Properties of function algebras in terms of their orthogonal measures

Jan Čerych

Abstract. In the present note, we characterize the pervasive, analytic, integrity domain and the antisymmetric function algebras respectively, defined on a compact Hausdorff space X, in terms of their orthogonal measures on X.

Keywords: compact Hausdorff space X, the sup-norm algebra $C(X)$ of all complex-valued continuous functions on X, its closed subalgebras called function algebras, pervasive (analytic, integrity domain, antisymmetric) function algebra, measure orthogonal to a function algebra

Classification: 46J10

Let X be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra, consisting of all continuous complex-valued functions on X (with respect to usual point-wise algebraic operations) endowed with the sup-norm.

By a <u>function algebra</u> on X we mean any closed subalgebra of $C(X)$ which contains constant functions on X and which separates points of X.

A function algebra A on X is said to be:
<u>pervasive</u> whenever for any nonvoid proper closed subset F of X, the algebra A/F of all restrictions of functions in A to the set F is dense in $C(F)$ with respect to $|\cdot|_F$, the sup-norm on F;
<u>analytic</u> whenever any function f in A which vanishes on a nonvoid open subset of X vanishes identically;
<u>integrity domain</u> whenever A has no nontrivial divisors of zero, i.e. whenever $fg = 0$ for f, g in A implies either $f = 0$ or $g = 0$;
<u>antisymmetric</u> provided it satisfies the following condition: any function in A, which is real-valued, is a constant one.

The mentioned notions are due to Helson, Quigley [1] and Hoffman, Singer [2].

Denote by $M(X)$ the space of all complex Borel regular measures on X, i.e. by the Riesz Representation Theorem, the dual space of $C(X)$.

Whenever A is a closed subspace of $C(X)$, let $A^{\perp}$ be the <u>annihilator</u> of A, or the set of all measures m in $M(X)$ such that $\int f\,dm = 0$ for any f in A. The dual space A' of A is then canonically isomorphic to the quotient space $M(X)/A^{\perp}$.

Now endow $M(X)$ with the weak-star topology: it is well known that $M(X)$ becomes a locally convex topological linear space with the dual space $C(X)$.

Our aim here is to characterize the above-mentioned properties of a function algebra A by means of the properties of the measures in $A^{\perp}$.

We have dealt with a part of these problems in [3].

Theorem 1 (cf. [3])**.** *A function algebra A on X is pervasive if and only if the closed support of any nontrivial measure in $A^{\perp}$ is all X.*

Proof: Let A be pervasive. Fix an arbitrary m in $M(X)$ such that $\emptyset \neq \operatorname{spt} m = F \neq X$, and prove that m does not annihilate A. Without loss of generality, we may assume that $|m|$, the total variation of m on X, is equal to one. Take an f in $C(F)$ such that $\int_F f\,dm = 1$ and choose a g in A such that $|f-g|_F < 1$. Then

$$|1 - \int g\,dm| = |\int (f-g)\,dm| \leq |f-g|_F |m| < 1\,.$$

It follows that m does not annihilate g.

Let, on the contrary, A be not pervasive. Then there exists a closed nonvoid proper subset F of X and a function f in $C(F)$ which is not in the closure of A/F in the sup-norm on F. By the Hahn-Banach Separating Theorem there is an m in $M(F)$ such that $\int_F f\,dm \neq 0$ while $\int_F g\,dm = 0$ for any g in A. If $\mathbf{m}$ is the trivial extension of m from F to X, then $\mathbf{m}$ is in $A^{\perp}$. Indeed, $\int g\,d\mathbf{m} = \int_F g\,dm = 0$ for an arbitrary g in A. But $\operatorname{spt} \mathbf{m} = \operatorname{spt} m \subset F \neq X$ and, m being nontrivial, $\mathbf{m}$ is not trivial, too. $\square$

Remark. Let us mention that the proof above does also work in case of A being merely a function space, not necessarily an algebra.

For a Borel set $G \subset X$, denote by $M(G)$ the set $M(X)/G$ of all restrictions of measures on X to the set G; for m in $M(G)$ let us not distinguish between m and its trivial extension to X.

For a closed set $F \subset X$, denote by I_F the ideal of all functions in $C(X)$ which vanish on F.

Now, let us search for the dual space I'_F of I_F.

Any measure on F annihilates I_F, this implies $I_F^{\perp} \subset M(F)$. F is closed in X and then $M(F)$ is weak-star closed in $M(X)$. Suppose that there is an m in $I_F^{\perp}$ which is not supported by F and take, by the Mazur Separation Theorem for locally convex spaces, an f in $C(X)$ such that $\int f\,dm = 1$ while $\int f\,dn = 0$ for any n in $M(F)$. But then f is in I_F, a contradiction.

So $I_F^{\perp} = M(F)$ and $m \to m + M(F)$ is a canonical isomorphism of $M(X-F)$ onto the quotient space $M(X)/M(F) = I'_F$. If we endow $M(X-F)$ with the w_F^+-topology, i.e. the weak-star topology generated by all functions in I_F, it becomes a locally convex space, and the ideal I_F becomes its dual space.

Proposition. *Let A be a function algebra on X and let F be a closed nonvoid subset of X. Then the following two conditions are equivalent.*

(i) *The intersection $I_F \cap A$ consists of the zero function only;*
(ii) *The set $A^{\perp}/X-F$ is dense in $M(X-F)$ with respect to the w_F^+-topology.*

Proof: Let (ii) be not valid. Take m in $M(X-F)$ but off $\overline{A^\perp/X-F}$, where the bar denotes the w_F^+-closure. Then there is an f in I_F such that $\int f\,dm = 1$ while $\int f\,dn = 0$ for all n in $A^\perp/X-F$.

Now take an arbitrary p in $A^\perp$. Then, f being equal to zero identically on F,

$$\int f\,dp = \int f\,d(p/X-F) = 0$$

and f is in A. But m does not annihilate f, hence $f \neq 0$. The condition (i) fails.

Let (i) be not valid. Take f in $I_F \cap A$ which does not vanish identically on X. Let m in $M(X/F)$ be such that $\int f\,dm = 1$ (e.g. a one-point mass). For any n in $A^\perp$, we have

$$0 = \int f\,dn = \int f\,d(n/X-F)\,,$$

so that f annihilates $A^\perp/X-F$, and, by the continuity, annihilates its closure, too. But f does not annihilate m, m is not in the closure of $A^\perp/X-F$, and the condition (ii) fails. □

Now, let us formulate two simple consequences of the Proposition.

Theorem 2. *A function algebra A on X is analytic if and only if the following condition is fulfilled:*

For an arbitrary open G, which is not dense in X, the set $A^\perp/G$ is dense in $M(G)$ with respect to the w^+_{X-G}-topology.

Theorem 3. *A function algebra A on X is an integrity domain if and only if the following condition is fulfilled:*

Whenever G_1, G_2, is a disjoint couple of open non-dense subsets of X, then the space $A^\perp/G_i$ is $w^+_{X-G_i}$-dense in $M(G_i)$ either for $i=1$ or for $i=2$.

Now, denote by Re $C(X)$, Re $M(X)$, the space of all real parts of the elements in $C(X)$, $M(X)$, respectively. Then Re $M(X)$ is the real dual space of Re $C(X)$. Endow Re $M(X)$ with the weak-star topology. Re $M(X)$ becomes a real locally convex space with the real dual space Re $C(X)$.

Finally, denote by M_0 the set of all measures m in $M(X)$ such that $\int\,dm = 0$; M_0 is then the annihilator of constant functions on X and Re M_0 is the real annihilator of real (or all, the same) constant functions on X.

Theorem 4. *Let A be a function algebra on X. Then A is antisymmetric if and only if the set* Re $A^\perp = \{\mathrm{Re}\, m, m \text{ in } A^\perp\}$ *is dense in* Re M_0 *with respect to the weak-star topology.*

Proof: Let m be in $A^\perp$. Then

$$0 = \int 1\,dm = \int d\,\mathrm{Re}\,m + i\int d\,\mathrm{Im}\,m$$

and both Re m, Im m, are in Re M_0, so Re $A^\perp \subset$ Re M_0.

Suppose Re $A^\perp$ is not dense in Re M_0 and fix an m in Re M_0 but off the weak-star closure of Re $A^\perp$. Take, by the Hahn-Banach Separating Theorem (or Mazur Theorem, more precisely) an f in Re $C(X)$ such that

$$\int f\,dm \neq 0 \text{ and } \int f\,d\mathrm{Re}\;n = 0 \text{ for any } n \text{ in } A^\perp\,.$$

If n is in $A^\perp$, then both Re n, Im n, are in Re $A^\perp$, so that

$$\int f\,dn = \int f\,d\mathrm{Re}\;n + i\int f\,d\mathrm{Im}\;n = 0$$

and f annihilates $A^\perp$ and so f lies in A. But m annihilates constants and does not annihilate f, this implies that f is not constant, so A is not antisymmetric.

Conversely, let f be a non-constant real function in A. There is a measure m in Re $M(X)$ such that

$$\int f\,dm \neq 0 \text{ and } \int dm = 0\,.$$

Then m is in M_0. For an arbitrary n in $A^\perp$,

$$\int f\,d\mathrm{Re}\;n = \mathrm{Re}\int f\,dn = 0\,,$$

so f annihilates Re $A^\perp$ and then, by a continuity reason, it annihilates the whole $\overline{\mathrm{Re}\;A^\perp}$ as well. But f does not annihilate m, thus m is not in Re $A^\perp$, consequently, Re $A^\perp$ is not dense in Re M_0, and Theorem 4 is proved. □

Remark Two. Both proofs, of Theorems 2 and 4, are valid also in the case of mere function spaces. Of course, the notion of "integrity domain" in case of a function space which is not an algebra does not make sense.

References

[1] Helson H., Quigley F., *Maximal algebras of continuous functions*, Proc. Amer. Math. Soc. **8** (1957), 111–114.

[2] Hoffman K., Singer I.M., *Maximal algebras of continuous functions*, Acta Math. **103** (1960), 217–241.

[3] Čerych J., *A word on pervasive function spaces*, Conference. Complex Analysis and Applications '80, Sofia, 1984, pp. 107–109.

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

(*Received* June 8, 1993)