \mu_2$, then $\delta S_1 \leq \delta S_2$. — (C) If an MD-space $S = \langle Q, d, \mu\rangle$ is not t.b., then $\delta S = \infty$ except when μ is the coarsest merotopy on Q, in which case $\delta S = 0$. — (D) Let $S = \langle Q, d, \mu\rangle$ be an MD-space and let $\langle Q, d\rangle$ be t.b. If μ is t.b. and $\mu \supset \mu[d]$, then $\delta S = \delta\langle Q, d, \mu[d]\rangle = \delta\langle Q, d\rangle$.

3.12. Proposition. *Let $S_i = \langle Q_i, d_i, \mu_i\rangle$, $i = 1, 2$, be* MD-*spaces. If $\mu_i \supset \mu[d_i]$, $i = 1, 2$, then $\delta(S_1 \times S_2) \leq \delta S_1 + \delta S_2$.*

Proof: It follows from 3.11 (C) that we can assume S_1 and S_2 to be t.b. Put $Q = Q_1 \times Q_2$, $d = d_1 \times d_2$, $\mu = \mu_1 \times \mu_2$. Since $\mu_i \supset \mu[d_i]$, the D-spaces $\langle Q_i, d_i\rangle$

and $\langle Q, d\rangle$ are also t.b. Hence, by 3.11 (D), $\delta S_i = \delta\langle Q_i, d_i\rangle$, $\delta(S_1 \times S_2) = \delta\langle Q, d\rangle$. By 1.17, this implies the proposition. □

4.

In this section, we introduce and examine the entropy $\delta\langle f, S\rangle$ for self-mappings on MD-spaces.

4.1. Notation. (A) A self-mapping $\langle f, S\rangle$, where $S = \langle Q, d, \mu\rangle$ will also be denoted by $\langle f; Q, d, \mu\rangle$. — (B) If $\langle f, S\rangle$ is a self-mapping of an MD-space $S = \langle Q, d, \mu\rangle$, then (1) for every $M \subset Q$ and every $\mathscr{U} \in \mu$, we put $\delta_r\langle f, S; M \mid \mathscr{U}\rangle = \overline{\lim}_{n\to\infty}(\delta\langle M, d^{(n)} | \mathscr{U}^{(n)}\rangle/n)$, (2) for every $M \subset Q$, $\delta_r\langle f, S; M\rangle$ will denote the supremum of all $\delta_r\langle f, S; M \mid \mathscr{U}\rangle$, $\mathscr{U} \in \mu$. — We do not introduce a special name for $\delta_r\langle \dots\rangle$, since it plays only an auxiliary role. The subscript r in $\delta_r\langle\dots\rangle$ stems from "raw entropy" (a possible name for δ_r).

4.2. Definition. Let $\langle f, S\rangle$ be a self-mapping of an MD-space $S = \langle Q, d, \mu\rangle$. Let $M \subset Q$. The pair $\langle\langle f, S\rangle, M\rangle$ will be denoted by $\langle f, S; M\rangle$ and $\delta\langle f, S; M\rangle$ will denote the supremum of all $\delta_r\langle f, S; X\rangle$, where $X \subset M$ is totally bounded in S. Instead of $\delta\langle f, S; Q\rangle$, we write $\delta\langle f, S\rangle$. — We call $\delta\langle f, S\rangle$ the <u>δ-entropy</u> of $\langle f, S\rangle$; $\delta\langle f, S; M\rangle$ will be called the <u>δ-entropy</u> of $\langle f, S\rangle$ <u>with respect to</u> M.

4.3. Theorem. *Let $S = \langle Q, \tau\rangle$ be a compact Hausdorff topological space and let $\langle f, S\rangle$ be a continuous self-mapping. Then the topological entropy $h\langle f, S\rangle$ is equal to the δ-entropy $\delta\langle f; Q, 1, \mu[\tau]\rangle$.*

This is an immediate consequence of 2.13.

4.4. Theorem. *Let $S = \langle Q, \varrho\rangle$ be a metric space and let $\langle f, S\rangle$ be a uniformly continuous self-mapping. Then the Bowen entropy $h^*\langle f, S\rangle$ is equal to the δ-entropy $\delta\langle f; Q, 1, \mu[\tau]\rangle$, where τ is the topology induced by ϱ.*

Proof: I. If $M \subset Q$ is t.b. in $\langle Q, \mu[\tau]\rangle$, then its closure is a compact subset of $\langle Q, \tau\rangle$. This is a consequence of the following simple fact, the proof of which can be omitted: if $\langle X, \tau\rangle$ is a regular topological space and $M \subset X$ is t.b. with respect to $\mu[\tau]$, then the closure of M is compact. — II. By 2.10 and 2.13, the Bowen entropy $h^*\langle f, S\rangle$ is equal to the supremum of all $\sup\{\overline{\lim}(\delta\langle K, 1 \mid \mathscr{U}_{(n)}\rangle/n) : \mathscr{U} \in \mu[\tau]\}$, where $K \subset S$ is compact. On the other hand, $\delta\langle f; Q, 1, \mu[\tau]\rangle$ is equal, by definition, to the supremum of all $\sup\{\overline{\lim}(\delta\langle M, 1 \mid \mathscr{U}^{(n)}\rangle/n) : \mathscr{U} \in \mu[\tau]\}$, where M is t.b. in $\langle Q, \mu[\tau]\rangle$. By I, these two suprema are equal. □

4.5. Along with the δ-entropy, there is another kind of entropy for self-mappings, which will be denoted by $\overline{\delta}$. The $\overline{\delta}$-entropy has some useful properties; under certain not too strong assumptions it coincides with the δ-entropy. On the other hand, some properties of $\overline{\delta}$ are highly unpleasant (see e.g. 4.9 below). Therefore, we will not consider $\overline{\delta}$ in any detail, except proving the proposition (see 4.15) on the coincidence of $\delta\langle f, S\rangle$ and $\overline{\delta}\langle f, S\rangle$ under certain conditions.

4.6. Definition. Let $\langle f, S\rangle$ be a self-mapping of an MD-space $S = \langle Q, d, \mu\rangle$. For any $X \subset Q$, put $\overline{\delta}_r\langle f, S; X\rangle = \overline{\lim}_{n\to\infty}(\delta\langle X, d^{(n)}, \mu^{(n)}\rangle/n)$, where $\mu^{(n)} = \bigvee(f^{-k}\mu : 0 \le k < n)$. If $M \subset Q$, then $\overline{\delta}\langle f, S; M\rangle$ will denote the supremum of all $\overline{\delta}_r\langle f, S; X\rangle$, where $X \subset M$ is totally bounded in S. Instead of $\overline{\delta}\langle f, S; Q\rangle$ we write $\overline{\delta}\langle f, S\rangle$. We say that $\overline{\delta}\langle f, S; M\rangle$ is the $\overline{\delta}$-entropy of $\langle f, S\rangle$ with respect to M, and we call $\overline{\delta}\langle f, S\rangle$ the $\overline{\delta}$-entropy of $\langle f, S\rangle$. Observer that we always have $\overline{\delta}\langle f, S; X\rangle \ge \delta\langle f, S; X\rangle$; this follows easily from the definitions.

4.7. One would expect the entropy of an identity mapping to be zero. Next we shall show that this is, in fact, true for the δ-entropy but generally fails for the $\overline{\delta}$-entropy (see 4.9). Hence the usefulness of $\overline{\delta}$ is rather limited.

4.8. Proposition. *Let $S = \langle Q, d, \mu\rangle$ be an* MD-*space. Then $\delta\langle \mathrm{id}, S\rangle = 0$, where* id *denotes the identity mapping.*

PROOF: Let $X \subset S$ be t.b. Then $\delta_r\langle \mathrm{id}, S; X\rangle$ is equal to the supremum of all $\delta_r\langle \mathrm{id}, S; X \mid \mathscr{U}\rangle$, where $U \in \mu$, and $\delta_r\langle \mathrm{id}, S; X \mid \mathscr{U}\rangle$ is equal to $\overline{\lim}(\delta\langle X, d \mid \mathscr{U}\rangle/n)$. Since X is t.b., we have, for every $\mathscr{U} \in \mu$, $\delta\langle X, d \mid \mathscr{U}\rangle < \infty$, hence $\delta_r\langle \mathrm{id}, S; X \mid \mathscr{U}\rangle = 0$. This implies $\delta_r\langle \mathrm{id}, S; X\rangle = 0$ and proves $\delta\langle \mathrm{id}, S\rangle = 0$, since X is an arbitrary t.b. subspace of S. □

4.9. Fact. Let $S = \langle Q, d, \mu\rangle$ be an MD-space. If $\delta T < \infty$ for every totally bounded $T \subset S$, then $\overline{\delta}\langle \mathrm{id}, S\rangle = 0$. If $\delta T = \infty$ for some totally bounded $T \subset S$, then $\overline{\delta}\langle \mathrm{id}, S\rangle = \infty$.

PROOF: If $T = \langle X, d, \mu\rangle$ is t.b., then $\overline{\delta}_r\langle \mathrm{id}, T\rangle = \overline{\lim}(\delta\langle X, d, \mu\rangle/n)$ and therefore $\overline{\delta}_r\langle \mathrm{id}, T\rangle = 0$ provided $\delta T < \infty$, whereas $\overline{\delta}_r\langle \mathrm{id}, T\rangle = \infty$ whenever $\delta T = \infty$. □

4.10. Among MD-spaces, those of the form $S = \langle Q, 1, \mu\rangle$ play an important role due, among other things, to the equalities stated in 4.3 and 4.4. On the other hand, it is easy to see that if $S = \langle Q, 1, \mu\rangle$, then, under quite weak assumptions on f, $\overline{\delta}\langle f, S\rangle = \infty$ or $\overline{\delta}\langle f, S\rangle = 0$, hence $\overline{\delta}\langle f, S\rangle \ne \delta\langle f, S\rangle$ whenever $0 < \delta\langle f, S\rangle < \infty$.

4.11. The situation is quite different for δ-regular MD-spaces of the form $\langle Q, d, \mu[d]\rangle$. For spaces of this form the equality $\overline{\delta} = \delta$ is fulfilled, and it may be easier to calculate (or estimate) $\overline{\delta}$ than δ.

In accordance with [K92a] we will use the following notation. Let S be a D-space $\langle Q, d\rangle$ or an MD-space of the form $\langle Q, d, \mu[d]\rangle$. Then, for every $t > 0$, we put $C_S(t) = \sup\{\delta\langle X, d\rangle : X \subset Q, d(X) \le t\}$. If $C_S(t) \to 0$ for $t \to 0$, we say that S is δ-regular. We are going to prove the equality $\overline{\delta} = \delta$ for δ-regular spaces. First we give some simple lemmas.

4.12. Lemma. *Let $S = \langle Q, d\rangle$ be a* D-*space. Then* (1) *for every dyadic expansion $\mathscr{S} = (S_x : x \in D)$ of S, $\delta S \le \delta(\mathscr{S}) + C_S(\varepsilon)$, where $\varepsilon = d(\mathscr{S}'')$,* (2) *for every $\varepsilon > 0$, $\delta S \le \delta\langle S \mid \varepsilon\rangle + C_S(\varepsilon)$.*

PROOF: Put $\vartheta = C_S(\varepsilon)$; we can assume $\vartheta < \infty$. Let b and c be arbitrary positive reals. For every $z \in D''$, $\delta(S_z) \le \vartheta$, and therefore there exists a d.e. $\mathscr{T}_z$ of S_z such that $d(\mathscr{T}_z) < \vartheta + b$, $d(\mathscr{T}''_z) < c$. Let $\mathscr{U}$ be the d.e. constructed from $\mathscr{S}$ and $\mathscr{T}_z$,

$z \in D''$, in the manner described in 1.11. Then $d(\mathscr{U}^n) < c$, $\delta(\mathscr{U}) \leq \delta(\mathscr{S}) + \vartheta + b$. This proves the assertion (1). The second assertion is an immediate consequence. □

4.13. Lemma. *Let $\langle f, S\rangle$ be a self-mapping of a* D*-space $S = \langle Q, d\rangle$. Let $n \in N$, $n > 0$. Then* (1) $\delta\langle Q, d^{(n)}\rangle \leq n\cdot\delta S$, (2) *if $M \subset Q$, then* $\delta\langle M, d^{(n)}\rangle \leq \sum(\delta\langle f^k M, d\rangle : 0 \leq k < n)$, (3) $C_Z \leq nC_S$, *where $Z = \langle Q, d^{(n)}\rangle$,* (4) $\delta\langle Q, d^{(n)}, \mu[d]\rangle \leq \delta\langle Q, d^{(n)} \mid \varepsilon\rangle + nC_S(\varepsilon)$ *for every $\varepsilon > 0$.*

Proof: I. For $x \in Q$ put $g(x) = (f^k x : 0 \leq k < n)$. Then g maps Q onto the subspace $\langle gQ, d^n\rangle$ of S^n and transforms $d^{(n)}$ into $d^n \restriction g(Q)$. Hence $\delta\langle Q, d^{(n)}\rangle \leq \delta(S^n)$ and therefore, by 1.17, $\delta\langle Q, d^{(n)}\rangle \leq n \cdot \delta S$. — II. If $M \subset Q$, then $g(M) \subset \prod(f^k M : 0 \leq k < n)$. This implies, by 1.17, the second assertion. — III. Let $t > 0$. Let $M \subset Q$, $d^{(n)}(M) \leq t$. Then $d(f^k M) \leq d^{(n)}(M)$ for $0 \leq k < n$ and therefore $\delta\langle f^k M, d\rangle \leq C_S(t)$. Hence, by (1), $\delta\langle M, d^{(n)}\rangle \leq n \cdot C_S(t)$. This proves $C_Z(t) \leq n$. — IV. Since $d^{(n)} \geq d$, $\mu[d^{(n)}]$ is finer than $\mu[d]$ and therefore $\delta\langle Q, d, \mu[d]\rangle \leq \delta\langle Q, d^{(n)}, \mu[d^{(n)}]\rangle = \delta\langle Q, d^{(n)}\rangle$ (see 3.8). For every $\varepsilon > 0$, we have, by 4.12, $\delta\langle Q, d^{(n)}\rangle \leq \delta\langle Q, d^{(n)} \mid \varepsilon\rangle + C_Z(\varepsilon)$, where $Z = \langle Q, d^{(n)}\rangle$. Hence, by (3), $\delta\langle Q, d^{(n)}, \mu[d]\rangle \leq \delta\langle Q, d^{(n)} \mid \varepsilon\rangle + nC_S(\varepsilon)$. □

4.14. Lemma. *Let $S = \langle Q, d, \mu[d]\rangle$ be an* MD*-space. Let $\langle f, S\rangle$ be a continuous self-mapping; let $X \subset Q$. Then, for every $\varepsilon > 0$, $\overline{\delta}_r\langle f, S; X\rangle \leq \delta_r\langle f, S; X\rangle + C_S(\varepsilon)$.*

Proof: Let $\varepsilon > 0$. Since f is continuous, we have $(\mu[d])^{(n)} = \mu[d]$ for all $n \in N$, $n > 0$. Hence, by 4.13, $\overline{\delta}_r\langle f, S; X\rangle = \overline{\lim}(\delta\langle X, d^{(n)}, \mu[d]\rangle/n) \leq \overline{\lim}(\delta\langle X, d^{(n)} \mid \mathscr{M}(d^{(n)}, \varepsilon)\rangle/n) + C_S(\varepsilon)$. It is easy to see that $\mathscr{M}(d^{(n)}, \varepsilon) = \mathscr{U}^{(n)}$, where $\mathscr{U} = \mathscr{M}(d, \varepsilon)$. Hence $\overline{\lim}(\delta\langle X, d^{(n)} \mid \mathscr{M}(d^{(n)}, \varepsilon)\rangle/n) = \overline{\lim}(\delta\langle X, d^{(n)} \mid \mathscr{U}^{(n)}\rangle/n) = \delta_r\langle f, S; X \mid \mathscr{U}\rangle \leq \delta_r\langle f, S; X\rangle$. This proves that $\overline{\delta}_r\langle f, S; X\rangle \leq \delta_r\langle f, S; X\rangle + C_S(\varepsilon)$. □

4.15. Proposition. *Let $\langle Q, d\rangle$ be a δ-regular* D*-space. Put $S = \langle Q, d, \mu[d]\rangle$. Let $\langle f, S\rangle$ be a continuous self-mapping. Then $\delta\langle f, S; M\rangle = \overline{\delta}\langle f, S; M\rangle$ for every $M \subset Q$; in particular, $\delta\langle f, S\rangle = \overline{\delta}\langle f, S\rangle$.*

This is an immediate consequence of 4.14.

4.16. It is a well-known and almost trivial fact that the entropies h and h^* satisfy the equalities $h\langle f^p, S\rangle = p \cdot h\langle f, S\rangle$, $h^*\langle f^p, S\rangle = p \cdot h^*\langle f, S\rangle$ for $p \in N$. It turns out that, under certain conditions, the corresponding equality $\delta\langle f^p, S\rangle = p\delta\langle f, S\rangle$ is also valid, see 4.20 below. On the other hand, in many fairly simple cases (see 5.10), the set of all $\delta\langle f^p, S\rangle$, $p \in N$, is bounded whereas $\delta\langle f, S > 0\rangle$ and therefore $\delta\langle f^p, S\rangle = p \cdot \delta\langle f, S\rangle$ cannot hold for all $p \in N$.

4.17. Lemma. *Let $\langle f, S\rangle$ be a continuous self-mapping of an* MD*-space $S = \langle Q, d, \mu\rangle$. Let $p \in N$, $p > 0$. Then $\delta\langle f^p; Q, d^{(p)}, \mu\rangle = p \cdot \delta\langle f; Q, d, \mu\rangle$.*

Proof: The case p=1 is trivial. In what follows, we assume $p = 2$; for $p > 2$ the proof is completely analogous, but the notation is more complicated.

Let us write e instead of $d^{(2)}$, g instead of $f^{(2)}$. Let $X \subset Q$ be t.b. It is easy to see that, for any $n \in N$, $n > 0$, and any $Y \subset Q$, $e^{(n)}(Y)$ is equal to $\max\{d(f^jY) : 0 \le j < 2n\}$, hence to $d_f^{(2n)}(Y)$. If $\mathscr{U} \in \mu$, then, with $\mathscr{V} = \mathscr{U} \vee f^{-1}\mathscr{U}$, we have $\delta_r\langle g; X, e \mid \mathscr{V}\rangle = \overline{\lim}(\delta\langle X, e_g^{(n)} \mid \mathscr{V}_g^{(n)}\rangle/n) = \overline{\lim}(\delta\langle X, d_f^{(2n)} \mid \mathscr{U}_f^{(2n+1)}\rangle/n)$. Since

$$\delta\langle X, d_f^{(2n)} \mid \mathscr{U}_f^{(2n)}\rangle \le \delta\langle X, d_f^{(2n)} \mid \mathscr{U}_f^{(2n+1)}\rangle \le \delta\langle X, d_f^{(2n+1)} \mid \mathscr{U}_f^{(2n+1)}\rangle,$$

we get

$$2\,\overline{\lim}(x_{2n}/2n) \le \delta_r\langle g; X, e \mid \mathscr{V}\rangle \le 2\,\overline{\lim}(x_{2n+1}/(2n+1)),$$

where $x_k = \delta\langle X, d_f^{(k)} \mid \mathscr{U}_f^{(k)}\rangle$.

It is easy to see that, due to the fact that $(x_k : k \in N)$ is non-decreasing, we have $\overline{\lim}(x_{2n}/2n) = \overline{\lim}(x_{2n+1}/(2n+1)) = \overline{\lim}(x_n/n)$. This implies $\delta_r\langle g; X, e \mid \mathscr{V}\rangle = 2\delta_r\langle f; X, d \mid \mathscr{U}\rangle$.

Clearly, $\delta_r\langle g; X, e, \mu\rangle$ is equal to the supremum of all $\delta_r\langle g; X, e \mid \mathscr{V}\rangle$, where $\mathscr{V}$ is of the form $\mathscr{U} \vee f^{-1}\mathscr{U}$, $\mathscr{U} \in \mu$; this follows from the fact that f is continuous and therefore covers $\mathscr{U} \vee f^{-1}\mathscr{U}$ form a base for μ. Hence $\delta_r\langle g; X, e, \mu\rangle = 2\delta_r\langle f; X, d, \mu\rangle$. □

4.18. Definition. A self-mapping $\langle f, S\rangle$ of a D-space $\langle Q, d\rangle$ or an MD-space $\langle Q, d, \mu\rangle$ is called expanding if $d(fX) \ge d(X)$ for every $X \subset Q$.

4.19. Lemma. *Let $\langle f, S\rangle$ be an expanding self-mapping of an* MD-*space $S = \langle Q, d, \mu\rangle$. Let $p \in N$, $p \ge 1$. Put $g = f^p$, $e = d^{(p)}$. Then $\delta\langle g; Q, e, \mu\rangle = \delta\langle g; Q, d, \mu\rangle$.*

Proof: Let $X \subset Q$. Since f is expanding, we easily obtain that, for every $\mathscr{U} \in \mu$,

$$\delta\langle X, d^{(n)} \mid \mathscr{U}^{(n)}\rangle \le \delta\langle X, e^{(n)} \mid \mathscr{U}^{(n)}\rangle \le \delta\langle X, d^{(n+1)} \mid \mathscr{U}^{(n+1)}\rangle,$$

and therefore

$$\overline{\lim}(\delta\langle X, e^{(n)} \mid \mathscr{U}^{(n)}\rangle/n) = \overline{\lim}(\delta\langle X, d^{(n)} \mid \mathscr{U}^{(n)}\rangle/n).$$

This implies $\delta_r\langle g; X, e, \mu\rangle = \delta_r\langle g; X, d, \mu\rangle$ for every $X \subset Q$, which proves the assertion.

□

4.20. Proposition. *Let $\langle f, S\rangle$ be a continuous expanding self-mapping of an* MD-*space $S = \langle Q, d, \mu\rangle$. Then $\delta\langle f^p, S\rangle = p \cdot \delta\langle f, S\rangle$ for every $p \in N$.*

This is an immediate consequence of 4.17, 4.19, and 4.8.

5.

In this section, we introduce the δ-entropy for objects of the form $\langle P, S\rangle$, where S is an MD-space and $P \subset S^N$. The motivation for considering such objects can be explained as follows.

5.1. If Q is a set and $f : Q \to Q$ is a mapping, let $\Phi(f)$ denote the set of all $x = (x_k : k \in N) \in Q^N$ such that $x_{k+1} = f(x_k)$ for all $k \in N$; if, in addition, a set $X \subset Q$ is given, we put $\Phi(f, X) = \{x \in \Phi(f) : x_0 \in X\}$. The set $\Phi(f)$ can be interpreted as describing a fully determined process (with discrete time) whose course depends only on the initial value x_0. To describe a non-deterministic process with no probability intervening, we can take any non-void subset S of Q^N, interpreted as the set of all possible courses.

It will turn out that, for Q equipped with d and μ so as to obtain an MD-space $\langle Q, d, \mu\rangle$, it is possible to introduce an entropy for $\langle P, S\rangle$, denoted by $\delta\langle P, S\rangle$, in a reasonable way. In particular, we get $\delta\langle \Phi(f), S\rangle = \delta\langle f, S\rangle$ under certain fairly weak assumptions (see 5.7).

Since $\langle P, S\rangle$ can be considered as a "probability-free" analogue of a stochastic process (with discrete time), we will call it a polydromic process (from $\pi o\lambda\upsilon$-, many-, and $\delta\varrho o\mu o\varsigma$, which means course).

5.2. Definition. Let $S = \langle Q, d, \mu\rangle$ be an MD-space. If $\emptyset \neq P \subset Q^N$, then $\langle P, S\rangle$ will be called a polydromic process on S. — Sometimes $\langle P, S\rangle$ will be considered together with a given set $M \subset Q$; instead of $(\langle P, S\rangle, M)$, we will write $\langle P, S; M\rangle$. — If $\langle f, S\rangle$ is a self-mapping of an MD-space S, we will say that $\langle \Phi(f), S\rangle$ is the polydromic process associated with $\langle f, S\rangle$. Similarly, if $M \subset Q$ is given, we will say that $\langle \Phi(f, M), S\rangle$ is associated with $\langle f, S; M\rangle$.

5.3. Definition. The following notation (cf. 1.16, 3.2) will be used. If $\mathscr{U} \in \mathrm{Cov}\,(Q)$, then $\mathscr{U}^1 = \mathscr{U}$, $\mathscr{U}^{n+1} = \mathscr{U}^n \times \mathscr{U}$ for $n \in N$, $n > 0$. If $X \subset Q^N$, $n \in N$, $n > 0$, then $X \restriction n = \{x \restriction n : x \in X\}$. If $S = \langle Q, d, \mu\rangle$ is an MD-space, then $d^1 = d$, $\mu^1 = \mu$, $d^{n+1} = d^n \times d$, $\mu^{n+1} = \mu^n \times \mu$, thus $S^n = \langle Q^n, d^n, \mu^n\rangle$. Let $S = \langle Q, d, \mu\rangle$ be an MD-space and let $\langle P, S\rangle$ be a polydromic process. For every $\mathscr{U} \in \mu$ put $\delta_r\langle P, S \mid \mathscr{U}\rangle = \overline{\lim}_{n\to\infty}(\delta\langle P \restriction n, d^n \mid \mathscr{U}^n\rangle/n)$; put $\delta_r\langle P, S\rangle = \sup\{\delta_r\langle P, S \mid \mathscr{U}\rangle : \mathscr{U} \in \mu\}$. The supremum of all $\delta_r\langle X, S\rangle$, where $\emptyset \neq X \subset P$ and for every $n \in N$, $n > 0$, $X \restriction n$ is totally bounded in S^n, will be denoted by $\delta\langle P, S\rangle$ and will be called the δ-entropy of the polydromic process $\langle P, S\rangle$.

5.4. We are going to discuss the intuitive meaning of $\delta\langle P, S\rangle$. For the sake of simplicity, we will assume $S = \langle S, d, \mu\rangle$ to be totally bounded. The entropy $\delta\langle P, S\rangle$ is then equal to the supremum of all $\delta_r\langle P, S \mid \mathscr{U}\rangle = \overline{\lim}_{n\to\infty}(\delta\langle P \restriction n, d^n \mid \mathscr{U}^n\rangle/n)$, where $\mathscr{U} \in \mu$. Hence we can restrict ourselves to explaining the meaning of this limit from the standpoint introduced in [K92b] and developed in Section 1 of the present article.

5.5. It turns out that we have a situation related to that considered in 1.23. For $n \in N$, $n > 0$, let d_n be the diameter on P defined as follows: $d_n(X) = 0$ if $X \restriction n \subset V$ for some $V \in \mathscr{U}^n$, and $d_n(X) = d^n(X \restriction n)$ if $X \restriction n \subset V$ for $V \not\subset \mathscr{U}^n$; thus, $d_n(X) = (\mathscr{U}^n \odot d)(X \restriction n)$. By 1.19, we obtain $\delta\langle P \restriction n, d^n \mid \mathscr{U}^n\rangle = \delta\langle P, d_n\rangle$; it is easy to see that $d_n \leq d_{n+1}$ for all $n \in N$, $n > 0$.

Thus, every $\langle P, d_n\rangle$ is a subspace (II) (see 1.3) of $\langle P, d_{n+1}\rangle$; we have, similarly as in 1.23, successive transitions from one way of looking at possible courses (sequences $x \in Q^N$) to another one, which is more informative. More explicitly, we pass on from seeing the development on $\{0,\dots,n-1\}$ with a "sharpness" given by d_n to seeing it on $\{0,\dots,n\}$ with a greater sharpness given by d_{n+1}. The resulting information gain can be expressed as the relative entropy $\delta\langle P, d_{n+1}\rangle - \delta\langle P, d_n\rangle$.

5.6. The average information gain after n steps is equal to $\delta\langle P, d_n\rangle/n$ (we put $\delta\langle P, d_0\rangle = 0$). Since these values can fail to have a limit, we take the upper limit (or, which is the same, the upper Cesàro limit of relative entropies). This limit can be considered as expressing the speed with which the entropy increases when successively larger segments of the process are taken into account.

5.7. Theorem. *Let $\langle f, S\rangle$ be a bounded continuous self-mapping of an* MD-*space $S = \langle Q, d, \mu\rangle$. If $M \subset S$, then*

$$\delta\langle f, S; M\rangle = \delta\langle \Phi(f, M), S\rangle.$$

Proof: I. If $X \subset Q$, put $P_X = \Phi(f, X)$. We are going to show that, for every $X \subset Q$,

$$(*) \qquad \delta_r\langle f, S; X\rangle = \delta_r\langle P_X, S\rangle = \delta_r\langle \Phi(f, X), S\rangle.$$

To this end, it is sufficient to show that, for all $n \in N$, $n > 0$, and $\mathscr{U} \in \mu$, the following equality holds (as for $d^{(n)}$, $\mathscr{U}^{(n)}$, d^n, $\mathscr{U}^n$, see 2.4, 2.8, and 5.2):

$$(**) \qquad \delta_r\langle X, d^{(n)} \mid \mathscr{U}^{(n)}\rangle = \delta_r\langle P_X, d^n \mid \mathscr{U}^n\rangle.$$

For every $n \in N$, $n > 0$, put $g_n(x) = (f^k x : 0 \le k < n)$ for every $x \in X$. Clearly, $g_n : X \to P_X \restriction n$ is a bijection, which transforms $\mathscr{U}^{(n)} \restriction X$ into $\mathscr{U}^n \restriction (P_x \restriction n)$ and $d^{(n)} \restriction X$ into $d^n \restriction (P_x \restriction n)$. This implies $(**)$, hence also $(*)$. — II. Let $\mathscr{A}$ and $\mathscr{B}$ consist, respectively, of all t.b. $X \subset M$ and of all $Y \in P_M$ such that every $Y \restriction n$ is t.b. (in S^n). By definition (5.3) and by $(*)$, it is sufficient to show that $X \in \mathscr{A}$ implies $\Phi(f, X) \in \mathscr{B}$ whereas $T \in \mathscr{B}$ implies $\pi(T) \in \mathscr{A}$, $T \subset \Phi(f, \pi(T))$, where $\pi(x) = x_0$ for $x = (x_k : k \in N) \in S^N$. Indeed, if $X \in \mathscr{A}$, then, under our assumptions, $f^n X$ is t.b. for every n. Hence $P_x \restriction n = g_n(X)$ is also t.b., and therefore $P_x \in \mathscr{B}$. It is evident that $T \in \mathscr{B}$ implies $\pi(T) \in \mathscr{A}$ and $T \in \Phi(f, \pi(T))$. This proves the theorem. □

5.8. Fact. If S is an MD-space of the form $S = \langle Q, d, \mu[d]\rangle$, then $\delta\langle Q^N, S\rangle \le \delta S$. — This follows easily from 1.17.

5.9. Proposition. *If $\langle f, S\rangle$ is a bounded continuous self-mapping of an* MD-*space $S = \langle Q, d, \mu[d]\rangle$, then $\delta\langle f, S\rangle \le \delta\langle Q^N, S\rangle \le \delta S$.*

This is an immediate consequence of 5.7 and 5.8.

5.10. It follows from 5.9 that if $\langle f, S\rangle$ satisfies the conditions stated in 5.9, $\delta S < \infty$, and $\delta\langle f, S\rangle > 0$, then $\delta\langle f^p, S\rangle = p \cdot \delta\langle f, S\rangle$ cannot hold for large p. An example: $J = [0,1] \subset R$, $S = \langle J, d, \mu[d]\rangle$, where d is the usual diameter, $f(t) = 1 - |2t-1|$ for $t \in J$. It is easy to see that $1 \le \delta S \le 2$ (in fact, we have $\delta S = 2$) and that $\delta\langle f, S\rangle > 0$.

6.

In this section, we present some rather elementary examples. We also state some simple propositions indicating ceratin connections between the δ-entropy and some other characteristics of self-mappings.

6.1. In 6.2–6.6 below, we consider the self-mapping $f(x) = x+1$ of the real line R equipped (i) with the unit diameter or with the diameter $d(X) = \sup\{|x-y| : x, y \in X\}$ or else with $d^* = d \wedge 1$, (ii) with the merotopy $\mu_0 = \mu[d]$ or with the merotopy μ_1 induced (see 3.3 (B)) by the topology of the one-point compactification of R.

6.2. It can be shown that $\delta\langle f; R, 1, \mu_0\rangle = \infty$ whereas $\delta\langle f; R, 1, \mu_1\rangle = 0$. This is essentially well known, though in a different context; see the example at the end of Section 1 of [B73]. Therefore we omit the proof of $\delta\langle f; R, 1, \mu_0\rangle = \infty$ and prove only the second equality. — Let $J \subset R$ be a bounded interval. Let $\mathscr{G} \in \mu_1$ be a finite open cover. Then, for some $G_0 \in \mathscr{G}$, the set $R \setminus G_0$ is bounded and therefore there is $p \in N$ such that $f^{-t}(G_0) \supset J$ for all $t \in N$, $t \geq p$. This implies that, for $t \geq p$, $\{J \cap G : G \in \mathscr{G}^{(t)}\} \supset \{J \cap G : G \in \mathscr{G}^{(p)}\}$ and therefore $\delta\langle J, 1 \mid \mathscr{G}^{(t)}\rangle \leq \delta\langle J, 1 \mid \mathscr{G}^{(p)}\rangle$. Hence $\overline{\lim}(n^{-1}\delta\langle J, 1 \mid \mathscr{G}^{(p)}\rangle) = 0$, $\delta_r\langle f; J, 1 \mid \mathscr{G}\rangle = 0$. Since $\mathscr{G}$ is an arbitrary finite open cover belonging to μ_1, we have shown that $\delta_r\langle f; J, 1, \mu_1\rangle = 0$. This proves $\delta\langle f; R, 1, \mu_1\rangle = 0$.

Observe that $\overline{\delta}\langle f; R, 1, \mu_0\rangle = \overline{\delta}\langle f; R, 1, \mu_1\rangle = \infty$. This follows from the evident fact that $\delta\langle J, 1, \mu_0\rangle = \delta\langle J, 1, \mu_1\rangle = \infty$ for any non-degenerate interval $J \subset R$.

6.3. If 1 is replaced by d, then the value of $\overline{\delta}$ and δ are zero. This is a consequence of the following fact, the proof of which is easy and can be omitted. — Let $S = \langle Q, \mu, d\rangle$ be an MD-space. Let $\langle g, S\rangle$ be a continuous self-mapping such that $d(gX) = d(X)$ for all $X \subset Q$. Assume that $\delta\langle X, d, \mu\rangle < \infty$ for all totally bounded $X \subset S$. Then $\delta\langle g, S\rangle = 0$.

6.4. If $\beta = \langle b_n : n \in N\rangle$, $b_0 > 0$, $b_n \leq b_{n+1}$ for $n \in N$, we put $g_\beta(t) = 1$ if $t < 0$, and $g_\beta(t) = b_n$ if $n \leq t < n+1$, $n \in N$. Let ϱ_β be the metric on R defined by $\varrho_\beta(x, y) = \int_x^y g_\beta(t)\, dt$ whenever $x, y \in R$, $x < y$. Put $d_\beta = d[\varrho_\beta]$, $d^*_\beta = d_\beta \wedge 1$, $S_{\beta i} = \langle R, d_\beta, \mu_i\rangle$, $S^*_{\beta i} = \langle R, d^*_\beta, \mu_i\rangle$ for $i = 0, 1$.

6.5. By 4.17, we have $\delta\langle f, S_{\beta 0}\rangle = \overline{\delta}\langle f, S_{\beta 0}\rangle$ since $\mu_0 = \mu[d_\beta]$. As $\langle X, d_\beta, \mu_0\rangle$ is totally bounded iff so is $\langle X, d_\beta, \mu_1\rangle$ and $\mu_0 \restriction X = \mu_1 \restriction X$ for every bounded $X \subset R$, we have $\overline{\delta}\langle f, S_{\beta 1}\rangle = \overline{\delta}\langle f, S_{\beta_0}\rangle$.

We are going to show that $\delta\langle f, S_{\beta 0}\rangle = \infty$ if $\overline{\lim}(n^{-1}b_n) > 0$, and $\delta\langle f, S_{\beta 0}\rangle = 0$ if $n^{-1}b_n \to 0$. — Assume that $\overline{\lim}(n^{-1}b_n) > 0$. Let $m \in N$, $m > 1$, and put $J = [0, m] \subset R$. We have $\delta\langle J, d^{(n)}_\beta, \mu^{(n)}_0\rangle = \delta\langle J, d^{(n)}_\beta, \mu_0\rangle \geq \delta\langle J, b_n d, \mu_0\rangle$. Since $\delta\langle J, d\rangle \geq m$ (see 5.10), this implies $\delta\langle J, d^{(n)}_\beta, \mu_0\rangle \geq m b_n$. Hence $\overline{\delta}_r\langle f; J, d_\beta, \mu_0\rangle \geq m \cdot \overline{\lim}(n^{-1}b_n)$ and therefore $\overline{\delta}\langle f, S_{\beta 0}\rangle = \infty$. — If $n^{-1}b_n \to 0$ for $n \to \infty$, then $\delta\langle J, d^{(n)}_\beta, \mu_0\rangle \leq \delta\langle J, b_{m+n} d, \mu_0\rangle \leq 2mb^{m+n}$, since $\delta\langle [0,1], d\rangle \leq 2$ (see 5.10). This implies $\overline{\delta}_r\langle f, S_{\beta 0}; J\rangle = 0$ and proves $\overline{\delta}\langle f, S_{\beta 0}\rangle = 0$.

6.6. If d_β is replaced by d^*_β, then the values of δ and $\overline{\delta}$ are, in general, different from those for d_β. In particular, $\overline{\delta}\langle f, S^*_{\beta i}\rangle$, $i = 1, 2$, and $\delta\langle f, S^*_{\beta 0}\rangle$ coincide and are equal to $\overline{\lim}(n^{-1}\log b_n)$. The proof presents no serious difficulties and it is omitted.

It is easy to see that $\delta\langle f, S^*_{\beta 1}\rangle = 0$. Indeed, let $M \subset R$ be bounded. Then, for every finite open cover $\mathscr{U} \in \mu_1$, there is $p \in N$ such that $\mathscr{U}^{(n)} \restriction M \cong \mathscr{U}^{(p)} \restriction M$ whenever $n \geq p$. Hence, for every $n \geq p$, $\delta\langle M, d_\beta \mid \mathscr{U}^{(n)}\rangle \leq \delta\langle M, 1 \mid \mathscr{U}^{(n)}\rangle \leq h(\mathscr{U}^{(n)}, 1) = h(\mathscr{U}^{(p)}, 1) < \infty$. This implies $\delta_r\langle f, S^*_{\beta 1}; M\rangle = 0$ and proves $\delta\langle f, S^*_{\beta 1}\rangle = 0$.

6.7. We are going to prove some propositions which connect the δ-entropy $\delta\langle f, S^*\rangle$, where S^* is of the form $\langle Q, d \wedge t, \mu[d]\rangle$, $t > 0$, with the behavior of $h(\langle f^n X, d\rangle, t)$, $X \subset Q$, for $n \to \infty$, and also, for $S = R^p$, with the behavior of $\lambda(f^n X)$, where λ is the Lebesgue measure.

6.8. Lemma. *Let $S = \langle Q, d\rangle$ be a* D*-space. Let $t > 0$. Then* (1) $\delta\langle Q, d \wedge t\rangle \geq t \cdot h(S, t)$, (2) $\delta\langle Q, d \wedge t\rangle \leq t \cdot h(S, t) + t + C_S(t) \cdot \delta\langle Q, e\rangle < \infty$.

Proof: Put $e = d \wedge t$. — I. To prove (1), we can assume that $\delta\langle Q, e\rangle < \infty$. Let $c > \delta\langle Q, e\rangle$ be arbitrary. There exists a d.e. $\mathscr{S}(S_a : a \in A)$ of $\langle Q, e\rangle$ such that $\delta(\mathscr{S}) < t$ and $e(\mathscr{S}'') < t$. Clearly, there is $B \subset A$ such that $\mathscr{T} = (S_x : x \in B)$ is a d.e. of $\langle Q, e\rangle$, $e(\mathscr{T}'') < t$, and $e(T_x) \geq t$ whenever $x \in B'$. For every $b \in B'$, we have $|b| < c/t$, since otherwise we would have $\delta(\mathscr{T}) \geq c$, hence $\delta(\mathscr{S}) \geq c$. The inequality $|b| < c/t$ for $b \in B'$ implies card $B'' < \exp(c/t)$. Since $e(\mathscr{T}'') < t$ and therefore $d(\mathscr{T}'') < t$, we get $h(S, t) < c/t$, which proves the first assertion. — II. To prove (2), we can assume that $h(S, t) < \infty$. Choose a cover $\mathscr{U}$ of Q such that $d(\mathscr{U}) < t$ and card $\mathscr{U} = \exp(h(S, t))$. Then there exists a d.e. $\mathscr{S} = (S_a : a \in A)$ of S such that $\mathscr{S}'' = \mathscr{U}$ and $|a| \leq \mathrm{Log}\,(\text{card } \mathscr{U}) + 1$ for every $a \in A$. Put $T_a = \langle Q_a, e\rangle$, $\mathscr{T} = (T_a : a \in A)$. Then $\mathscr{T}$ is a d.e. of $\langle Q, e\rangle$ and $e(\mathscr{T}'') \leq t$. This implies $\delta(\mathscr{T}) \leq t \cdot \mathrm{Log}\,(\text{card } \mathscr{U}) + C_S(t) < t \cdot h(S, t) + t + C_S(t)$. □

6.9. Proposition. *Let $\langle Q, d\rangle$ be a* D*-space; put $\mu = \mu[d]$, $S = \langle Q, d, \mu\rangle$. Let $\langle f, S\rangle$ be an expanding and continuous self-mapping. Let $t > 0$ and let $C_S(t) < \infty$. Put $e = d \wedge t$, $S^* = \langle Q, e, \mu\rangle$. Then, for every totally bounded $X \subset Q$, $\overline{\delta}\langle f, S^*; X\rangle = t \cdot \overline{\lim}(n^{-1}h(\langle f^n X, d\rangle, t))$.*

Proof: We have $\overline{\delta}\langle f, S^*; X\rangle = \overline{\lim}(n^{-1}\delta\langle X, e^{(n)}, \mu^{(n)}\rangle)$. Since f is expanding and continuous, we have $e^{(n)} = e^n$, $\mu^{(n)} = \mu$, hence $\overline{\delta}\langle f, S^*; X\rangle = \overline{\lim}(n^{-1}\delta\langle X, e^n\rangle)$. It is easy to see that f^n transforms e^n on X into e on $f^n X$, and therefore $\overline{\delta}\langle f, S^*; X\rangle = \overline{\lim}(n^{-1}\delta\langle f^n X, e\rangle)$. By 6.8, with Q replaced with $f^n X$, this proves the proposition. □

6.10. Fact. Let $\langle Q, d\rangle$ be a D-space; put $\mu = \mu[d]$, $S = \langle Q, d, \mu\rangle$. Let $t > 0$, $C_S(t) < \infty$. Let $\langle f, S\rangle$ be an expanding continuous self-mapping. Put $S^* = \langle Q, d \wedge t, \mu\rangle$. Then, for any $X \subset Q$, $\overline{\delta}\langle f, S^*; X\rangle = \overline{\delta}\langle f, S^*; fX\rangle$.

Proof: Clearly, X is t.b. iff so is fX. Hence it is sufficient to prove the equality for X totally bounded. By 6.9, $\overline{\delta}\langle f, S^*; X\rangle = t \cdot \overline{\lim}(n^{-1}h(\langle f^n X, d\rangle, t))$, $\overline{\delta}\langle f, S^*; fX\rangle = t \cdot \overline{\lim}(n^{-1}h(\langle f^n(fX), d\rangle, t))$, from which the assertion follows. □

6.11. Fact. Let $\langle Q, d\rangle$ be a δ-regular D-space. Put $\mu = \mu[d]$, $S = \langle Q, d, \mu\rangle$. Let $t > 0$, $C_S(t) < \infty$. Put $S^* = \langle Q, d \wedge t, \mu\rangle$. Then (1) for any $X, Y \subset Q$, $\delta\langle f, S^*; X \cup Y\rangle = \delta\langle f, S^*; X\rangle \vee \delta\langle f, S^*; Y\rangle$, (2) if $X \subset Q$ and $n \in N$, $n > 0$, then $\delta\langle f, S^*; \bigcup(f^k X : 0 \leq k < n)\rangle = \delta\langle f, S^*; X\rangle$.

Proof: For $Z \subset Q$, put $\psi(Z) = \overline{\lim}(h(\langle f^n Z, d\rangle, t)/n)$. To prove the first assertion for the case of $X \cup Y$ totally bounded, it is, by 6.9 and 4.15, sufficient to show that $\psi(X \cup Y) = \psi(X) \vee \psi(Y)$. Evidently, $\nu(X \cup Y, t) \leq \nu(X < t) + \nu(Y, t)$ (for ν see 2.1); hence $h(X \cup Y, t) \leq 1 + h(X, t) \vee h(Y, t)$, which proves $\psi(X \cup Y) = \psi(X) \vee \psi(Y)$. If $X, Y \subset Q$ are arbitrary, then, writing $\varphi(Z)$ instead of $\delta\langle f, S^*; Z\rangle$, we have $\varphi(M) = \varphi(M \cap X) \vee \varphi(M \cap Y)$ for every totally bounded $M \subset X \cup Y$. This implies $\varphi(X \cup Y) = \varphi(X) \vee \varphi(Y)$. — The second assertion is an easy consequence of (1), 6.10, and 4.15. □

6.12. Lemma. *Let $\langle Q, d\rangle$ be a complete and locally compact metric space; put $d = d[\varrho]$, $\mu = \mu[d]$, $S = \langle Q, d, \mu\rangle$. Assume that $\langle Q, d\rangle$ is δ-regular; let $t > 0$, $C_S(t) < \infty$; put $S^* = \langle Q, d \wedge t, \mu\rangle$. Let $\langle f, S\rangle$ be an expanding continuous self-mapping. Assume that $\bigcup(f^k G : k \in N) = Q$ for every open $G \neq \emptyset$. Then $\delta\langle f, S^*\rangle = \delta\langle f, S^*; X\rangle$ for every totally bounded non-meager $X \subset Q$.*

Proof: Let $X_i \subset Q$, $i = 0, 1$, be t.b. and non-meager. It is easy to show that $\delta\langle f, S^*; \overline{X}_i\rangle = \delta\langle f, S^*; X_i\rangle$. Let G_i denote the interior of $\overline{X}_i$. Since, clearly, f is a surjective homeomorphism, all $f^k G$ are open and therefore, for some n, $\overline{X}_i \subset \bigcup(f^k G_{1-i} : 0 \leq k < n)$, $i = 0, 1$. By 6.11, this proves $\delta\langle f, S^* : X_0\rangle = \delta\langle f, S^*; X_1\rangle$, from which the assertion follows. □

6.13. In what follows, λ denotes the Lebesgue measure on R^p, $p \in N$ fixed, $p > 0$; $\lambda(X)$ denotes the (outer) Lebesgue measure of X. The letter ϱ denotes the metric induced by the norm $|x| = \max\{|x_i| : 0 \leq i < p\}$ for $x = (x_i : 0 \leq i < p) \in R^p$. We put $d = d[\varrho]$, $S = \langle R^p, d, \mu\rangle$, $S^* = \langle R^p, d \wedge 1, \mu\rangle$, where $\mu = \mu[d]$.
Remark: the results stated below remain valid, with appropriate changes, if ϱ is replaced by any metric induced by a norm on R^p.

Proposition. *If f is an expanding continuous self-mapping of $S = \langle R^p, d, \mu\rangle$, then* (1) *for any $X \subset R^p$,*

$$\overline{\lim}(n^{-1} \log \lambda(f^n X)) \leq \delta\langle f, S^*; X\rangle;$$

(2) *if X is totally bounded non-meager, then*

$$\overline{\lim}(n^{-1} \log \lambda(f^n X)) \leq \delta\langle f, S^*\rangle.$$

Proof: It is easy to see that $\log \lambda(Y) \leq h(\langle Y, d\rangle, 1)$ for every $Y \subset R^p$. Hence $\overline{\lim}(n^{-1} \log \lambda(f^n X)) \leq \overline{\lim}(n^{-1} h(\langle f^n X, d\rangle, 1))$. Since, clearly, $\langle R^p, d\rangle$ is δ-regular and $C_S(1) < \infty$, we have, by 6.11, $\overline{\lim}(n^{-1} h(\langle f^n X, d\rangle, 1)) = \delta\langle f, S^*; X\rangle$. The second assertion follows from 6.12. □

6.14. Recall that, for a mapping $g : R^r \to R^r$, the Jacobian of g, denoted $J(g)$, is defined as follows: if all partial derivatives $\partial g_i/\partial x_j$ exist and are continuous, then $J(g)$ is the determinant of the matrix $[\partial g_i/\partial x_j]_{i,j=0,\dots,p-1}$.

6.15. Proposition. *If f is an expanding self-mapping of $\langle R^p, d, \mu\rangle$ such that the Jacobian $J(f)$ exists, then*
(1) *for any measurable $X \subset R^k$,*

$$\overline{\lim}(n^{-1}\log(\int_X |J(f^n)|\,d\lambda)) \le \delta\langle f, S^*; X\rangle.$$

(2) *if X is totally bounded, non-meager measurable, then*

$$\overline{\lim}(n^{-1}\log(\int_X |J(f^n)|\,d\lambda)) \le \delta\langle f, S^*\rangle.$$

PROOF: This is an immediate consequence of 6.13 and the well-known equality $\lambda(gY) = \int_Y |J(g)|\,d\lambda$. □

6.16. We will not investigate the conditions under which the inequalities can be replaced, in the above proposition, by equalities. We state only the following simple result.

Fact. Let f be an expanding self-mapping of $\langle R^p, d, \mu\rangle$ such that the Jacobian $J(f)$ exists. Assume that, with $B = \{x \in R^p : |x| \le 1\}$, the set of all $\lambda\{x \in R^p : \varrho(x, f^nB) \le 1\}/\lambda(f^nB)$, $n \in N$, is bounded. Then $\delta\langle f, S^*\rangle = \overline{\lim}(n^{-1}\log\lambda(f^nB))= \overline{\lim}(n^{-1}\log(\int_B |J(f^n)|\,d\lambda))$.

PROOF: We can assume that all $h(\langle f^nX, d\rangle, 1)$ are finite. Let $n \in N$. There is a set $M \subset f^nB$ such that card $M = \exp(h^*(\langle f^nB, d\rangle, 1/2))$, $\varrho(x,y) > 1/2$ for $x, y \in M$, $x \ne y$. For $x \in M$, put $A_x = \{z \in R^p : \varrho(z,x) \le 1/4\}$. Then A_x are disjoint, $A_x \subset \{z \in R^p : \varrho(z, f^nB) \le 1/4\}$, $\lambda(A_x) = \exp(-p)$. This implies $h^*(\langle f^nB, d\rangle, 1/2) - p \le \log\lambda\{z \in R^p : \varrho(z, f^nB) \le 1/4\}$, hence, by 2.2 (D), $h(\langle f^nB, d\rangle, 1) \le \log\lambda\{z \in R^p : \varrho(z, f^nB) \le 1/4\} + p$. By the assumptions made, we obtain $\overline{\lim}(n^{-1}h(\langle f^nB, d\rangle, 1)) \le \overline{\lim}(n^{-1}\log\lambda(f^nB))$. By 6.9 and 4.15, this implies $\delta\langle f, S^*; B\rangle \le \overline{\lim}(n^{-1}\log(\int_B |J(f^n)|\,d\lambda))$, which proves the assertion by 6.13 and 6.12. □

6.17. Proposition. *Let $f : R^p \to R^p$ be an expanding linear mapping. Then $\delta\langle f; R^p, d\wedge 1, \mu\rangle = \log|J(f)|$.*

PROOF: I. Evidently, $|f^{-1}|$, the norm of f^{-1}, does not exceed 1. First we assume that $|f^{-1}| = c < 1$. Put $B = \{x \in R^p : |x| \le 1\}$; for $X \subset R^p$, $\varepsilon > 0$, put $V(X,\varepsilon) = \{y \in R^p : \varrho(y, X) \le \varepsilon\}$. Let $\vartheta > 0$. Choose $m \in N$ such that $c^m < \vartheta$. Let $n \ge m$. If $x \in V(f^nB, 1)$, then there is $z \in f^nB$ with $|z - x| \le 1$. We have $|f^{-n}z - f^{-n}x| \le c^n$, $f^{-n}z \in B$, hence $f^{-n}x \in V(B, \vartheta)$. This implies $V(f^nB, 1) \subset f^n(V(B,\vartheta))$, from which we obtain $\lambda(V(f^nB,1)) \le \lambda(f^n(V(B,\vartheta))) = |J(f^n)|\lambda(V(B,\vartheta)) \le |J(f^n)|(1+2\vartheta)^p\lambda(B) = (1+2\vartheta)^p\lambda(f^nB)$. From this inequality we get, by 6.16, $\delta\langle f, S^*; B\rangle = \log|J(f)|$. — II. Let $|f^{-1}| = 1$. For $t \geq 1$ put $f_t = tf$. By I, $\delta\langle f_t, S^*\rangle = \log|J(f_t)|$ for $t > 1$. Hence, to prove $\delta\langle f, S^*; B\rangle = \log|J(f)|$, it is sufficient to show that $\delta\langle f_t, S^*; B\rangle \to \delta\langle f, S^*; B\rangle$ for $t \to 1$. By 6.9, $\delta\langle f_t, S^*; B\rangle = \overline{\lim}(n^{-1}h(\langle t^n f^n B, d\rangle, 1)) = \overline{\lim}(n^{-1}h(\langle f^n B, d\rangle, t^{-n})) \leq \overline{\lim}(n^{-1}h(\langle f^n B, d\rangle, 1)) + \overline{\lim}(n^{-1}h(B, t^{-n}))$. Clearly, $h(B, t^{-n}) \leq p\log(t^n + 1)$. This proves, by 6.9, that $\delta\langle f_t, S^*; B\rangle \leq \delta\langle f, S^*; B\rangle + p\log t$. Evidently, $\delta\langle f^t, S^*; B\rangle \leq \delta\langle f_t, S^*; B\rangle$; hence we get $\delta\langle f_t, S^*; B\rangle \to \delta\langle f, S^*; B\rangle$, which proves the proposition. □

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