Copies of l^1 and c_o in Musielak-Orlicz sequence spaces

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Abstract. Criteria in order that a Musielak-Orlicz sequence space l^Φ contains an isomorphic as well as an isomorphically isometric copy of l^1 are given. Moreover, it is proved that if $\Phi = (\Phi_i)$, where Φ_i are defined on a Banach space, X does not satisfy the δ_2^o-condition, then the Musielak-Orlicz sequence space $l^\Phi(X)$ of X-valued sequences contains an almost isometric copy of c_o. In the case of $X = \mathbb{R}$ it is proved also that if l^Φ contains an isomorphic copy of c_o, then Φ does not satisfy the δ_2^o-condition. These results extend some results of [A] and [H2] to Musielak-Orlicz sequence spaces.

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0. Introduction

Two Banach spaces X, Y are said to be $(1+\varepsilon)$-isometric provided there exists a linear isomorphism $P : X \xrightarrow{\text{onto}} Y$ such that $\|P\| \ \|P^{-1}\| \leq 1+\varepsilon$. It is easy to see that P is a $(1+\varepsilon)$-isometry if

$$\|x\|_X \leq \|Px\|_Y \leq (1+\varepsilon)\|x\|_X$$

for any $x \in X$. We say a Banach space X contains an almost isometric (isomorphic) copy of Y if for any $\varepsilon > 0$ (for some $\varepsilon > 0$) there exists a subspace Z in X such that Z, Y are $(1+\varepsilon)$-isomorphic. We say a Banach space X contains an isomorphically isometric (shortly isometric) copy of Y if there exist a subspace Z of X and a linear isomorphism P from Z onto Y such $\|Px\|_Y = \|x\|_X$ for any $x \in Z$.

In the sequel X denotes a real Banach space and $\mathbb{N}, \mathbb{R}, \mathbb{R}_+$ and $\mathbb{R}_+^e$ stand for the set of natural numbers, the set of reals, the set of nonnegative reals, and for $\mathbb{R}_+ \cup +\infty$, respectively. A map $\Phi : X \to \mathbb{R}_+^e$ is said to be an <u>Orlicz</u> <u>function</u> if it is convex, even, vanishing and continuous at 0, lower semicontinuous on the whole X and

$$(*) \qquad \inf\{\Phi(x) : \|x\| = r\} \to \infty \quad \text{as} \quad r \to \infty.$$

We define a Musielak-Orlicz function Φ to be a sequence (Φ_i) of Orlicz functions (we write then $\Phi = (\Phi_i)$). Given a Banach space X, we denote by $l^o(X)$ the real space of all X-valued sequences $x = (x_n)$. We write shortly l^o instead of $l^o(\mathbb{R})$.

Given an arbitrary Musielak-Orlicz function $\Phi = (\Phi_i)$ we define a functional $I_\Phi : l^o(X) \to \mathbb{R}^e_+$ by

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x_i),$$

which is even and convex, $I_\Phi(0) = 0$ and for any $x \in l^o(X)$ the condition $I_\Phi(\lambda x) = 0$ for any $\lambda > 0$ yields $x = 0$.

Musielak-Orlicz space $l^\Phi(X)$ generated by a Musielak-Orlicz function Φ is defined as the set of all $x \in l^o(X)$ such that $I_\Phi(\lambda x) < \infty$ for some $\lambda > 0$ (cf. [T] and in the scalar case also [KR], [L], [M] and [RR]).

The subspace $h^\Phi(X)$ of $l^\Phi(X)$ is defined to be the closure in $l^\Phi(X)$ of the space $h(X)$ of all x in $l^o(X)$ which have only finite number of coordinates different from 0. The space $l^\Phi(X)$ can be equipped with the norm

$$\|x\|_\Phi = \inf\{\varepsilon > 0 : I_\Phi(x/\varepsilon) \le 1\},$$

called the Luxemburg norm (cf. [M] and in the case of Orlicz spaces also [KR], [L] and [RR]). The space $h^\Phi(X)$ will be considered with the norm $\| \ \|_\Phi$ induced from $l^\Phi(X)$. $l^\Phi(X)$ and $h^\Phi(X)$ equipped with the norm $\| \ \|_\Phi$ are Banach spaces (cf. [T]).

We say that a Musielak-Orlicz function $\Phi = (\Phi_i)$ satisfies the δ_2^o-condition (we write $\Phi \in \delta_2^o$) if there are positive constants k and a, a sequence (c_i) with $c_i \in \mathbb{R}^e_+$ such that $\sum\limits_{i=j}^{\infty} c_i < \infty$ for some $j \in \mathbb{N}$ and

$$\Phi_i(2x) \le \Phi_i(x) + c_i$$

for any $i \in N$ and $x \in X$ satisfying $\Phi_i(x) \le a$.

If $\Phi = (\Phi_i)$ satisfies the δ_2^o-condition with $j = 1$ we say that Φ satisfies the δ_2-condition. Of course, for any Musielak-Orlicz function $\Phi = (\Phi_i)$ with finite-valued Φ_i for any $i \in \mathbb{N}$ the δ_2-condition is equivalent to the δ_2^o - condition (cf. [DH] and [K]).

Let us define for any Musielak-Orlicz function $\Phi = (\Phi_i)$ the sequence $\lambda = (\lambda_i)$ in $\mathbb{R}_+$, where

$$\lambda_i = \sup\{u \in \mathbb{R}_+ : \Phi_i \quad \text{is linear on } [0,u] \text{ and } \Phi_i(u) \le 1\}$$

for $i = 1, 2, \ldots$.

1. Results

We start with the following theorem:

Theorem 1. *Let $\Phi = (\Phi_i)$ be a Musielak-Orlicz function with finite-valued Φ_i defined on $\mathbb{R}$ for any $i \in \mathbb{N}$. Then $l^\Phi = (l^\Phi, \| \ \|_\Phi)$ contains an isometric copy of l^1 if and only if:*

(i) *Φ does not satisfy the δ_2-condition.*

(ii) $\sum\limits_{i=1}^{\infty} \Phi_i(\lambda_i) = \infty$.

Proof: Sufficiency. Under our assumptions concerning Φ, the conditions δ_2 and δ_2^o are equivalent. Therefore, if Φ satisfies condition (i), then $\Phi \notin \delta_2^o$. This yields that l^Φ contains an isometric copy of l^∞ (cf. [K]) and so also an isometric copy of l^1.

Assume now that Φ satisfies the condition (ii). Define i_1 to be the largest natural number satisfying

$$\sum_{i=1}^{i_1} \Phi_i(\lambda_i) \leq 1.$$

Then

$$\sum_{i=1}^{i_1+1} \Phi_i(\lambda_i) > 1.$$

There is a number $\alpha_i \in [0, \lambda_i)$ such that

$$\sum_{i=1}^{i_1} \Phi_i(\lambda_i) + \Phi_{i_1+1}(\alpha_1) = 1.$$

We have

$$\sum_{i=i_1+2}^{\infty} \Phi_i(\lambda_i) = \infty.$$

Define $i_2 \geq i_1 + 2$ to be the largest natural number such that

$$\sum_{i=i_1+2}^{i_2} \Phi_i(\lambda_i) \leq 1.$$

Then

$$\sum_{i=i_1+2}^{i_2+1} \Phi_i(\lambda_i) > 1.$$

There is a number $\alpha_2 \in [0, \lambda_{i_2+1})$ such that

$$\sum_{i=i_1+2}^{i_2} \Phi_i(\lambda_i) + \Phi_{i_2+1}(\alpha_2) = 1.$$

Proceeding in such a way by induction we find sequences (i_k) of natural numbers and (α_k) of numbers from the intervals $[0, \lambda_{i_k+1})$ such that

$$\sum_{i=i_{k-1}+2}^{i_k} \Phi_i(\lambda_i) + \Phi_{i_k+1}(\alpha_k) = 1$$

for $k = 1, 2, \ldots$, where $i_o = -1$ by definition. Denote

$$A_k = \{i_{k-1} + 1, \ldots, i_k, i_k + 1\}, \quad k = 1, 2, \ldots .$$

The sets A_k are pairwise disjoint and $\bigcup\limits_{k=1}^{\infty} A_k = \mathbb{N}$. Define a new sequence $d = (d_i)$ by

$$d_i = \begin{cases} \lambda_i & \text{if } \ i \in A_k \setminus \{i_k + 1\} \\ \alpha_k & \text{if } \ i = i_k + 1 \end{cases}$$

for $k = 1, 2, \ldots$. Define now

$$f_k = \sum_{i \in A_k} d_i e_i,$$

where e_i is the i-th unit vector, i.e. $e_i = (0, \ldots, 0, 1, 0, \ldots)$ with 1 on the i-th place. We have

$$I_\Phi(f_k) = 1 \quad \text{for} \ \ k = 1, 2, \ldots .$$

We have also $f_k \perp f_l$ (i.e. the sequences f_k and f_l have disjoint supports) if $k \neq l$. Moreover, the coordinates of $f_k (k = 1, 2, \ldots)$ belong to the intervals on which the respective Orlicz functions Φ_i are linear. Define an operator P from l^1 into l^Φ by the formula

$$Px = \sum_{k=1}^{\infty} x_k f_k \qquad (\forall\, x = (x_k) \in l^1).$$

It is obvious that P is linear. Moreover

$$\begin{aligned} I_\Phi\left(\frac{Px}{\|x\|_{l^1}}\right) &= \sum_{k=1}^{\infty} I_\Phi\left(\frac{x_k f_k}{\|x\|_{l^1}}\right) \\ &= \sum_{k=1}^{\infty} \frac{|x_k|}{\|x\|_{l^1}} I_\Phi(f_k) = \sum_{k=1}^{\infty} \frac{|x_k|}{\|x\|_{l^1}} = 1. \end{aligned}$$

Hence

$$\|\frac{Px}{\|x\|_{l^1}}\|_\Phi = 1, \text{i.e. } \|Px\|_\Phi = \|x\|_{l^1}.$$

This means that P is an isometry between l^1 and a closed subspace $P(l^1)$ of l^Φ.

Necessity. Assume that Φ satisfies the δ_2-condition and $\sum\limits_{i=1}^{\infty}(\lambda_i) < +\infty$. Then there is $n \in \mathbb{N}$, $n \geq 2$, such that $\sum\limits_{i=1}^{\infty} \Phi_i(\lambda_i) \leq n$. We will prove that l^Φ is non-l^1_n, i.e. for any elements $x^1, \ldots, x^n$ from the unit sphere $\mathcal{S}(l^\Phi)$ of l^Φ there holds

$$\|\frac{1}{n}(x^1 \pm x^2 \pm \cdots \pm x^n)\| < 1$$

for some choice of signs, which yields that l^1 can not be isometrically embedded into l^Φ.

Take arbitrary $x^1, \ldots, x^n \in \mathcal{S}(l^\Phi)$. Then $I_\Phi(x^1) = \cdots = I_\Phi(x^k) = 1$ for $i = 1, 2, \ldots, n$. Define the set

$$A = \{i \in \mathbb{N} : \sum_{k=1}^{n} \Phi_i(x_i^k) > \Phi_i(\lambda_i)\}.$$

We will prove that for any $i \in A$

$$\Phi_i\left(\frac{(x_i^1 \pm \cdots \pm x_i^n)}{n}\right) < \frac{1}{n}\sum_{k=1}^{n} \Phi_i(x_i^k) \tag{1}$$

for some choice of signs ± 1, dividing the proof into two cases.

I. $\max\{|x_i^k| : k = 1, \ldots, n\} \le \lambda_i$ and $i \in A$. Then at least two numbers among $\Phi_i(x_i^k), k = 1, \ldots, n$, must be positive. Assume that this is not true, i.e. that there is only one positive number $\Phi_i(x_i^j)$ among these numbers. Then

$$\sum_{k=1}^{n} \Phi_i(x_i^k) = \Phi(x_i^j) \le \Phi_i(\lambda_i),$$

which contradicts the fact that $i \in A$. Therefore,

$$\max\{\Phi_i(x_i^k) : k = 1, \ldots, n\} < \sum_{k=1}^{n} \Phi_i(x_i^k).$$

It is evident that

$$|x_i^1 \pm \cdots \pm x_i^n| \le \max\{|x_i^k| : k = 1, \ldots, \} \tag{2}$$

for some choice of signs ± 1, whence

$$\begin{aligned}\Phi_i\left(\frac{1}{n}(x_i^1 \pm \cdots \pm x_i^n)\right) &\le \Phi\left(\frac{1}{n}\max_k |x_i^k|\right)\\ &= \frac{1}{n}\max \Phi_i(x_i^k) < \frac{1}{n}\sum_{k=1}^{n}\Phi_i(x_i^k).\end{aligned}$$

This means that inequality (1) holds true in case I.

II. $i \in A$ and $\max\{|x_i^k| : k = 1, \ldots, n\} > \lambda_i$. Applying (2), we get for a choice of signs ± 1

$$\begin{aligned}\Phi_i\left(\frac{1}{n}(x_i^1 \pm \cdots \pm x_i^n)\right) &\le \Phi_i\left(\frac{1}{n}\max_k |x_i^k|\right) < \frac{1}{n}\Phi_i(\max_k |x_i^k|)\\ &= \frac{1}{n}\max_k \Phi_i(x_i^k) \le \frac{1}{n}\sum_{k=1}^{n}\Phi_i(x_i^k).\end{aligned}$$

Combining both cases I and II we get inequality (1) for some choice of signs ± 1. For the remaining $2^{n-1}-1$ choices of signs ± 1, we have by the convexity of Φ,

$$\Phi_i\left(\frac{1}{n}(x_i^1 \pm \cdots \pm x_i^n)\right) \le \frac{1}{n}\sum_{k=1}^{n} \Phi_i(x_i^k) \quad (\forall\, i \in A). \tag{3}$$

Combining (1) and (3), we get

$$\sum_{\pm 1} \Phi_i\left(\frac{1}{n}(x_i^1 \pm \cdots \pm x_i^n)\right) < \frac{2^{n-1}}{n}\sum_{k=1}^{n} \Phi_i(x_i^k) \quad (\forall\, i \in A).$$

Summing up both-side of the last inequality over $i \in A$, we have

$$\sum_{\pm 1} I_\Phi\left(\frac{1}{n}(x^1 \pm \cdots \pm x^n)\chi_A\right) < \frac{2^{n-1}}{n}\sum_{k=1}^{n} I_\Phi(x^k \chi_A),$$

where χ_A denotes the characteristic function of A.
Hence it follows that

$$\begin{aligned}
&2^{n-1} - \sum_{\pm 1} I_\Phi\left(\frac{1}{n}(x^1 \pm \cdots \pm x^n)\right)\\
&= \frac{2^{n-1}}{n}\sum_{k=1}^{n} I_\Phi(x^k) - \sum_{\pm 1} I_\Phi\left(\frac{1}{n}(x^1 \pm \cdots \pm x^n)\right)\\
&\ge \frac{2^{n-1}}{n}\sum_{k=1}^{n} I_\Phi(x^k\chi_A) - \sum_{\pm 1} I_\Phi\left(\frac{1}{n}(x^1 \pm \cdots \pm x^n)\chi_A\right) > 0,
\end{aligned}$$

i.e.

$$\sum_{\pm 1} I_\Phi\left(\frac{1}{n}(x^1 \pm \cdots \pm x^n)\right) < 2^{n-1}\,.$$

Therefore

$$I_\Phi\left(\frac{1}{n}(x^1 \pm \cdots \pm x^n)\right) < 1$$

for at least one choice of signs ± 1. Since Φ_i are finite-valued by the assumption and $\Phi \in \delta_2$, we get

$$\left\|\frac{1}{n}(x^1 \pm \cdots \pm x^n)\right\| < 1$$

for at least one choice of signs (cf. [DH] and [K]), i.e. l^Φ is non-l_n^1. This means that l^1 cannot be embedded isometrically into l^Φ, and the proof is finished. $\square$

Theorem 2. *Let $\Phi = (\Phi_i)$ be an arbitrary Musielak-Orlicz function defined on $\mathbb{R}$. Then $l^\Phi = (l^\Phi, \| \ \|_\Phi)$ contains an isomorphic copy of l^1 if and only if Φ or Φ^* (= the complementary function to Φ in the sense of Young) does not satisfy the δ_2^o-condition, i.e. if and only if l^Φ is not reflexive.*

PROOF: Sufficiency. If $\Phi \notin \delta_2^o$, then l^Φ contains an isometric copy of l^∞ (cf. [K]) and so of l^1 as well. Assume now that $\Phi \in \delta_2^o$ but $\Phi^* \notin \delta_2^o$. Then the dual of l^Φ is isomorphically isometric to l^{Φ^*} equipped with the Orlicz norm $\| \ \|_{\Phi^*}^o$ (cf. [M] and in the case of Orlicz spaces also [KR], [L] and [RR]). Therefore l^{Φ^*} contains an isomorphic copy of l^∞ (cf. again [K]), whence it follows that l^Φ contains an isomorphic copy of l^1.

Necessity. Assume that both functions Φ and Φ^* satisfy the δ_2^o-condition. Then l^Φ is reflexive and so l^1 can not be embedded isomorphically into l^Φ as a nonreflexive space. □

Theorem 3. *Let X be an arbitrary Banach space and $\Phi = (\Phi_i)$ be a Musielak-Orlicz function defined on X. Then:*

(i) *if Φ does not satisfy the δ_2^o-condition, then $h^\Phi(X) = (h^\Phi(X), \| \ \|_\Phi)$ contains an almost isometric copy of c_o;*

(ii) *if $X = \mathbb{R}$ and $(h^\Phi = h^\Phi, \| \ \|_\Phi)$ contains an isomorphic copy of c_o, then Φ does not satisfy the δ_2^o-condition.*

PROOF: (i). Let

$$c_i^{k,\varepsilon} = \sup\{\Phi_i((1+\varepsilon)x) - 2^{k+1}\Phi_i(x) : \Phi_i(x) \le 2^{-k-1}\} (\forall\, i,k \in \mathbb{N}, \varepsilon > 0).$$

We have that $\Phi \in \delta_2^o$ if and only if there exists $\varepsilon > 0$ and $m,k \in \mathbb{N}$ such that $\sum\limits_{i=m}^{\infty} c_i^{k,\varepsilon} < \infty$ (cf. [DH] and [H1]).

Define

$$\alpha_i^{k,\varepsilon} = \sup\{\Phi_i((1+\varepsilon)x) : \Phi_i(x) \le 2^{-k-1} \quad \text{and} \quad \Phi_i((1+\varepsilon)x) - 2^{-k-1}\Phi_i(x) \ge 0\}.$$

Since $\Phi_i(0) = 0 < 2^{-k-1}$, so 0 belongs to the set of these x over which the supremum in the definition of $c_i^{k,\varepsilon}$ is taken. Moreover,

$$\Phi((1+\varepsilon)0) - 2^{k+1}\Phi_i(0) = 0.$$

Hence it follows that $c_i^{k,\varepsilon} \ge 0$. Therefore, we can restrict ourselves in the definition of $c_i^{k,\varepsilon}$ to these x for which $\Phi_i((1+\varepsilon)x) - 2^{k+1}\Phi_i(x) \ge 0$. Hence we get $c_i^{k,\varepsilon} \le d_i^{k,\varepsilon}$ for every $i,k \in \mathbb{N}$ and $\varepsilon > 0$. We have by the assumption that $\Phi \notin \delta_2^o$. Hence it follows that

$$\sum_{i=m}^{\infty} d_i^{k,\varepsilon} = \infty \qquad (\forall\, m,k \in \mathbb{N}, \varepsilon > 0),$$

so we have among others $\sum\limits_{i=1}^{\infty} d_i^{k,\varepsilon} = \infty$. Define i_1 as the largest natural number such that

$$\sum_{i=1}^{i_1} d_i^{k,\varepsilon} \leq 1,$$

whenever $d_1^{k,\varepsilon} \leq 1$ and $i_1 = 0$ whenever $d_1^{k,\varepsilon} > 1$. Then

$$\sum_{i=1}^{i_1+1} d_i^{k,\varepsilon} > 1.$$

Define in the first step $N_1 = \{1, \ldots, i_1 + 1\}$. Define i_2 as the largest natural number such that

$$\sum_{i=i_1+2}^{i_2} d_i^{k,\varepsilon} \leq 1$$

if $d_{i_1+2}^{k,\varepsilon} \leq 1$ and $i_2 = i_1 + 2$ if $d_{i_1+2}^{k,\varepsilon} > 1$.
Then

$$\sum_{i=i_1+2}^{i_2+1} d_i^{k,\varepsilon} > 1.$$

Put $N_2 = \{i_1 + 2, \ldots, i_2 + 1\}$. Proceeding in such a way by induction we can find a sequence (i_k) of nonnegative integers such that the sequence of pairwise disjoint sets (N_k) in $\mathbb{N}$ defined by

$$N_k = \{i_{k-1} + 2, \ldots, i_k + 1\}, i_o := -1 \quad (k = 1, 2, \ldots)$$

satisfies

$$\sum_{i \in N_k \setminus \{i_k+1\}} d_i^{k,\varepsilon} \leq 1, \tag{4}$$

$$\sum_{i \in N_k} d_i^{k,\varepsilon} > 1. \tag{5}$$

In view of the definition of $d_i^{k,\varepsilon}$ and inequality (5), it follows that for any $\varepsilon > 0$ and $k \in \mathbb{N}$ there are $x_i \in X (x \in N_k)$ such that

$$\sum_{i \in N_k} \Phi_i((1 + \varepsilon)x_i) > 1, \tag{6}$$

$$\Phi_i(x_i) \leq 2^{-k-1} \quad \text{and} \quad \Phi_i((1 + \varepsilon)x_i) \geq 2^{k+1}\Phi_i(x_i) \quad (\forall\, i \in N_k). \tag{7}$$

Applying (6) and (7) we get

$$\sum_{i\in N_k}\Phi_i(x_i)\le 2^{-k-1}\sum_{i\in N_k\setminus\{i_k+1\}} d_i^{k,\varepsilon}+2^{-k-1}\le 2^{-k-1}+2^{-k-1}=2^{-k}. \tag{8}$$

Define $y_k=\sum\limits_{i=N_k} x_ie_i$. Then y_k have pairwise disjoint supports. In virtue of (6) and (8) we have

$$I_\Phi(y_k)=\sum_{i\in N_k}\Phi_i(x_i)\le 2^{-k}\qquad (\forall\, k\in\mathbb{N}), \tag{9}$$

$$I_\Phi((1+\varepsilon)y_k)=\sum_{i\in\mathbb{N}_k}\Phi_i((1+\varepsilon)x_i)>1\qquad (\forall\, k\in\mathbb{N}). \tag{10}$$

Define now an operator $P_1: c_o\to h^\Phi(X)$ by

$$P_1u=\sum_{k=1}^{\infty}u_ky_k\quad (\forall\, u=(u_k)\in c_o).$$

It is obvious that P_1 is linear. We will prove now that $P_1u\in h^\Phi(X)$ for any $u\in c_o$. We need to prove that there is a sequence (y^l) in $l^o(X)$ such that y^l has only finite number of coordinates different from zero and $\|P_1u-y^l\|_\Phi\to 0$ as $l\to\infty$, i.e.

$$I_\Phi(\lambda(P_1u-y^l))\to 0\quad \text{as}\ \ l\to\infty\quad (\forall\,\lambda>0). \tag{11}$$

Take arbitrary $\lambda,\varepsilon>0$ and choose $k_o\in\mathbb{N}$ such that $\sum\limits_{k=k_o}^{\infty}2^{-k}<\varepsilon$. Define

$$y^l=\sum_{k=1}^{l}u_ky_k\qquad (l=1,2,\dots).$$

Obviously any y^l has only finite number of coordinates different from zero. Let $l_o\ge k_o$ be such that $|u_k|\lambda\le 1$ for any $k\ge l_o$. Such a number l_o exists because $u=(u_k)\in c_o$. We have for any $l\ge l_o$

$$\begin{aligned} I_\Phi(\lambda(P_1u-u^l)) &= I_\Phi\Big(\sum_{k=l+1}^{\infty}\lambda u_ky_k\Big)\le I_\Phi\Big(\sum_{k=l+1}^{\infty}y_k\Big)\\ &=\sum_{k=l+1}^{\infty}I_\Phi(y_k)\le\sum_{k=l+1}^{\infty}2^{-k}<\varepsilon, \end{aligned}$$

i.e. condition (11) holds. This means that $P_1 u \in h^\Phi$ for any $u \in c_o$. Applying (9), we get for any $u \in c_o$,

$$\begin{aligned} I_\Phi(P_1 u/\|u\|_\infty) &= \sum_{k=1}^{\infty} I_\Phi(u_k y_k/\|u\|_\infty) \\ &\le \sum_{k=1}^{\infty} I_\Phi(y_k) \le \sum_{k=1}^{\infty} 2^{-k} = 1, \end{aligned}$$

whence it follows that

$$\|P_1 u\|_\Phi \le \|u\|_\infty \qquad (\forall\, u \in c_o). \tag{12}$$

Let $k_o \in N$ be such that $|u_{k_o}| = \|u\|_\infty$. Then, in view of (10), we get

$$\begin{aligned} I_\Phi\big((1+\varepsilon)\|u\|_\infty^{-1} P_1 u\big) &\ge I_\Phi\big((1+\varepsilon)\|u\|_\infty^{-1} u_{k_o} y_{k_o}\big) \\ &= I_\Phi\big((1+\varepsilon) y_{k_o}\big) > 1, \end{aligned}$$

whence it follows that

$$\|P_1 u\|_\Phi \ge \frac{1}{1+\varepsilon}\|u\|_\infty \qquad (\forall\, u \in c_o).$$

Defining now

$$Pu = (1+\varepsilon) P_1 u \qquad (\forall\, u \in c_o),$$

we get a linear operator from c_o into $h^\Phi(X)$ satisfying

$$\|u\|_\infty \le \|Pu\|_\Phi \le (1+\varepsilon)\|u\|_\infty \quad (\forall\, u \in c_o),$$

which means that P is a $(1+\varepsilon)$-isometry. Since $\varepsilon > 0$ is arbitrary this means that c_o is embedded almost isometrically into h^Φ and the proof of statement (i) is finished.

(ii). Assume that $\Phi \in \delta_2^o$ and $X = \mathbb{R}$. Then $h^\Phi = l^\Phi$ and l^Φ is the dual space of h^{Φ^*}, where Φ^* is the Orlicz function complementary in the sense of Young to Φ (cf. [HY]). Assume that h^Φ contains an isomorphic copy of c_o. Then it contains (as a dual space) a copy of l^∞ (cf. [BP]). But this contradicts the fact that the norm $\|\ \|_\Phi$ is order continuous in h^Φ. This contradiction finishes the proof of statement (ii) and so of Theorem 3 as well. □

Recall that a Banach lattice E is said to be a KB-space whenever every increasing bounded in the norm sequence of nonnegative elements in E is norm convergent to an element of E (cf. [AB] and [KA]).

Remark. It is known (cf. [AB, p. 227]) that if E is a Banach lattice that is not KB-space, then c_o is embeddable in E and conversely. The space h^Φ is a Banach lattice that is KB-space if and only if $\Phi \in \delta_2^o$. Therefore, h^Φ contains an isomorphic copy of c_o if and only if $\Phi \notin \delta_2^o$. It is worth to notice that if X is an arbitrary Banach space, then $h^\Phi(X)$ need not be a Banach lattice. However, in one direction an analogous result (cf. Theorem 3) still holds true.

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