On the existence of weak solutions of integral equations in Banach spaces

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Abstract. In this paper we investigate weakly continuous solutions of some integral equations in Banach spaces. Moreover, we prove a fixed point theorem which is very useful in our considerations.

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1. Introduction

The purpose of this paper is to prove an existence theorem for weakly continuous solutions of the Hammerstein integral equation

$$x(t) = g(t) + \lambda \int_I K(t,s) f(s, x(s))\, ds \tag{1}$$

and a Kneser-type theorem for weakly continuous solutions of the Volterra integral equation

$$x(t) = g(t) + \lambda \int_0^t K(t,s) f(s, x(s))\, ds, \tag{2}$$

where "$\int$" denotes the weak Riemann integral.

A similar existence theorem for strongly continuous solutions of (1) was proved in [8].

In our considerations we apply the following fixed point

Theorem 1. *Let D be a closed and convex subset of a Hausdorff locally convex space such that $0 \in D$, and let G be a continuous mapping of D into itself. If the implication*

$$(V = \operatorname{conv} G(V) \ \text{ or } \ V = G(V) \cup \{0\}) \Longrightarrow \ V \text{ is relatively compact} \tag{3}$$

holds for every subset V of D, then G has a fixed point.

Proof: In our proof we use some ideas from the paper of Daneš [3]. Define a sequence (y_n) by the formulas $y_0 = 0$, $y_{n+1} = G(y_n)$ $(n = 0, 1, \dots)$. Let

$Y = \{y_n : n = 0, 1, \dots\}$. Since $Y = G(Y) \cup \{0\}$, so from (3) it is clear that the set Y is relatively compact. Denote by Z the set of all limit points of the sequence (y_n). It can be verified that $Z = G(Z)$. Now, let $R(X) = \operatorname{conv} G(X)$ for $X \subset D$ and let Ω denote the family of all subsets X of D such that $Z \subset X$ and $R(X) \subset X$. Since D is convex, so $\operatorname{conv} G(D) \subset \operatorname{conv} D = D$. Moreover, D is closed, so it contains all limit points of Y. Hence $D \in \Omega$. Denote by V the intersection of all sets of the family Ω. Because $Z \subset V$, so V is nonempty. Moreover, $Z = G(Z) \subset R(Z) \subset R(V)$ and $R(V) \subset R(X) \subset X$ for every $X \in \Omega$, so $R(V) \subset V$ and, consequently, $R(V) \in \Omega$. Hence $V = R(V)$, i.e. $V = \operatorname{conv} G(V)$. By (3), this implies that $\overline{V}$ is a compact subset of D. Applying now the Schauder-Tychonoff fixed point theorem, we conclude that the mapping G has a fixed point. The proof is completed. $\square$

Our main condition that guarantees the existence of weak solutions of (1) and (2) will be formulated in terms of measure of weak noncompactness ω introduced by De Blasi in [4]. Let us recall that for any nonvoid, bounded subset X of a Banach space E, $\omega(X) = \inf\{t > 0 :$ there exists a weakly compact set C such that $X \subset C + tB\}$, where B is the norm unit ball.

2. Hammerstein integral equation

Let $I = [0, a]$ be a compact interval in $\mathbb{R}$ and let E_1, E_2 be Banach spaces. We assume that E_1 is weakly sequentially complete and

1° $g : I \to E_1$ is a weakly continuous function;
2° $f : I \times E_1 \to E_2$ is a weakly-weakly continuous function such that for any $r > 0$ there exists $m_r > 0$ such that $\|f(s, x)\| \le m_r$ for all $s \in I$ and $\|x\| \le r$;
3° K is a continuous function from I^2 into the space $\mathscr{L}(E_2, E_1)$ of continuous linear functions $E_2 \to E_1$.

Now we shall prove the following existence theorem for equation (1).

Theorem 2. *If* 1°–3° *hold and there exists* $M > 0$ *such that*

$$\omega(f(I \times X)) \le M\omega(X) \tag{4}$$

for each bounded subset X *of* E_1*, then there exists* $\varrho > 0$ *such that for any* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \varrho$ *the equation* (1) *has at least one weakly continuous solution* (*for simplicity, we denote by the same symbol* ω *the measures of weak noncompactness in* E_1 *and* E_2).

Proof: Let $\varrho = \min(\sup_{r>0} \frac{r-c}{aLm_r}, \frac{1}{r(H)})$, where $c = \sup_{t\in I} \|g(t)\|$, $L = \sup_{t,s\in I} \|K(t, s)\|$, and let $r(H)$ be the spectral radius of the integral operator H defined by

$$(Hu)(t) = \int_I M\|K(t, s)\|u(s)\, ds \qquad (u \in C(I, \mathbb{R}),\ t \in I).$$

Fix $\lambda \in \mathbb{R}$ with $|\lambda| < \varrho$, and choose $b > 0$ in such a way that $c + |\lambda| aKm_b < b$. Denote by $C_w(I, E_1)$ the space of weakly continuous functions $I \to E_1$, endowed with the topology of weak uniform convergence, and by $\widetilde{B}$ the set of all weakly continuous functions $I \to B_b$, where $B_b = \{z \in E_1 : \|z\| \leq b\}$. We shall consider $\widetilde{B}$ as a topological subspace of $C_w(I, E_1)$. Put $G(x)(t) = g(t) + F(x)(t)$, where

$$F(x)(t) = \lambda \int_I K(t,s) f(s, x(s))\, ds \quad \text{for} \quad x \in \widetilde{B} \quad \text{and} \quad t \in I.$$

From the inequalities

$\|F(x)(t) - F(x)(\tau)\| \leq |\lambda| \int_I \|K(t,s) - K(\tau, s)\| m_b \, ds$

$\|F(x)(t)\| \leq |\lambda| aLm_b \quad (x \in \widetilde{B},\ t, \tau \in I)$

it is clear that G maps $\widetilde{B}$ into itself and the set $F(\widetilde{B})$ is strongly equicontinuous. Moreover, by using the Krasnoselskii-type

Lemma 1. *Let E be a Banach space. For any $\phi \in E^*$, $\varepsilon > 0$ and $x \in \widetilde{B}$ there exists a weak neighbourhood U of 0 in E such that $|\phi(f(t, x(t)) - f(t, w(t)))| \leq \varepsilon$ for $t \in I$ and $w \in \widetilde{B}$ such that $w(s) - x(s) \in U$ for all $s \in I$,*

it can be shown that G is continuous. Before passing to further considerations, we shall quote the following known lemmas:

Lemma 2. *Let $V \subset C_w(I, E_1)$. Put $V(t) = \{u(t) : u \in V\}$ and $V(T) = \{u(t) : u \in V,\ t \in T\}$. If V is strongly equicontinuous and uniformly bounded, then*

(i) *the function $t \to \omega(V(t))$ is continuous on I;*

(ii) *$\omega(V(T)) = \sup\{\omega(V(t)) : t \in T\}$ for each compact subset T of I.*

Lemma 3. *For any continuous mapping $A : I \to \mathscr{L}(E_2, E_1)$ and for each bounded subset Z of E_2 we have*

$$\omega(\{A(s)z : s \in I,\ z \in Z\}) \leq \max_{s \in I} \|A(s)\| \omega(Z)$$

(*cf.* [1]).

Now we shall show that G satisfies (3). It is clear that the set $\widetilde{B}$ is convex and closed. Let V be a subset of $\widetilde{B}$ such that $V \subset \overline{\mathrm{conv}}\,(G(V) \cup \{0\})$. Put $W = F(V)$, $v(t) = \omega(V(t))$ and $w(t) = \omega(W(t))$ for $t \in I$. Using the properties of ω we get

(5)

$$v(t) \leq \omega(\overline{\mathrm{conv}}\,(G(V)(t) \cup \{0\})) = \omega(G(V)(t)) = \omega(F(V)(t)) = w(t) \quad \text{for} \quad t \in I$$

and, similarly,

$$w(V(T)) \leq w(W(T)) \quad \text{for each subinterval } T \text{ of } I. \tag{6}$$

As W is strongly equicontinuous and uniformly bounded, by Lemma 2 the function $s \to w(s)$ is continuous on I. Fix $t \in I$ and $\eta > 0$. Since I is compact and the functions $s \to w(s)$, $s \to \|K(t,s)\|$ are continuous, so there exists $r > 0$ such that $\|K(t,s)\| \le r$ and $|\lambda| M w(s) \le r$ for $s \in I$. Choose $\delta > 0$ in such a way that

$$\big|\|K(t,s)\| - \|K(t,\tau)\|\big| \le \frac{\eta}{2r} \quad \text{and} \quad M|\lambda|\,|w(s) - w(\tau)| \le \frac{\eta}{2r} \tag{7}$$

for $s, \tau \in I$ such that $|s - \tau| \le \delta$. Divide the interval I into n parts $0 = t_0 < t_1 < \dots < t_n = a$ in such a way that $t_i - t_{i-1} \le \delta$ for $i = 1, \dots, n$. Let $T_i = [t_{i-1}, t_i]$, $i = 1, \dots, n$. By Lemma 2 there exists $\tau_i \in T_i$ such that

$$\omega(W(T_i)) = w(\tau_i), \qquad i = 1, \dots, n. \tag{8}$$

By the mean value theorem we obtain

$$\begin{aligned} &F(x)(t) = \\ &= \sum_{i=1}^{n} \lambda \int_{T_i} K(t,s) f(s, x(s))\, ds \in \lambda \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{\operatorname{conv}}\, (K(t,T_i) f(T_i \times V(T_i))), \end{aligned}$$

where $K(t,T_i) f(T_i \times V(T_i)) = \{K(t,s) f(s,x(s)) : x \in V,\ s \in T_i\}$.

By (4) and Lemma 3 we have $\omega(K(t,T_i) f(T_i \times V(T_i))) \le \|K(t,s_i)\| M \omega(V(T_i))$ for some $s_i \in T_i$. Hence, by (6) and (8)

$$\begin{aligned} w(t) &\le |\lambda| \sum_{i=1}^{n} (t_i - t_{i-1}) \omega(\operatorname{conv}\, K(t,T_i) f(T_i \times V(T_i))) \\ &\le |\lambda| \sum_{i=1}^{n} (t_i - t_{i-1}) \|K(t,s_i)\| M \omega(V(T_i)) \\ &\le |\lambda| \sum_{i=1}^{n} (t_i - t_{i-1}) \|K(t,s)\| M \omega(W(T_i)) \\ &= |\lambda| \sum_{i=1}^{n} (t_i - t_{i-1}) \|K(t,s_i)\| M w(\tau_i). \end{aligned}$$

By (7) we obtain

$$|\lambda| M (t_i - t_{i-1}) \|K(t,s_i)\| w(\tau_i) \le |\lambda| \int_{T_i} M \|K(t,s)\| w(s)\, ds + (t_i - t_{i-1})\eta.$$

Thus

$$w(t) \le |\lambda| \int_I M \|K(t,s)\| w(s)\, ds + a\eta.$$

Since the above inequality holds for every $\eta > 0$, so

$$w(t) \le |\lambda| \int_I M \|K(t,s)\| w(s)\, ds.$$

As $|\lambda| r(H) < 1$, it follows that $w(t) = 0$ and, consequently, by (5), $v(t) = 0$ for $t \in I$. Hence $V(t)$ is relatively compact for $t \in I$ and, by Ascoli's theorem, V is relatively compact in $C_w(I, E_1)$. Applying now Theorem 1, we deduce that there exists $u \in \widetilde{B}$ such that $u = G(u)$. This ends the proof of Theorem 2. $\square$

3. A Kneser-type theorem

In this section we shall consider the equation (2) and we shall prove the following

Theorem 3. *If* $1°$–$3°$ *and* (4) *hold, then there exists an interval* $J = [0, d]$ *such that the set* S *of all weakly continuous solutions of* (2)*, defined on* J*, is nonempty, compact and connected in the space* $C_w(J, E_1)$.

PROOF: The proof of this theorem is based on some ideas from [7]. Let $\varrho = \sup_{r>0} \frac{r-c}{Lm_r}$, where $c = \sup_{t\in J} \|g(t)\|$ and $L = \sup_{t,s\in J} \|K(t,s)\|$. Fix $e < \varrho$ and choose $b > 0$ in such a way that $c + Lm_b e < b$. Let $d = \min(a, e)$ and $J = [0, d]$. Denote by $\widetilde{B}$ the set of all weakly continuous functions $J \to B_b$, where $B_b = \{z \in E : \|z\| \le b\}$. We will consider $\widetilde{B}$ as a topological subspace of $C_w(J, E_1)$. Put $G(x)(t) = g(t) + F(x)(t)$, where

$$F(x)(t) = \int_0^t K(t,s) f(s, x(s))\, ds \quad \text{for} \quad x \in \widetilde{B} \quad \text{and} \quad t \in J.$$

From the inequalities

$\|F(x)(t) - F(x)(\tau)\| \le \int_0^\tau \|K(\tau, s)\| m_b\, ds + (t-\tau) L m_b$

$\|F(x)(t)\| \le L d m_b \quad (x \in \widetilde{B},\ 0 \le \tau \le t \le d)$

it is clear that $G(\widetilde{B}) \subset \widetilde{B}$ and the set $F(\widetilde{B})$ is strongly equicontinuous. By Lemma 1 we can prove that G is continuous. For any positive integer n set

$$G_n(x)(t) = \begin{cases} g(t) & \text{if } 0 \le t \le \frac{d}{n}, \\ g(t) + \int_0^{t-d/n} K(t,s) f(s, x(s))\, ds & \text{if } \frac{d}{n} \le t \le d. \end{cases}$$

Analogously as for G, it can be shown that G_n maps continuously $\widetilde{B}$ into itself. Moreover,

$$\|G_n(x)(t) - G(x)(t)\| \le \frac{d}{n} L m_b \quad \text{for} \quad x \in \widetilde{B} \quad \text{and} \quad t \in J. \tag{9}$$

Further, it can be easily verified that there exists a unique element $x_n \in \widetilde{B}$ such that $x_n = G_n(x_n)$. From the above it is clear that there exists a sequence (u_n) such that $u_n \in \widetilde{B}$ and

$$\lim_{n\to\infty} \sup_{t\in J} \|u_n(t) - G(u_n)(t)\| = 0. \tag{10}$$

Let $V = \{u_n : n \in \mathbb{N}\}$ and let $W = F(V)$. Arguing similarly as in Section 2, it can be shown that V is relatively compact in $C_w(J, E_1)$. Hence (u_n) has a limit point. From (10) and the continuity of G, it follows that $u = G(u)$. This proves that the set S is nonempty.

Further, since G is continuous, so S is closed. As $S = G(S)$, so $\omega(S(t)) = \omega(F(S)(t))$ for $t \in J$. Using again similar arguments as in Section 2, we can show that S is compact in $C_w(J, E_1)$.

Now we shall prove that S is connected. Suppose that it is not connected. Thus there exist nonempty compact sets S_0, S_1 such that $S = S_0 \cup S_1$ and $S_0 \cap S_1 = \emptyset$. Since $C_w(J, E_1)$ is a completely regular space, so there exists (cf. [6, § 41, II, Remark 3]) a continuous function $w : C_w(J, E_1) \to [0, 1]$ such that $w(x) = 0$ for $x \in S_0$ and $w(x) = 1$ for $x \in S_1$. Fix $u_0 \in S_0$, $u_1 \in S_1$ and a positive integer n. Set

$$a_n(t) = r(u_1 - G_n(u_1)) + (1 - r)(u_0 - G_n(u_0)) \qquad (0 \le r \le 1).$$

By (9), we have

$$\|a_n(r)(t)\| \le \frac{d}{n} L m_b \ \text{ for } \ t \in J \ \text{ and } \ 0 \le r \le 1. \tag{11}$$

Hence

$$\|a_n(r)(t) + G_n(x)(t)\| \le \|a_n(r)(t)\| + \|G_n(x)(t)\| \le \frac{d}{n} L m_b + c + (d - \frac{d}{n}) L m_b < b \tag{12}$$

for $x \in \widetilde{B}$, $t \in J$ and $0 \le r \le 1$.

Fix $r \in [0, 1]$. Define a sequence of functions x_i, $i = 1, \ldots, n$, by the formulas

$$x_1(t) = a_n(r)(t) + g(t) \qquad \text{for } \ 0 \le t \le \frac{d}{n}$$

$$\overline{x}_i(t) = \begin{cases} x_i(t) & \text{for } \ 0 \le t \le \frac{i}{n} d, \\ x_i(\frac{i}{n} d) & \text{for } \ \frac{i}{n} d \le t \le d, \end{cases}$$

$$x_{i+1}(t) = \begin{cases} x_i(t) & \text{for } \ 0 \le t \le \frac{i}{n} d, \\ a_n(r)(t) + G_n(\overline{x}_i)(t) & \text{for } \ \frac{i}{n} d \le t \le \frac{i+1}{n} d. \end{cases}$$

Put $u_{nr} = x_n$. From the above definitions and (12) it follows that $u_{nr} \in \widetilde{B}$ and $u_{nr} = a_n(r) + G_n(u_{nr})$.

Now we shall show that u_{nr} depends continuously on r. Since

$$\begin{aligned} &\|a_n(p)(t) - a_n(r)(t)\| = \|p(u_1(t) - G_n(u_1)(t)) + (1 - p)(u_0(t) - G_n(u_0)(t)) - \\ &- r(u_1(t) - G_n(u_1)(t)) + (1 - r)(u_0(t) - G_n(u_0)(t))\| \le \\ &\le |p - r|(\|u_1(t) - G_n(u_1)(t)\| + \|u_0(t) - G_n(u_0)(t)\|) = \\ &= |p - r|(\|G(u_1)(t) - G_n(u_1)(t)\| + \|G(u_0)(t) - G_n(u_0)(t)\|) \le \\ &\le |p - r| \frac{2}{n} d L m_b \quad \text{for } \ 0 \le p \le 1 \ \text{ and } \ t \in J, \end{aligned}$$

so $\lim_{p \to r} u_{np}(t) = u_{nr}(t)$ uniformly on $[0, \frac{d}{n}]$. Thus $\lim_{p \to r} \overline{u}_{np}(t) = \overline{u}_{nr}(t)$ uniformly on J. By the continuity of G_n

$$\lim_{p \to r} \phi(G_n(\overline{u}_{np})(t) - G_n(\overline{u}_{nr})(t)) = 0$$

uniformly on J, so $\lim_{p\to r}\phi(u_{np}(t)-u_{nr}(t))=0$ uniformly on $[\frac{1}{n}d,\frac{2}{n}d]$ and, consequently, $\lim_{p\to r}\phi(u_{np}(t)-u_{nr}(t))=0$ uniformly on $[0,\frac{2}{n}d]$ for $\phi\in E_1^*$. Repeating this argument, we deduce that

$$\lim_{p\to r}\phi(u_{np}(t)-u_{nr}(t))=0 \text{ uniformly on } J$$

for $\phi\in E_1^*$. Hence u_{nr} depends continuously on r and, consequently, the mapping $r\to w(u_{nr})$ is continuous on $[0,1]$. Moreover, $u_{n0}=u_0$ and $u_{n1}=u_1$, so $w(u_{n0})=0$ and $w(u_{n1})=1$. From this we deduce that there exists $r_n\in[0,1]$ such that

$$w(u_{nr_n})=\frac{1}{2}\,. \tag{13}$$

For simplicity put $v_n=u_{nr_n}$. As $\lim_{n\to\infty}a_n(r)=0$ uniformly on r, we get

$$\lim_{n\to\infty}(v_n-G(v_n))=\lim_{n\to\infty}(a_n(r)+G_n(v_n)-G(v_n))=0. \tag{14}$$

Using once more similar arguments as in Section 2, we conclude that the sequence (v_n) has a limit point v. In view of (14) and the continuity of G, we infer that $v\in S$, so $w(s)=0$ or $w(s)=1$. On the other hand, from (13) it is clear that $w(v)=\frac{1}{2}$, which yields a contradiction. $\square$

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