On tempered convolution operators

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Abstract. In this paper we show that if S is a convolution operator in $\mathscr{S}'$, and $S*\mathscr{S}' = \mathscr{S}'$, then the zeros of the Fourier transform of S are of bounded order. Then we discuss relations between the topologies of the space $\mathscr{O}'_c$ of convolution operators on $\mathscr{S}'$. Finally, we give sufficient conditions for convergence in the space of convolution operators in $\mathscr{S}'$ and in its dual.

Keywords: tempered distribution, convolution operator, Fourier transform, convergence of sequences

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Introduction

Convolution equations in the space $\mathscr{S}'$ of tempered distributions were investigated by Sznajder and Zielezny [6]. They were interested in characterizing convolution operators S which satisfy the equation $S * \mathscr{S}' = \mathscr{S}'$. They have shown that if $S * \mathscr{S}' = \mathscr{S}'$, then S, the Fourier transform of S, satisfy the following equivalent conditions:

(I) For every integer k there exist an integer $m \geq 0$ and constants $c, M \geq 0$ such that
$$\sup_{\substack{|\alpha|\leq m,\ s\in\mathbb{R}^n \\ |s-\xi|\leq(1+\xi)^{-k}}} |D^\alpha \widehat{S}(s)| \geq |\xi|^{-c},$$
where $\xi \in \mathbb{R}^n$, and $|\xi| \geq M$.

(II) If u is a convolution operator in $\mathscr{S}'$ and $S * u \in \mathscr{S}$, then u is in $\mathscr{S}$.

The problem of characterizing the convolution operator S for which $S * \mathscr{S}' = \mathscr{S}'$ is an interesting one and still open. Sznajder and Zielezny [3] conjectured that, if the order of the zeros of S is bounded, then conditions (I) and (II) are equivalent to the equality $S*\mathscr{S}' = \mathscr{S}'$. In this paper we show that if $S*\mathscr{S}' = \mathscr{S}'$ then the zeros of $\widehat{S}$ are of bounded order. This together with the above result of Sznajder and Zielezny prove the necessity part of their conjecture. We also give an example of a convolution operator S so that $S * \mathscr{S}' \neq \mathscr{S}'$.

Next, we consider convergence questions in $\mathscr{O}'_c$ and $\mathscr{O}_c$. It is known that if (S_j) is a sequence which converges to 0 in $\mathscr{O}'_c$, then $(S_j * \phi)$ converges to 0 in $\mathscr{S}$, for every ϕ in $\mathscr{S}$. This implies that $(S_j * \phi)$ converges to 0 in $\mathscr{O}'_c$. Here we prove the converse, if (S_j) is a sequence in $\mathscr{O}'_c$ and $(S_j * \phi)$ converges to 0 in $\mathscr{O}'_c$ for every ϕ in $\mathscr{S}$, then (S_j) converges to 0 in $\mathscr{O}'_c$. Similar questions of convergence were discussed by Keller [5]. Among other things, he has shown that if (T_j) is a sequence in $\mathscr{S}'$ such that $(T_j * \phi)$ converges to 0 in $\mathscr{S}'$ for all ϕ in $\mathscr{S}$, then (T_j) converges to 0 in $\mathscr{S}'$. We use Keller's result to prove ours. Moreover, it will be shown that if (ψ_j) is a sequence in $\mathscr{O}_c$ such that $(\psi_j * \phi)$ converges to 0 in $\mathscr{O}_c$ for every ϕ in $\mathscr{S}$, then (ψ_j) converges to 0 in $\mathscr{O}_c$. Most of the topological properties of $\mathscr{O}_c'$ are proved when it is provided with the strong dual topology sdt. In the proof of the result on convergence in $\mathscr{O}_c'$, we work with $\mathscr{O}_c'$ with the topology τ_b which is induced by $L_b(\mathscr{S},\mathscr{S})$.

By $\mathscr{S}$ we denote the space of all C^∞-functions in $\mathbb{R}^n$ such that

$$\sup_{\substack{|\alpha|\le k\\ x\in\mathbb{R}^n}} (1+|x|)^k |D^\alpha \phi(x)| < \infty, \quad k = 0,1,2,3,\ldots$$

We denote by $\mathscr{S}'$ the space of tempered distributions which is the strong dual of $\mathscr{S}$. Since the Fourier transform is an isomorphism from $\mathscr{S}$ onto itself, the same is true for $\mathscr{S}'$. The space of all convolution operators in $\mathscr{S}'$ will be denoted by $\mathscr{O}_c'(\mathscr{S}',\mathscr{S}')$. An $S\in\mathscr{S}'$ is in $\mathscr{O}_c'(\mathscr{S}',\mathscr{S}')$ if and only if the map $\phi \to S*\phi$ from $\mathscr{S}$ into itself is continuous, where $(S*\phi)(x) = \langle S_y, \phi(x-y)\rangle$. And for u in $\mathscr{S}'$, $S*u$ is given by

$$\langle S*u,\phi\rangle = \langle u, \check{S}*\phi\rangle\,, \quad \phi \text{ in } \mathscr{S}.$$

We denote by $\mathscr{O}_M(\mathscr{S}',\mathscr{S}')$ the space of all C^∞-functions f such that, for every multi-index α there exists $k = 0,1,2,\ldots$, such that

$$D^\alpha f(x) = O(1+|x|)^k \text{ as } |x| \longrightarrow \infty.$$

If S is in $\mathscr{O}_c'(\mathscr{S}',\mathscr{S}')$, its Fourier transform $\widehat{S}$ is in $\mathscr{O}_M(\mathscr{S}',\mathscr{S}')$, i.e. a multiplier of $\mathscr{S}'$. For such S and u in $\mathscr{S}$ one has $\widehat{S*u} = \widehat{S}\widehat{u}$.

For k in $\mathbb{N}$ we denote by $\mathscr{S}_k$ the space of all infinitely differentiable functions ψ such that, for each α in $\mathbb{N}^n$ and positive ε, there exists a positive ϱ such that

$$\left|(1+|x|^2)^{-k} D^\alpha \psi(x)\right| \le \varepsilon \text{ for all } |x| \text{ greater than } \varrho.$$

The space $\mathscr{S}_k$ is provided with the topology generated by the semi-norms

$$q_{k,\alpha}(\psi) = \sup_{x\in\mathbb{R}^n} \left|(1+|x|^2)^{-k} D^\alpha \psi(x)\right|\,, \quad \alpha\in\mathbb{N}^n\,.$$

We denote by $\mathscr{O}_c(\mathscr{S}',\mathscr{S}')$ the union of the spaces $\mathscr{S}_k$ provided with the inductive limit topology. It follows that $\mathscr{O}_c$ is a Hausdorff locally convex space and $\mathscr{O}_c'$ is its strong dual. It follows that $\mathscr{O}_c' = \bigcap_{k=0}^\infty \mathscr{S}_k'$, where $\mathscr{S}_k'$ is the strong dual of $\mathscr{S}_k$. The strong dual topology sdt on $\mathscr{O}_c'$ is the topology of uniform convergence on bounded subsets of $\mathscr{O}_c$.

The results

Theorem 1. *Let $S \in \mathscr{O}'_c(\mathscr{S}', \mathscr{S}')$, if $S * \mathscr{S}' = \mathscr{S}'$, then the zeros of $\widehat{S}$ are of bounded order.*

Proof: Suppose $S * \mathscr{S}' = \mathscr{S}'$ and $\widehat{S}$ has a zero x of unbounded order. Without loss of generality we can assume that $x = 0$. Hence $\widehat{S}(x) = \sigma(|x|^m)$ for all $m \geq 0$ and all x in the unit ball $B(0,1)$. By hypothesis there exists $u \in \mathscr{S}'$ such that $S * u = 1$, hence $\widehat{S}\widehat{u} = \delta$. From the structure theorem of tempered distributions it follows that one can represent $\widehat{u}$ as a finite sum $\sum_{|\alpha| \leq k} D^\alpha u_\alpha$ of derivatives of continuous functions growing at infinity slower than some polynomial. Let $\phi \in \mathscr{D}(B(0,1))$ such that $\phi(0) = 1$. Let $\phi_\varepsilon(x) = \phi(x/\varepsilon)$. Then one has

$$
\begin{aligned}
\Big|\langle \widehat{S}\widehat{u}, \phi_\varepsilon\rangle\Big| &= \Big|\sum_{|\alpha|\leq k} \langle D^\alpha u_\alpha, \widehat{S}\phi_\varepsilon\rangle\Big| = \Big|\sum_{|\alpha|\leq k} (-1)^\alpha \langle u_\alpha, D^\alpha(\widehat{S}\phi_\varepsilon)\rangle\Big| \\
&= \Big|\sum_{|\alpha|\leq k}\sum_{\beta\leq\alpha} C_\beta (-1)^\alpha \langle u_\alpha, D^\beta \widehat{S} D^{\alpha-\beta}\phi_\varepsilon\rangle\Big| \\
&\leq \sum_{|\alpha|\leq k}\sum_{\beta\leq\alpha} |C_\beta| \int |D^\beta \widehat{S}(x)|\, |u_\alpha(x)|\, |D^{\alpha-\beta}\phi_\varepsilon(x)|\, dx.
\end{aligned}
$$

Since $\widehat{S}$ has 0 as zero of unbounded order, it follows that the same is true for its derivatives, hence $D^\beta \widehat{S}(x) = \sigma(|x|)^m$, for all $\beta \leq \alpha$. Hence one has

$$
\begin{aligned}
\Big|\langle \widehat{S}\widehat{u}, \phi_\varepsilon\rangle\Big| &\leq \sum_{|\alpha|\leq k}\sum_{\beta\leq\alpha} |C_\beta| \int |x|^m (1+|x|)^{k(\alpha)} |D^{\alpha-\beta}\phi(x/\varepsilon)|\, dx \\
&\leq \sum_{|\alpha|\leq k} C_\alpha \varepsilon^{m+k(\alpha)-k} \leq C_\alpha \varepsilon^{m+k(\alpha)-k}\,,
\end{aligned}
$$

where C_α is a constant which depends on α but not the same in all estimates. Since the above estimate holds for all $m \geq 0$, by taking m large enough and letting ε go to 0, it follows that $\langle \widehat{S}\widehat{u}, \phi_\varepsilon\rangle \to 0$. On the other hand, $\langle \delta, \phi_\varepsilon\rangle = \phi_\varepsilon(0) = 1$ for all ε. The contradiction proves the theorem. $\square$

Remark. In the above proof we could have used the local structure of u. In a small neighborhood of 0 one can represent u as the derivative of a continuous function of compact support ([1, Theorem 2.21]).

Example 1. We give an example of convolution operator on $\mathscr{S}'$ which is not invertible because the zeros of its Fourier transform are not of bounded order. Let

$$
f(x) = \begin{cases} \exp(-1/|x|^2), & x \neq 0, \\ 0 & x = 0. \end{cases}
$$

The function f is infinitely differentiable and has the origin as zero of unbounded order. Moreover, $f \in \mathscr{O}_M(\mathscr{S}', \mathscr{S}')$. Hence f is the Fourier transform of some S in $\mathscr{O}'_c$. From the theorem it follows that S is not invertible in $\mathscr{S}'$. Also, one can verify easily that f satisfies condition (I) of Sznajder and Zielezny.

Example 2. Consider the infinite product

$$f(z) = \prod_{n=1}^{\infty} \cos(z/n^2), \quad z = x + iy \in \mathbb{C}.$$

One can verify that the infinite product is convergent, hence $f(z)$ is an entire function. We show that f satisfies the Paley-Wiener estimate ([1, Theorem 4.12]). Since $|\cos(z/n^2)| \leq e^{(y/n^2)}$, it follows that

$$|f(z)| \leq e^{(Ay)} \leq e^{(A|\operatorname{Im} z|)}$$

where $A = \sum_{n=1}^{\infty}(1/n^2)$. Hence

$$|f(z)| \leq C(1 + |z|)^N \, e^{(A|\operatorname{Im} z|)}$$

where $C = 1$ and $N = 1$. Thus f is a Fourier transform of some distribution S of compact support. Hence $S \in \mathscr{O}'_c(\mathscr{S}', \mathscr{S}')$. From the remark which follows Lemma 2 of [3] it follows that $S * \mathscr{S}' = \mathscr{S}'$. The zeros of $\widehat{S}$ are isolated, and since $\widehat{S}$ is an entire function which is not identically zero, its zeros are of bounded order.

Now, we examine the topologies which $\mathscr{O}'_c$ will be equipped with to get the convergence results. Since $\mathscr{O}'_c$ is a subset of $L_b(\mathscr{S}, \mathscr{S})$, the space of all continuous linear maps from $\mathscr{S}$ into itself provided with the topology of uniform convergence on bounded subsets of $\mathscr{S}$, we can provide $\mathscr{O}'_c$ with this topology and will denote it by τ_b. Similarly, we will provide $\mathscr{O}'_c$ with the topology τ'_b which is induced by $L_b(\mathscr{S}', \mathscr{S}')$, τ'_b is the topology of uniform convergence on bounded subsets of $\mathscr{S}'$. The topologies τ_b and τ'_b are equal. Indeed, let

$$W(B, U) = \left\{S \in \mathscr{O}'_c : S * \phi \in U \quad \text{for all } \phi \text{ in } B\right\}$$

be a member of 0-neighborhood base in τ_b, where U is a neighborhood of 0 in $\mathscr{S}$ and B is a bounded subset of $\mathscr{S}$. We can assume that $U = (B')^\circ$, the polar of B' a bounded subset of $\mathscr{S}'$. One gets

$$\begin{aligned} W(B, U) &= \{S \in \mathscr{O}'_c : |\langle S * \phi, T\rangle| < 1 \quad \text{for all} \quad \phi \in B \quad \text{and} \quad T \in B'\} \\ &= \{S \in \mathscr{O}'_c : |\langle \breve{S} * T, \phi\rangle| < 1 \quad \text{for all} \quad \phi \in B \quad \text{and} \quad T \in B'\} \\ &= V(\breve{B}', (\breve{B})^\circ). \end{aligned}$$

$V(\breve{B}', (\breve{B})^\circ)$ is a member of 0-neighborhood base in τ_b. Since all the above equalities are reversible, the proof is complete.

□

Theorem 2. *The topology τ_b of $\mathscr{O}_c'$ is less fine than the strong dual topology.*

PROOF: Let

$$W(B,U) = \{S \in \mathscr{O}_c' : S * \phi \in U \text{ for all } \phi \text{ in } B\}$$

be a member of 0-neighborhood base in τ_b, where U is a neighborhood of 0 in $\mathscr{S}$, $U = (B')^\circ$ the polar of a bounded subset of $\mathscr{S}'$. Since the bilinear map $(\phi, S) \to \phi * S$ from $\mathscr{S} \times \mathscr{S}'$ into $\mathscr{O}_c$ is separately continuous, it follows from the Banach-Steinhaus theorem that $\breve{B} * B'$ is bounded in $\mathscr{O}_c$. We claim that $W(B,U) = (\breve{B} * B')^\circ$. For

$$\begin{aligned} W(B,U) &= \{S \in \mathscr{O}_c' : |\langle S * \phi, T\rangle| < 1 \text{ for all } \phi \in B,\ T \in B'\} \\ &= \{S \in \mathscr{O}_c' : |\langle S, \breve{\phi} * T\rangle| < 1 \text{ for all } \phi \in B,\ T \in B'\} \\ &= (\breve{B} * B')^\circ. \end{aligned}$$

This completes the proof of the theorem. □

In the proof of the next result we will use $\mathscr{O}_c'$ with the topology τ_b, and in the one after it will be provided with the strong dual topology.

Theorem 3. *Let (S_j) be a sequence in $\mathscr{O}_c'$ such that $(S_j * \phi)$ converges to 0 in $\mathscr{O}_c'$ for every ϕ in $\mathscr{S}$, then (S_j) converges to 0 in $\mathscr{O}_c'$.*

PROOF: Let B be a bounded subset of $\mathscr{S}$, we show that $S_j * \phi \to 0$ in $\mathscr{S}$ uniformly in $\phi \in B$. Since $\mathscr{S}$ is reflexive and $\mathscr{S}'$ is Montel, all what we need to show is that, for every T in $\mathscr{S}'$, the sequence $(\langle S_j * \phi, T\rangle)$ converges to 0 uniformly in $\phi \in B$. For this, let $\Psi \in \mathscr{S}$. From the hypothesis one has $(S_j * T) * \Psi = (S_j * \Psi) * T \to 0$ in $\mathscr{S}'$. From Theorem 1 of [5] it follows that $S_j * T \to 0$ in $\mathscr{S}'$. Hence $\langle S_j * \phi, T\rangle = \langle \breve{S}_j * T, \phi\rangle \to 0$ uniformly in $\phi \in B$. This completes the proof. □

Theorem 4. *Let (Ψ_j) be a sequence in $\mathscr{O}_c$ such that the sequence $(\Psi_j * \phi)$ converges to 0 in $\mathscr{O}_c$ for every ϕ in $\mathscr{S}$, then (Ψ_j) converges to 0 in $\mathscr{O}_c$.*

PROOF: Since $\mathscr{O}_c'$ is Montel, it suffices to show that for any $S \in \mathscr{O}_c'$ the sequence $(\langle \Psi_j, S\rangle)$ converges to 0. Let (ϕ_k) be a sequence in $\mathscr{D}$ converging to δ in $\mathscr{E}'$. Since the bilinear map $(\Psi, S) \to \Psi * S$ is separately continuous, it follows from the hypothesis that, for fixed k,

$$(**) \qquad \lim_{j\to\infty} \langle \Psi_j, \breve{\phi}_k * S\rangle = \lim_{j\to\infty} \langle \Psi_j * \phi_k, S\rangle = \lim_{j\to\infty} ((\Psi_j * \phi_k) * S)(0) = 0.$$

The hypothesis implies that $\Psi_j \to 0$ weakly in the dual of $\mathscr{S}$ considered as a subspace of $\mathscr{O}_c'$. Since $\mathscr{S}$ is dense in $\mathscr{O}_c'$, one can show that $\mathscr{S}$ with the relative topology of $\mathscr{O}_c'$ is Montel. Thus $(**)$ implies that $(\Psi_j) \to 0$ strongly in the dual of $\mathscr{S}$ with the relative topology of $\mathscr{O}_c'$. Since the set $\{S * \phi_k : k = 1, 2, \ldots\}$ is bounded in $\mathscr{S}$ with the relative topology of $\mathscr{O}_c'$, it follows that the convergence in $(**)$ is uniform in k. Hence

$$\lim_{j\to\infty}\langle\Psi_j,S\rangle=\lim_{j\to\infty}\lim_{k\to\infty}\langle\Psi_j,\phi_k*S\rangle=\lim_{k\to\infty}\lim_{j\to\infty}\langle\Psi_j*\phi_k,S\rangle.$$

The proof is complete. □

The following problem ([4, Problem 8, p. 425]) is useful to show equality of the topologies of $\mathscr{O}_c'$.

Problem (Horvath)**.** The strong dual topology is the least fine topology on $\mathscr{O}_c'$ such that for any nonnegative integer k, the map $S\to(1+|x|^2)^kS$, from $\mathscr{O}_c'$ into $\mathscr{S}_0'$ is continuous.

Remark. If we assume the truth of the above problem, we can show that on $\mathscr{O}_c'$ the strong dual topology is less fine than τ_b'. Indeed, since the strong dual topology is the least fine topology such that the maps $S\to(1+|x|^2)^kS$ from $\mathscr{O}_c'$ into $\mathscr{S}_0'$ are continuous, it suffices to show that these maps are continuous when we provide $\mathscr{O}_c'$ with τ_b'. Since τ_b' is equal to τ_b and $(\mathscr{O}_c',\tau_b)$ is bornologic (see [2, Chapter 2, Theorem 16]), we show that the maps are sequentially continuous. Fix $k\in\mathbb{N}$, let (S_j) be a sequence in $\mathscr{O}_c'$ converging to 0 in τ_b'. Let B be any bounded subset of $\mathscr{S}_0$. The set $(1+|x|^2)^kB$ is bounded in $\mathscr{S}_k\hookrightarrow\mathscr{O}_c$, hence bounded in $\mathscr{O}_c$. Thus $(1+|x|^2)^kB$ is bounded in $\mathscr{E}$. Let $\Psi\in\mathscr{S}_0$, we claim that the map Λ_Ψ from $(\mathscr{O}_c',\tau_b')$ into $\mathscr{E}$ which maps S to $\Psi*S$ is bounded. Indeed, let B' be a bounded subset of $(\mathscr{O}_c',\tau_b')$, let $(B_e')^\circ$, B_e' is a bounded subset of $\mathscr{E}'$, be a member of 0-neighborhood base in $\mathscr{E}$. We find $\lambda>0$ such that $\lambda(\Psi*B')$ is contained in $(B_e')^\circ$. Since τ_b' is less fine than the sdt and $\mathscr{E}'$ is continuously embedded in $(\mathscr{O}_c',\mathrm{sdt})$, it follows that B_e' is bounded in τ_b'. Since $\mathscr{F}(\mathscr{E}')$, the Fourier transform of $\mathscr{E}'$, is continuously embedded in $\mathscr{O}_M$, it follows that B_e' and B' are bounded in $\mathscr{O}_M$. Hence $B_e'\cdot B'$ is bounded in $\mathscr{O}_M$ (see [6, p. 248]). This implies that $B_e'*B'$ is bounded in $\mathscr{O}_c'$ with the sdt. Thus there exists a constant $c>0$ such that $|\langle\Psi,S*T\rangle|<c$ for all $S\in B'$ and $T\in B'$. Thus $(1/c)(\Psi*B')$ is contained in $(B_e')^\circ$. This proves the claim. Since $\mathscr{S}_k$ is of second category (being a complete metric space), it follows from the Banach-Steinhaus theorem that the set $\{S_j*(1+|x|^2)^kf:f\in B,\ j=1,2,\dots\}$ is bounded in $\mathscr{S}_k\subset\mathscr{E}$. Let (ϕ_i) be a sequence in $\mathscr{D}$ which converges to δ in $\mathscr{E}'$. One has

$$\lim_{j\to\infty}\langle S_j,(1+|x|^2)^kf\rangle=\lim_{j\to\infty}\langle S_j*(1+|x|^2)^kf,\delta\rangle=$$
$$=\lim_{j\to\infty}\lim_{i\to\infty}\langle S_j*(1+|x|^2)^kf,\phi_i\rangle.$$

Since the inner limit converges uniformly in j, one can interchange the limits and get

$$\lim_{j\to\infty}\langle S_j,(1+|x|^2)^kf\rangle=\lim_{i\to\infty}\lim_{j\to\infty}\langle S_j*(1+|x|^2)^kf,\phi_i\rangle=0,$$

where the convergence is uniform in $f\in B$. This completes the proof of the assertion. □

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