Two results on a partial ordering of finite sequences

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Abstract. In the first part of the paper we are concerned about finite sequences (over arbitrary symbols) u for which $Ex(u,n) = O(n)$. The function $Ex(u,n)$ measures the maximum length of finite sequences over n symbols which contain no subsequence of the type u. It follows from the result of Hart and Sharir that the containment $ababa \prec u$ is a (minimal) obstacle to $Ex(u,n) = O(n)$. We show by means of a construction due to Sharir and Wiernik that there is another obstacle to the linear growth.

In the second part of the paper we investigate whether the above containment of sequences is wqo. It is trivial that it is not but we show that the smaller family of sequences whose alternate graphs contain no k-path is well quasiordered by that containment.

Keywords: : Davenport-Schinzel sequence, extremal problem, linear growth, minimal obstacle to linearity, well quasiordering, alternate graph

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1. Introduction.

Throughout this paper $\mathcal{S}$ denotes the set of all finite sequences over a fixed infinite universum of symbols S. For any sequence u of $\mathcal{S}$ we use $S(u)$ to denote the set of all symbols occurring in u. The quasiordering $(\mathcal{S}, \prec)$ which is the main subject of the paper is defined as follows. We say that a sequence $u = a_1a_2\dots a_m$ is *contained* in another sequence $v = b_1b_2\dots b_n$ and write $u \prec v$ iff there is an increasing mapping $f : \{1,\dots,m\} \to \{1,\dots,n\}$ and an injection $g : S(u) \to S(v)$ such that $g(a_i) = b_{f(i)}$ for all $i = 1,\dots,m$. In other words: some subsequence of v differs from u only in names of its symbols.

There are at least two reasons for investigating $(\mathcal{S}, \prec)$: one is that finite sequences (words) belong to the most basic mathematical concepts and the second is that so called Davenport-Schinzel sequences (from now DS sequences) which play an important role in computational geometry can be naturally defined in terms of $\prec$. Our results on $(\mathcal{S}, \prec)$ are:

1) Let $Ex(u,n)$ be a general extremal function measuring the maximum length of sequences over n symbols not containing a forbidden sequence u and let Lin be the set of all sequences u for which $Ex(u,n) = O(n)$. The elements of Lin are called *linear* sequences, the nonelements are called *nonlinear* sequences. Exact definitions are given at the beginning of Section 2. It is easy to show that the set Lin is a lower ideal in $(\mathcal{S}, \prec)$. Hence Lin is completely determined by the set B of all minimal (to $\prec$) nonlinear sequences. The result of Hart and Sharir [6] yields $ababa \in B$. We show that the construction [15] of Sharir and Wiernik implies

Theorem A. *There are at least two elements in B: $u_1 = ababa$ and $u_2 \prec abcbadadbcd$.*

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Problem. Is B finite?

2) If $(\mathcal{S}_k, \prec)$ were well quasiordering then B as well as all antichains in $(\mathcal{S}_k, \prec)$ would be finite. But there are infinite antichains in $(\mathcal{S}_k, \prec)$ and hence the wqo method fails. Nevertheless we prove

Theorem B. *The quasiordering $(\mathcal{S}_k, \prec)$ is wqo for any k.*

Here $\mathcal{S}_k$ consists of all sequences $u \in \mathcal{S}$ with the property that the graph $G(u)$ contains no path of the length k. The vertex set of $G(u)$ is $S(u)$, $\{a, b\}$ is an edge of $G(u)$ iff $abab$ or $baba$ is a subsequence of u. Theorem B implies

Consequence. $B \cap \mathcal{S}_k$ is finite for any $k \geq 1$.

2. Minimum nonlinear sequences.

For any sequence $u = a_1 a_2 \dots a_m$ of $\mathcal{S}$ the symbol $\|u\|$ denotes the cardinality of $S(u)$ and $|u|$ stands for the length of u. Clearly $\|u\| \leq |u|$ for all u. The sequence u is called *k-regular* if $a_i = a_j, i > j$ implies $i - j \geq k$. We define:

$$Ex(u, n) = \max\{|v| \mid u \not\prec v, \|v\| \leq n, v \text{ is } \|u\|\text{-regular}\}.$$

The function $Ex(u, n)$ was introduced in [1] and investigated in [9].

The primary question of Davenport and Schinzel [3] was (though in different notation) the growth rate of the functions $Ex(ababa\dots, n)$ where $ababa\dots$ is a fixed alternating sequence over two symbols.

They proved $Ex(abab, n) = 2n-1$ (this is not difficult and is recommended to the reader as an exercise), $Ex(ababa, n) = O(n \log n / \log\log n)$ and $Ex(ababa\dots, n) = O(n.exp(\sqrt{n}))$ for any fixed alternating sequence ([3] and [4]). This was later improved by Szemerédi [14] to $O(n \log^* n)$ but no result excluding $Ex(ababa\dots, n) = O(n)$ was known.

Hart and Sharir [6] proved that $Ex(ababa, n) = \Theta(n.\alpha(n))$ where $\alpha(n)$ is the functional inverse to the Ackermann function and grows to infinity extremely slowly. Their method was later generalized and sharp upper and lower bounds on the functions $Ex(ababa\dots, n)$ were found [2], [13]. In [8] their method was used to obtain a strong upper bound of this kind for any function $Ex(u, n)$.

We recall two lemmas of [9].

Lemma 2.1. *$Ex(u, n)$ is finite for any fixed sequence u and any integer $n \geq 1$.*

Lemma 2.2. *The set Lin is a lower ideal in $(\mathcal{S}, \prec)$: if $v \in Lin$ and $u \prec v$ then $u \in Lin$.*

According to [6], $ababa \notin Lin$. On the other hand, it is easy to see that $baba$, $aaba$, $abba$ and $abaa$ are linear. Thus $ababa \in B$. In the rest of this section we show that there is another obstacle to linearity. The powerful tool that will be used is a simple but ingenious construction of [15]. We recall it briefly and then we prove Theorem A.

We shall define, by double induction on the integers $i, j \geq 1$, 2-regular sequences $u(i,j) \in \mathcal{S}$. A sequence $v \in \mathcal{S}$ is called *j-block* if $v = x_1 x_2 \dots x_j$ for j distinct symbols x_i. Sequences $u = u(i,j)$ satisfy $u = b^1 c^1 b^2 c^2 \dots b^k c^k$ where any b^r is a j-block, c^r is an intermediate (possibly empty) sequence and $k = k(i,j)$ is an integer valued function which will be defined later. Moreover, it is required

$$S(u) = \bigcup_{r=1}^{k} S(b^r) \text{ and } S(b^1 c^1 \dots b^{r-1} c^{r-1}) \cap S(b^r) = \emptyset, r = 2 \dots k.$$

Observe that $\|u(i,j)\| = j.k(i,j)$. Let $b^r = b_0^r y^r$ where y^r is the last occurrence in the block b^r. Let

$$d(u) = b_0^1 y^1 y^1 c^1 b_0^2 y^2 y^2 c^2 \dots b_0^k y^k y^k c^k$$

denote the sequence obtained from u by doubling last occurrences in all j-blocks. The construction proceeds as follows.

1. If $i = 1, j \geq 1$ then $u(1,j) = b^1 = x_1 x_2 \dots x_j$ and $k(1,j) = 1$.
2. If $i > 1, j = 1$ then $u(i,1) = u(i-1,2)$ and the only change is that the 2-blocks in $u(i-1,2)$ are viewed now as pairs of neighbouring 1-blocks in $u(i,1)$. Hence $k(i,1) = 2.k(i-1,2)$.
3. If $i > 1, j > 1$ then put $J = k(i,j-1)$, $K = k(i-1,J)$, $u = u(i,j-1)$ and $v = u(i-1,J) = B^1 C^1 \dots B^K C^K$ where$B^r = x_1^r \dots x_J^r$ is the r-th J-block of v. The sequences $u_1^*, u_2^*, \dots, u_K^*$ are K disjoint copies of the sequence $d(u)$, all are disjoint of v. Let $u_r^* = b_0^1 y^1 y^1 c^1 \dots b_0^J y^J y^J c^J$ where $b_0^s y^s$ is the copy of the s-th $(j-1)$-block of u. Then define

   $$u_r = b_0^1 y^1 x_1^r y^1 c^1 \dots b_0^J y^J x_J^r y^J c^J.$$

   The J old $(j-1)$-blocks in u_r^* and $x_1^r, x_2^r, \dots, x_J^r$ yield J new j-blocks in u_r. Finally

   $$u(i,j) = u_1 x_J^1 C^1 u_2 x_J^2 C^2 \dots u_K x_J^K C^K$$

   and the j-blocks in $u(i,j)$ are the JK new blocks in $u_1, \dots, u_K$. Hence

   $$k(i,j) = J.K = k(i,j-1).k(i-1,k(i,j-1)).$$

   Briefly spoken, sufficiently many copies of $d(u)$ and a copy of $d(v)$ are merged together so that the order is preserved and so that the resulting sequence is again 2-regular.

For the proof of the following lemma we refer to [15].

Lemma 2.3. *$|u(i,j)|/\|u(i,j)\| > i - 2/j$ for all $i, j \geq 1$. Moreover: there is an increasing sequence $\{j_i\}_{i=1}^{\infty}$ of integers such that $|u(i,j_i)| \geq c.\|u(i,j_i)\|.\alpha(\|u(i,j_i)\|)$ for $i = 1, 2 \ldots$ and an absolute constant $c > 0$.*

Suppose $u \in \mathcal{S}$ is a sequence. We define the digraph $D(u) = (V, E)$ by $V = S(u)$ and $(a, b) \in E$ iff there is a b-occurrence in u which is not the first b-occurrence in u and which lies between two a-occurrences. Briefly: either $baba$ or $abba$ is a subsequence of u. Now we generalize the argument of Sharir and Wiernik (they considered only the case $w = ababa$).

Theorem 2.4. *Suppose $w \in \mathcal{S}$ is 2-regular and such that the digraph $D(w)$ is strongly connected. Then w is nonlinear and moreover $Ex(w, n) = \Omega(n.\alpha(n))$.*

Proof: We prove by double induction that $w \not\prec u(i,j)$ for all $i, j \geq 1$. Cases $i = 1$ or $j = 1$ are obvious. Now consider the sequence $u(i,j) = u_1 x_J^1 C^1 u_2 x_J^2 C^2 \ldots u_K x_J^K C^K$ where i and j are greater than 1. We use the above notation. Suppose on the contrary that $w \prec u(i,j)$ and that w^* is the subsequence of $u(i,j)$ which differs from w only in names of symbols. For any $x \in S(w^*)$ the symbol x is an element either of some $S(u_r^*)$ or of $S(d(v))$. In the former case x is called local and in the latter case global.

It is an easy observation that if (a, b) is an edge in $D(w^*) = D(w)$ and a is local then b is local too and all b-occurrences appear in the same u_r^* as those of a. Because of the strong connectivity of $D(w)$ either all symbols in $S(w^*)$ are local or all of them are global. In the former case w^* is a subsequence of some u_r^*, thus $w \prec d(u)$ and $w \prec u = u(i, j-1)$ which is a contradiction. In the latter case $w \prec d(v)$ and $w \prec v = u(i-1, J)$ which is a contradiction again.

We are not done yet because $u(i,j)$ are 2-regular and we need them to be $\|w\|$-regular. Let $k = \|w\|$. The sequence $ababa$ clearly satisfies the hypothesis of the theorem and thus $ababa \not\prec u(i,j)$ for all i and j. Let $Ex(ababa, k-1) = h$ (Lemma 2.1). We apply on $u(i,j) = a_0 a_1 \ldots a_m$ the following greedy procedure.

First we put $v(i,j) = a_0$ and we try to add elements a_i to $v(i,j)$. If the sequence $v(i,j)a_i$ is k-regular then we put $v(i,j) := v(i,j)a_i$ and we try to add a_{i+1}. If not then a_i is omitted and we continue also with a_{i+1}. We obtain a k-regular subsequence $v(i,j)$ of $u(i,j)$ satisfying

$$|v(i,j)| \geq \frac{|u(i,j)|}{h+1}$$

because any interval in $u(i,j)$ consisting of omitted elements has length at most h. The previous lemma implies $Ex(w, n) = \Omega(n \cdot \alpha(n))$ for infinitely many values n. It is not too difficult to prove that $Ex(w, n) = \Omega(n \cdot \alpha(n))$ for all n, one has to use the superaditivity of $Ex(w, n)$ and the definition of the numbers $\{j_i\}_{i=1}^{\infty}$ of the previous lemma. See [2] for similar calculation. □

Theorem A. *There are at least two elements in B: $u_1 = ababa$ and $u_2 \prec abcbadadbcd$.*

Proof: We know already that $ababa \in B$. Now consider the sequence $v_1 = abcbadadbcd$. Again $Ex(v_1, n) = \Omega(n \cdot \alpha(n))$ according to Theorem 2.4 because there is a Hamiltonian cycle $abdc$ in $D(v_1)$. But an easy check shows that $ababa \not\prec v_1$. Hence there must be a sequence $u_2 \prec v_1, u_2 \neq ababa, u_2 \in B$. $\square$

3. $(\mathcal{S}, \prec)$ **and wqo.**

First we demonstrate an infinite antichain in $(\mathcal{S}, \prec)$. Let $u \in \mathcal{S}$ be a sequence. The graph $G = (V, E)$ is defined by $V = S(u)$ and by $\{a, b\} \in E$ iff $abab$ or $baba$ is a subsequence of u. It is well-known that there are infinite antichains in the set of all finite graphs $(\mathcal{G}, \subset)$ ordered by the relation "be a subgraph": for instance all i-cycles $C_i, i \geq 3$. It is an immediate observation that $u \prec v$ implies $G(u) \subset G(v)$. The fact that $(\mathcal{G}, \subset)$ is not wqo reflects back to $(\mathcal{S}, \prec)$:

$$u_3 = abacbcac, u_4 = abacbcdcdad, u_5 = abacbcdcdedeae, \ldots$$

is an infinite antichain in $(\mathcal{S}, \prec)$ because $G(u_i) = C_i$. However, it is not difficult to prove [12] that the smaller family $(\mathcal{G}_k, \subset)$, where $\mathcal{G}_k$ consists of all k-path free graphs (no path of k edges), is wqo. It is interesting that this property reflects back to $(\mathcal{S}, \prec)$ as well. We now recall some things about wqo and after that Theorem B will be proved. For the proofs and for more basics we refer to [10].

Any binary relation $(Q, \leq_Q)$ which is transitive and reflexive is called a *quasiordering* or, shortly, *qo*. Notation $x <_Q y$ means that $x \leq_Q y \& y \not\leq_Q x$. A qo $(Q, \leq_Q)$ is a *well quasiordering* or, shortly, *wqo* if it has the property characterized by the following lemma.

Lemma 3.1. *Suppose $(Q, \leq_Q)$ is a qo. Then the following conditions are equivalent.*

1. *For any infinite sequence $(q_0, q_1, \ldots) \subseteq Q$ there are indices $i < j$ such that $q_i \leq_Q q_j$.*
2. *For any infinite sequence $(q_0, q_1, \ldots) \subseteq Q$ there are indices $0 \leq i_0 < i_1 < \ldots$ such that $q_{i_0} \leq_Q q_{i_1} \leq_Q \ldots$.*
3. *There is no strictly descending infinite chain $x_0 >_Q x_1 >_Q \ldots$ in Q and no infinite antichain.*

Sequences satisfying 1. are called *good*, other sequences are called *bad*. Thus the definition of wqo can be stated in this form: a qo$(Q, \leq_Q)$ is wqo iff there is no bad (infinite) sequence in Q. A strict partial ordering $(Q, <^*)$ is called *well founded* iff there are no infinite descending chains in $(Q, <^*)$. We say that $(Q, <^*)$ is *stronger* than a qo $(Q, \leq_Q)$ if $x \leq_Q y$ whenever $x <^* y$. We prove Theorem B by means of the following fundamental lemma.

Lemma 3.2 Nash-Williams [11]**.** *Suppose a well founded strict partial ordering $(Q, <^*)$ is stronger than a qo $(Q, \leq_Q)$ which is not wqo. Then there is an infinite sequence $A = (q_0, q_1, \ldots) \subseteq Q$ such that*

1. *A is bad in $(Q, \leq_Q)$.*
2. *$(W_A, \leq_Q)$ is wqo where $W_A = \{x \in Q \mid x <^* q_i \text{ for some } i\}$.*

Sequence A is called a minimum bad sequence.

For finite structures, $<_Q$ is usually well founded and one can put $<^*=<_Q$. This is the case here and in sequel we take tacitly $<^*=<_Q$. Now we give an overview of basic constructions for creating new wqo's. Suppose $(Q_0, \leq_{Q_0})$ and $(Q_1, \leq_{Q_1})$ are qo. The product qo $(Q_0 \times Q_1, \leq_{pr})$ is defined by

$$(a_0, a_1) \leq_{pr} (b_0, b_1) \text{ iff } a_i \leq_{Q_i} b_i \text{ for } i = 0, 1.$$

The sum qo $(Q_0 + Q_1, \leq_+)$ is defined by

$$Q_0 + Q_1 = (Q_0 \times \{0\}) \cup (Q_1 \times \{1\}),\ (a, i) \leq_+ (b, j) \text{ iff } i = j \text{ and } a \leq_{Q_i} b.$$

An easy consequence of Lemma 3.1 is that if $(Q_i, \leq_{Q_i}), i = 0, 1$ are wqo then both the product qo and the sum qo are wqo as well.

We shall use in sequel the wqo $N = (N, \leq)$ consisting of positive integers with the standard order and the trivial discrete wqo T_n consisting of n elements which are mutually incomparable.

Suppose $(Q, \leq_Q)$ is a qo. The elements of the structure $(SEQ(Q), \leq_H)$ are all finite sequences over Q. More specifically, elements of $SEQ(Q)$ are of the form (I, p) where I is a finite linear ordering and $p : I \to Q$ is a mapping.

We put $(I, p) \leq_H (J, r)$ (Higman ordering) iff there is an increasing injection $f : I \to J$ such that $p(x) \leq_Q r(f(x))$ for any $x \in I$. We shall need the following classical result which easily follows from Lemma 3.2.

Theorem 3.3 Higman [7]**.** *If $(Q, \leq_Q)$ is wqo then $(SEQ(Q), \leq_H)$ is wqo as well.*

To prove Theorem B it is convenient to work with a generalization of $(\mathcal{S}, \prec)$. Let $(Q, \leq_Q)$ be a qo. Recall that S is a fixed infinite universum of symbols.

Definition 3.4. *We define $R(Q)$ as consisting of the triples $u = (I, p, q)$ where I is a finite linear ordering and $p : Dom(p) \to S$ and $q : Dom(q) \to Q$ are two labelings whose domains partition I. The qo in $(R(Q), \leq_R)$ is defined by*

$$(I, p, q) \leq_R (J, r, s) \text{ iff } (Dom(p), p) \prec (Dom(r), r) \text{ via } f\lceil Dom(p) \text{ and}$$
$$(Dom(q), q) \leq_H (Dom(s), s) \text{ via } f\lceil Dom(q)$$

for some increasing injection $f : I \to J$.

We use $S(u)$ to denote (V, E) is defined, for an element $u = (I, p, q)$ of $R(Q)$, by $V = S(u) = Rng(p)$ and $\{a, b\} \in E$ iff $p(x) = p(z) = a, p(y) = p(t) = b$ or $p(x) = p(z) = b, p(y) = p(t) = a$ for some four elements $x < y < z < t$ of I. The set $R(Q, k)$ consists of all triples u of $R(Q)$ for which $G(u) \in \mathcal{G}_k$. To prove Theorem B we shall need two easy graph lemmas.

Lemma 3.5. *If $G = (V, E)$ is a connected graph whose longest path P has length k then the graph $H = G\lceil(V(G)\backslash V(P))$ belongs to $\mathcal{G}_k$.*

Proof: Suppose Q is a k-path in H and T is a P-Q path joining P and Q in G. These three paths contain obviously a $(2.\lceil\frac{k}{2}\rceil + 1)$-path which is a contradiction.

□

Suppose $u = (I, p, q) \in R(Q)$ is a sequence such that $\min I \in Dom(p)$, let $p(\min I) = v$. Let H be the component in $G(u)$ containing v. We define the *graph decomposition* of I as $I = J^0 \cup K^0 \cup J^1 \cup K^1 \cup \ldots \cup J^r \cup K^r$ where $J^0 < K^0 < J^1 < K^1 < \ldots$ and any J^j is a maximum nonempty interval in I such that $J^j \subset Dom(p)$ and $p(x) \in V(H)$for any $x \in J^j$. Let u^j (resp. v^j) be u restricted on I^j (resp. K^j).

Lemma 3.6. *Let u and the graph decomposition be as above. Then $S(v^j)$ are mutually disjoint for $j = 0, 1, \ldots, r$.*

PROOF: Let $w \in S(v^i) \cap S(v^j)$ for some $0 \leq i < j \leq r$. Consider the subsets of $V(H)$

$$X = \bigcup_{a=0}^{a=i} S(u^a) \cup \bigcup_{a=j+1}^{a=r} S(u^a) \text{ and } Y = \bigcup_{a=i+1}^{a=j} S(u^a).$$

There exists a $t \in X \cap Y$ otherwise there would be no edge in $G(u)$ between X and Y. Thus $\{t, w\}$ is an edge and $w \in V(H)$ which is a contradiction. $\square$

We prove Theorem B in the following general form.

Theorem 3.7. *$(R(Q, k), \leq_R)$ is wqo for any wqo $(Q, \leq_Q)$ and any positive integer k.*

PROOF: We shall proceed by double induction on k and $(Q, \leq_Q)$. Let $k = 1$ and let $(Q, \leq_Q)$ be an arbitrary wqo. Suppose $(R(Q, 1), \leq_R)$ is not wqo. Consider the minimum bad sequence

$$A = (u_0, u_1, \ldots) \subset R(Q, 1), \quad u_i = (I_i, p_i, q_i)$$

which is ensured by Lemma 3.2. Denote $x_i = \min I_i$. One can suppose that either $x_i \in Dom(q_i)$ for all i or $x_i \in Dom(p_i)$ for all i. In the former case consider triples $v_i = (J_i, p_i^*, q_i^*)$ where$J_i = I_i \backslash \{x_i\}, p_i^* = p_i \lceil J_i$ and $q_i^* = q_i \lceil J_i$. The sequence

$$(\ (q_0(x_0), v_0), (q_1(x_1), v_1), \ldots) \subset Q \times W_A$$

is a good sequence because $Q \times W_A$ is wqo and thus A is good as well which is a contradiction.

In the latter case consider the corresponding graph decomposition $I_i = J_i^0 \cup K_i^0 \cup J_i^1 \cup K_i^1 \cup \ldots \cup J_i^{r_i} \cup K_i^{r_i}$. The component H is now just a single point. Let v_i^j be the restriction of u_i to K_i^j. According to the previous lemma $v_i^j, j = 0, 1, \ldots, r_i$ can be treated independently. We define $s_i = (\{0, 1, \ldots, r_i\}, n_i) \in SEQ(N \times W_A)$ by $n_i(j) = (|J_i^j|, v_i^j)$. The sequence

$$(s_0, s_1, \ldots) \subset SEQ(N \times W_A)$$

is, according to Higman theorem, a good sequence. It is not difficult to see that this implies that A is a good sequence as well. This is a contradiction again. We conclude that $(R(Q, 1), \leq_R)$ is wqo.

Now suppose that $k > 1$, that $(R(Q,k-1), \leq_R)$ is wqo for any wqo $(Q, \leq_Q)$ and that $(R(Q,k), \leq_R)$ is not wqo for some wqo $(Q, \leq_Q)$. Let

$$A = (u_0, u_1, \dots) \subset R(Q,k), u_i = (I_i, p_i, q_i)$$

be a minimum bad sequence. One can suppose that $\min I_i = x_i \in Dom(p_i)$ for all i, the other possibility is treated as for $k = 1$.

Let $H_i \subset S(u_i)$ be the component in $G(u_i)$ which contains $p_i(x_i)$. Let $W_i \subset V(H_i)$ be the vertex set of the longest path in H_i. Consider the graph decomposition $I_i = J_i^0 \cup K_i^0 \cup J_i^1 \cup K_i^1 \cup \ldots \cup J_i^{r_i} \cup K_i^{r_i}$. Let v_i^j be the restriction of u_i to K_i^j.

Again $v_i^j, j = 0, 1, \dots, r_i$ are independent each to the other. The sequence u_i is transformed into the sequence u_i^* in the following manner. Any K_i^j is contracted into one point k_i^j which is labeled by v_i^j. The wqo Q^* is defined by $Q^* = T_k + W_A$. The elements of the trivial wqo T_k are the vertices of W_i. Now they are viewed as labels for the q-labeling. Formally:

$$u_i^* = (I_i^*, p_i^*, q_i^*) \text{ where } I_i^* = J_i^0 \cup \{k_i^0\} \cup J_i^1 \cup \{k_i^1\} \cup \ldots \cup J_i^{r_i} \cup \{k_i^{r_i}\} \text{ and}$$
$$J_i^0 < \{k_i^0\} < J_i^1 < \{k_i^1\} < \dots .$$

Further

$$Dom(p_i^*) = p_i^{-1}(V(H_i) \backslash W_i) \text{ and}$$
$$Dom(q_i^*) = I_i^* \backslash Dom(p_i^*) = p_i^{-1}(W_i) \cup \{k_i^0\} \cup \ldots \cup \{k_i^{r_i}\}.$$

Finally

$$p_i^*(x) = p_i(x),\ q_i^*(x) = p_i(x) \text{ if } x \in p_i^{-1}(W_i) \text{ and } q_i^*(x) = v_i^j \text{ if } x = k_i^j.$$

Clearly, according to the Lemma 3.5, $u_i^* = (I_i^*, p_i^*, q_i^*) \in R(Q^*, k-1)$. The sequence

$$(u_0^*, u_1^*, \dots) \subset R(Q^*, k-1)$$

is good according to the induction hypothesis. This implies that A is good as well contradicting our assumption. □

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