Weakly Picard mappings

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Abstract. In this paper we generalize the well known converse to the contraction principle due to C. Bessaga, dropping the uniqueness of the fixed point from its hypotheses. Some properties of weakly Picard mappings are given.

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1. Introduction.

Let X be a nonempty set and $f : X \to X$ a mapping. We will use the notation

$$F_f := \{x \in X \mid f(x) = x\},$$
$$I(f) := \{A \subset X \mid f(A) \subset A,\ A \neq \emptyset\},$$

and

$$O(f;x) \quad \text{—} \quad \text{the } f\text{-orbit through } x.$$

In order to characterize different kinds of contractivity of mappings, the following new classes of mappings are given in [15] (see also [16]–[19]):

Definition 1. Let (X,d) be a metric space. A continuous mapping $f : X \to X$ is a (strict) Picard mapping if there exists $x^* \in X$ such that $F_f = \{x^*\}$ and $\big(f^n(x_0)\big)_{n\in N}$ converges (uniformly) to x^* for all $x_0 \in X$.

Definition 2. Let (X,d) be a metric space. A continuous mapping $f : X \to X$ is (strict) weakly Picard mapping if $\big(f^n(x_0)\big)_{n\in N}$ converges (uniformly) for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of f.

Definition 3. Let X be a nonempty set. A mapping $f : X \to X$ is a Bessaga mapping if there exists $x^* \in X$, such that

$$F_{f^n} = \{x^*\},$$

for all $n \in N$.

Definition 4. Let X be a nonempty set. A mapping $f : X \to X$ is a Janoš mapping if

$$\bigcap_{n\in N} f^*(X) = \{x^*\}, \text{ with } x^* \in X.$$

It is clear that the Picard and weakly Picard mappings are metric dependent, whereas the Bessaga and Janoš mappings are set-theoretic notions.These definitions have their roots in the following results:

Theorem A (Picard, Banach, Cacciopoli). *Let (X,d) be a complete metric space and $f : X \to X$ an a-contraction, with $0 \le a < 1$. Then*

(i) $F_f = \{x^*\}$;
(ii) $F_f = F_{f^n}$, *for all* $n \in N$;
(iii) $\left(f^n(x_0)\right)_{n\in N}$ *converges to* x^* *for all* $x_0 \in X$.

Theorem B (Bessaga). *Let X be a nonempty set and $f : X \to X$ a mapping such that*

$$F_f = F_{f^n} = \{x^*\},$$

for all $n \in N$. Let $a \in]0,1[$. Then there exists a metric d on X such that

(i) (X,d) *is a complete metric space;*
(ii) $f : (X,d) \to (X,d)$ *is an a-contraction.*

Theorem C (Janoš). *Let (X,d) be a compact metric space and $f : X \to X$ a mapping. We suppose that*

(a) f *is continuous;*
(b) $\bigcap_{n\in N} f^*(X) = \{x^*\}$.

Then for each $a \in]0,1[$ there exists a metric ϱ on X such that:

(i) d *and* ϱ *are equivalent;*
(ii) $f : (X,\varrho) \to (X,\varrho)$ *is an a-contraction.*

Some results on these types of mappings have been given in [15]–[20] (see also [1], [2], [4], [5], [6], [10], [21], [22]). Some open problems were formulated in [17]. Problem 3 in [17] is the following:

Let X be a nonempty set and $f : X \to X$ a mapping such that

$$\emptyset \neq F_f = F_{f^n},$$

for all $n \in N$. Let $a \in]0,1[$. Does there exist a metric d on X such that:

(i) (X,d) is a complete metric space;
(ii) $d(f^2(x), f(x)) \le a \cdot d(x, f(x))$, for all $x \in X$?

The main aim of the paper is to give a positive answer to this problem.

2. Weakly Picard mappings.

Our main result is the following

Theorem 1. *Let X be a nonempty set and $f : X \to X$ a mapping. The following statements are equivalent:*

(i) *there exists a complete metric d on X such that $f : (X,d) \to (X,d)$ is a weakly Picard mapping;*
(ii) $F_f \neq \emptyset$ *and* $F_f = F_{f^n}$, *for all* $n \in N$;
(iii) *there exists a partition of X, $X = \bigcup_{i\in I} X_i$, such that:*
 (1) $X_i \in I(f)$, *for each* $i \in I$;
 (2) $f|_{X_i}$ *is a Bessaga mapping for each* $i \in I$;

(iv) *for each* $a \in]0,1[$, *there exists a complete metric* d *on* X *and a partition of* X, $X = \bigcup_{i\in I} X_i$, *such that:*
(1) $d(f^2(x), f(x)) \leq a \cdot d(x, f(x))$, *for all* $x \in X$;
(2) $X_i \in I(f)$ *and* $\operatorname{card}(F_f \cap X_i) = 1$;
(3) $f|_{X_i} : (X_i, d) \to (X_i, d)$ *is continuous;*

(v) *for each* $a \in]0,1[$, *there exists a complete metric* d *on* X, *such that:*
(1) $d(f^2(x), f(x)) \leq ad(x, f(x))$, *for all* $x \in X$;
(2) *the mapping* $\varphi : (X, d) \to R_+$, $\varphi(x) = d(x, f(x))$ *is* f*-orbitally lower semi-continuous (see* [13]*);*

(vi) *there exists a complete metric* d *on* X *such that* $f|_{\overline{O(f;x)}}$ *is a Picard mapping, for all* $x \in X$.

Proof: (i) $\Rightarrow$ (ii). The definition of weakly Picard mapping implies that $F_f \neq \emptyset$. The convergence of all sequences of successive approximation with the limits in F_f, implies that $F_{f^n} = F_f$, for all $n \in N$.

(ii) $\Rightarrow$ (iii). If $\operatorname{card} F_f = 1$, then f is a Bessaga mapping. We suppose that $\operatorname{card} F_f \geq 2$. Let $x_0 \in F_f$. We take $X_x := f^{-1}\{x\}$, for $x \in F_f \setminus \{x_0\}$, and $X_{x_0} := f^{-1}\{x_0\} \cup (X \setminus \bigcup_{x\in F_f\setminus\{x_0\}} X_x)$. Thus, $X = \bigcup_{x\in F_f} X_x$.

(iii) $\Rightarrow$ (iv). By Bessaga's theorem for $f : X_i \to X_i$, there exists a complete metric d_i on X_i, such that $f : (X_i, d_i) \to (X_i, d_i)$ is an a-contraction, for some $a \in]0,1[$. Now we define a complete metric on $X = \bigcup_{i\in i} X_i$. Let $x_i^0 \in X_i$, $i \in I$. We take $d(x,y) = d_i(x,y)$ if $x, y \in X_i$ and $d(x,y) := 1 + d_i(x, x_i^0) + d_j(y, x_j^0)$, if $x \in X_i$, $y \in X_j$, $i \neq j$. The mapping d is a metric on X. The completeness of (X, d) follows from the following remark:

$$d(x,y) < 1 \Rightarrow \exists\, i \in I,\ x, y \in X_i.$$

It is clear that we have (1), (2) and (3).

(iv) $\Rightarrow$ (v). If $f|_{X_i} : (X_i, d) \to (X_i, d)$ is continuous, then the mapping φ is f-orbitally lower semi-continuous.

(v) $\Rightarrow$ (vi). Follows from the Theorem 1 in [13].

(vi) $\Rightarrow$ (i). Obvious. □

3. Generalized metric spaces.

In what follows, by a generalized metric on a set X we mean a mapping $d : X \times X \to R_+ \cup \{+\infty\}$, which satisfies the Fréchet's axioms. For such spaces we have

Lemma 1 (Jung; see [7] or [14] or [15])**.** *Let* (X, d) *be a generalized metric space. Then there exists a partition of* X, $X \bigcup_{i\in I} X_i$, *such that* $d(x,y) < +\infty$, *for all* $x, y \in X_i$, $i \in I$. *Moreover,* (X, d) *is a complete metric space if and only if* (X_i, d) *is a complete metric space for each* $i \in I$.

Now we have another characterization of the weakly Picard mappings.

Theorem 2. *Let X be a nonempty set and $f : X \to X$ a mapping. The following statements are equivalent:*

(i) *there exists a complete metric d on X such that $f(X, f) \to (X, f)$ is a weakly Picard mapping;*

(ii) *for each $a \in]0,1[$, there exists a complete generalized metric ϱ on X such that:*
(1) *$f : (X, \varrho) \to (X, \varrho)$ is an a-contraction;*
(2) *$\varrho(x, f(x)) < +\infty$, for all $x \in X$;*

(iii) *for each $a \in]0,1[$, there exists a generalized complete generalized metric ϱ on X such that:*
(1) *$\varrho(f^2(x), f(x)) \leq a\varrho(x, f(x))$ for all $x \in X$;*
(2) *$\varrho(x, f(x)) < +\infty$, for all $x \in X$;*
(3) *$f : (X, \varrho) \to (X, \varrho)$ is continuous;*

(iii′) *there exist $a \in]0,1[$ and a generalized complete metric ϱ on X such that we have* (1), (2) *and* (3) *in* (iii).

The proof of this theorem is analogous to that of Theorem 1.

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