Some remarks about the p-Dirichlet integral*

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Abstract. We discuss variational problems for the p-Dirichlet integral, p non integer, for maps between manifolds, illustrating the role played by the geometry of the target manifold in their weak formulation.

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Let $\mathcal{X}$ and $\mathcal{Y}$ be two compact Riemannian manifolds of dimension respectively n and m. Suppose that $\mathcal{Y}$ be without boundary and isometrically embedded in $\mathbf{R}^N$ as a submanifold. For a given domain Ω in $\mathcal{X}$ consider the variational problem

$$\begin{aligned} \mathcal{D}_p(u) \;&:=\; \int_\Omega |Du|^p\,dx \;\to\; \min \\ u \;:\; \Omega \;&\to\; \mathcal{Y}\,,\;\; u = \varphi \text{ on } \partial\Omega \end{aligned} \tag{1}$$

where $\varphi\,:\,\overline{\Omega} \to \mathcal{Y}$ is a given smooth map and p is a real number with $1 < p < \underline{n} := \min(n, m)$.

It is usual to seek for a minimizer of (1) in the Sobolev class

$$\begin{aligned} W^{1,p}_\varphi(\Omega, \mathcal{Y}) := \{\; u \in W^{1,p}(\Omega, \mathbf{R}^N) \;\mid\; u(x) \in \mathcal{Y} \;\text{ for a.e. }\; x \in \Omega,\\ u = \varphi \text{ on } \partial\Omega \;\}\,. \end{aligned}$$

However, in the generic situation in which the geometry of $\mathcal{Y}$ is non trivial a *gap phenomenon* appears, i.e. we have

$$\begin{aligned} &\inf\{\mathcal{D}_p(u) \mid u \in W^{1,p}_\varphi(\Omega, \mathcal{Y})\} \;< \\ &<\; \inf\{\mathcal{D}_p(u) \mid u \in C^1(\Omega, \mathcal{Y}) \cap C^0_\varphi(\overline{\Omega}, \mathcal{Y})\}\,, \end{aligned} \tag{2}$$

compare [9], [10]. Moreover, in the weak limit process of sequences of smooth maps with equibounded $\mathcal{D}_p$-energies *concentrations* are produced in such a way that

$$\int_\Omega |Du|^p\,dx$$

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is not the relaxed energy of $\mathcal{D}_p$, at least if p is an integer. Those phenomena are primarily due to the fact that maps in $W^{1,p}(\Omega, \mathcal{Y})$ lack the fundamental homological property of having boundaryless graphs in $\Omega \times \mathcal{Y}$ enjoyed instead by smooth maps, compare [6], [5], [8]. In order to overcome those difficulties we proposed in [5], [8] to replace in the generalized approach to (1) the Sobolev classes with the class of *Cartesian currents* and in general with the class of (r, ℓ)-*currents*, still in the case of integer p's.

The situation is slightly different if p is not an integer, and this note aims to state a few remarks in this case. If p is not an integer, no concentration is produced in the weak limit procedure of sequences of smooth maps with equibounded $\mathcal{D}_p$-energies, and the gap is not anymore due to the energy associated to concentrations. Correspondingly, the limit graphs of sequences of smooth graphs do not contain *vertical parts* and they may be identified as a strict subclass of $W^{1,p}(\Omega, \mathcal{Y})$.

Introducing in fact the class

$$RW^{1,p}(\Omega, \mathcal{Y}) := \{\, u \in W^{1,p}(\Omega, \mathcal{Y}) \;\mid\; \partial G_u \llcorner \Omega \times \mathbf{R}^N = 0 \,\} \tag{3}$$

where G_u is the current carried by the graph of u in the sense of (r, ℓ)-currents, and denoting by

$$\begin{aligned} H^{1,p}(\Omega, \mathcal{Y}) \;:=\; & \text{sequential weak closure of} \\ & C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y}) \text{ in } W^{1,p}(\Omega, \mathcal{Y}) \end{aligned} \tag{4}$$

we shall see that

$$H^{1,p}(\Omega, \mathcal{Y}) \subset RW^{1,p}(\Omega, \mathcal{Y}) \subset W^{1,p}(\Omega, \mathcal{Y})\,. \tag{5}$$

Moreover,

$$RW^{1,p}(\Omega, \mathcal{Y}) \underset{\neq}{\subset} W^{1,p}(\Omega, \mathcal{Y}) \tag{6}$$

if $\mathcal{Y}$ has a non trivial homology in a suitable sense.

By a result of Bethuel [1] we also know, still when p is not an integer, that $H^{1,p}(\Omega, \mathcal{Y})$ agrees with the strong closure of $C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y})$ in $W^{1,p}(\Omega, \mathcal{Y})$, and under quite restrictive assumptions on $\mathcal{Y}$ that $RW^{1,p}(\Omega, \mathcal{Y}) = H^{1,p}(\Omega, \mathcal{Y})$, see [2], though in general

$$H^{1,p}(\Omega, \mathcal{Y}) \underset{\neq}{\subset} RW^{1,p}(\Omega, \mathcal{Y})\,. \tag{7}$$

In conclusion we may summarize the situation, and hopefully make it clearer by comparison with the much simpler case of scalar maps, as follows, see for more details [4]. Denote by $\mathcal{A}^p(\Omega, \mathbf{R})$ the class of functions $u \in L^p(\Omega, \mathbf{R})$, $1 \le p < \infty$, which are almost everywhere approximately differentiable in Ω with approximate differential ap Du in L^p. Then, in terms of distributional derivatives, the condition

$$\partial G_u \llcorner \Omega \times \mathbf{R} = 0$$

for $u \in \mathcal{A}^p(\Omega)$ is equivalent to $u \in W^{1,p}(\Omega, \mathbf{R})$. If $p > 1$, $W^{1,p}(\Omega, \mathbf{R})$ agrees with the sequential weak closure of $C^1 \cap W^{1,p}(\Omega, \mathbf{R})$ in $W^{1,p}$, if $p = 1$, the sequential weak closure of smooth functions with equibounded $\mathcal{D}_1$-energies is instead $BV(\Omega, \mathbf{R})$. In the vector valued case $W^{1,p}(\Omega, \mathcal{Y})$ plays an analogous role as $\mathcal{A}^p(\Omega, \mathbf{R})$ in the scalar case; the space $W^{1,p}(\Omega, \mathbf{R})$ has a natural substitute in the class of Cartesian currents or (r, ℓ)-currents (according to which we are in the rectifiable case $p = \underline{n}$ or in the non-rectifiable case $p < \underline{n}$) if p is an integer or in the class $RW^{1,p}(\Omega, \mathcal{Y})$ if p is not an integer. Notice that in the vector valued case, if $p \in \mathbf{N}$ and the geometry of $\mathcal{Y}$ is not trivial, with respect to the analogy to the scalar case we are close to the BV-situation more that to $W^{1,p}$-situation. Apart from some specific and particular cases, compare [7], [2], and [11], the problem of characterizing "strong" and "sequential weak" closure is still largely open.

As in the sequel the structure of manifold for $\mathcal{X}$ is irrelevant, from now on we shall assume that Ω is a bounded open set in $\mathbf{R}^n$. The relevant geometry of $\mathcal{Y}$ will be expressed in terms of the cohomology groups of $\mathcal{Y}$. We assume that for values ℓ to be specified later the De Rham cohomology group of order ℓ, $H^{\ell}_{DR}(\mathcal{Y}, \mathbf{R})$ be non-zero and denote by $[\sigma_1], \dots [\sigma_{\overline{s}}]$ a basis of $H^{\ell}_{DR}(\mathcal{Y}, \mathbf{R})$, where $\sigma_1, \dots \sigma_{\overline{s}}$ are ℓ-forms regarded as ℓ-forms in $\mathbf{R}^N$, or better in a neighborhood of $\mathcal{Y}$ in $\mathbf{R}^N$. Coordinates in $\mathbf{R}^n$ and $\mathbf{R}^N$ with respect to the standard bases $(e_1, \dots, e_n)$, $(\varepsilon_1, \dots, \varepsilon_N)$ are denoted by $(x^1, \dots, x^n)$ and $(y^1, \dots, y^N)$.

For $r \leq \min(n, N)$ and $\ell \leq r$ we denote by $\mathcal{D}^{r,\ell}(\Omega \times \mathbf{R}^N)$ the space of smooth and compactly supported r-forms in $\Omega \times \mathbf{R}^N$ with at most ℓ differentials in the variables y. In coordinates any $\omega \in \mathcal{D}^{r,\ell}(\Omega, \times \mathbf{R}^N)$ can be written as

$$\omega = \sum_{\substack{|\alpha|+|\beta|=r \\ |\beta| \leq \ell}} \omega_{\alpha\beta}(x, y)\, dx^{\alpha} \wedge dy^{\beta} . \tag{8}$$

The dual space of $\mathcal{D}^{r,\ell}(\Omega \times \mathbf{R}^N)$ will be referred as the space of (r, ℓ)-*currents* and denoted by $\mathcal{D}_{r,\ell}(\Omega \times \mathbf{R}^N)$, see [8]. Given a map $u \in W^{1,p}(\Omega, \mathbf{R}^N)$, $p < n$, the (n, ℓ)-*graph* or simply the *graph* of u is defined as the (n, ℓ)-current, ℓ being the integer part of p, $\ell := [p]$, given by

$$G_u(\omega) := \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta| \leq \ell}} \sigma(\alpha, \overline{\alpha}) \int_{\Omega} \omega_{\alpha\beta}(x, u(x))\, M^{\beta}_{\overline{\alpha}}(Du(x))\, dx$$

where ω is given by (8) and $M^{\beta}_{\overline{\alpha}}(Du(x))$ denotes the $(\beta, \overline{\alpha})$ minor of the approximate differential matrix $Du(x)$, $\overline{\alpha}$ is the complement of α in $\{1, 2, \dots, n\}$ and

$\sigma(\alpha, \overline{\alpha})$ denotes the sign of the permutation which reorders in increasing way the multiindex $(\alpha, \overline{\alpha})$.

Similarly we denote by $\mathcal{D}^{r,\ell}(\Omega\times\mathcal{Y})$, $\ell \leq r$, $r \leq \min(n,m)$, the space of r-forms in $\Omega \times \mathcal{Y}$ with at most ℓ differentials in $\mathcal{Y}$. The immersion $i : \mathcal{Y} \to \mathbf{R}^N$ induces a map

$$(\mathrm{id} \bowtie i)^{\#} \; : \; \mathcal{D}^{r,\ell}(\Omega \times \mathbf{R}^N) \;\to\; \mathcal{D}^{r,\ell}(\Omega \times \mathcal{Y})$$

which is onto. The space of (r,ℓ)-*currents in* $\Omega\times\mathcal{Y}$ is then defined as the subspace of (r,ℓ)-currents in $\mathcal{D}_{r,\ell}(\Omega, \times\mathbf{R}^N)$ which vanish on $\ker(\mathrm{id} \bowtie i)^{\#}$. It is easily checked that, if $u \in W^{1,p}(\Omega, \mathcal{Y})$, and $\ell = [p]$, then the (n,ℓ)-graph of u is an (n,ℓ)-current in $\Omega \times \mathcal{Y}$, $G_u \in \mathcal{D}_{n,\ell}(\Omega \times \mathcal{Y})$.

With the previous notations we now set

Definition 1. The *reduced Sobolev* class $RW^{1,p}(\Omega, \mathcal{Y})$ is given by

$$\begin{aligned} RW^{1,p}(\Omega, \mathcal{Y}) := \{\; u \in W^{1,p}(\Omega, \mathcal{Y}) \;|\; & G_u(\pi^{\#} d\varrho \wedge \widehat{\pi}^{\#}\sigma_s) = 0 \\ & \text{for all } s \text{ and for all } \varrho \in \mathcal{D}^{n-\ell-1}(\Omega) \;\}. \end{aligned}$$

Here π and $\widehat{\pi}$ denote the orthogonal projections of $\Omega \times \mathbf{R}^N$ into Ω and $\mathbf{R}^N$ respectively.

Of course, if

$$d\varrho \;=\; \sum_{|\alpha|=n-\ell} \varrho_\alpha(x)\, dx^\alpha\,, \qquad \sigma^s \;=\; \sum_{|\beta|=\ell} \psi^{(s)}_\beta(y)\, dy^\beta,$$

we have

$$\begin{aligned} & G_u(\pi^{\#} d\varrho \wedge \widehat{\pi}^{\#}\sigma_s) \;= \\ & = \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=\ell}} \sigma(\alpha,\overline{\alpha}) \int_\Omega \varrho_\alpha(x) \psi^{(s)}_\alpha(u(x)) M^{\beta}_{\overline{\alpha}}(Du(x))\, dx\,. \end{aligned} \tag{9}$$

Remark 1. For all s one can define the $(n-\ell)$-current in Ω

$$\mathbf{D}_s(u) \;:=\; \pi_{\#}(G_u \llcorner \widehat{\pi}^{\#}\sigma_s)$$

and the $(n-\ell-1)$-current in Ω

$$\mathbf{P}_s(u) \;:=\; \partial \mathbf{D}_s(u)\,.$$

One can see, compare [8, p. 348], that, while $\mathbf{D}_s(u)$ depends on the representative σ_s of $[\sigma_s]$, $\mathbf{P}_s(u)$ depends only on the cohomology class $[\sigma_s]$. Moreover the whole system of conditions

$$\mathbf{P}_s(u) = 0 \qquad s = 1, \dots, \overline{s}$$

depends only on the group $H^\ell_{DR}(\mathcal{Y})$ and not on the chosen basis of $H^\ell_{DR}(\mathcal{Y})$, so that

$$RW^{1,p}(\Omega, \mathcal{Y}) \;=\; \{\, u \in W^{1,p}(\Omega, \mathcal{Y}) \;|\; \mathbf{P}_s(u) = 0 \;\forall\; s\,\}$$

is a subclass of $W^{1,p}(\Omega, \mathcal{Y})$ which is fixed by the cohomology group $H^\ell_{DR}(\mathcal{Y})$.

Remark 2. We notice that the system of conditions $\mathbf{P}_s(u) = 0 \ \forall\ s$ reads

$$\int_\Omega d\alpha \wedge u^{\#}(\sigma^s) = 0 \quad \forall\ s = 1-n\,, \quad \forall\ \alpha \in \mathcal{D}^{n-\ell-1}(\Omega)\,.$$

Since the forms σ^s generate all closed forms modulo exact forms and

$$\int_\Omega d\alpha \wedge u^{\#}(\beta) = 0$$

for any exact form $\beta \in \mathcal{D}^\ell(\mathcal{Y})$, then

$$RW^{1,p}(\Omega, \mathcal{Y}) = \{\ u \in W^{1,p}(\Omega, \mathcal{Y}) \ \mid\ u^{\#}(\beta) = 0 \\ \text{for any closed form } \beta \in \mathcal{D}^\ell(\mathcal{Y})\,\}\,.$$

Compare [2].

Remark 3. One can introduce a notion of boundary of (r, ℓ)-currents, compare [8] and in particular Propositions 2.1 and 3.2. Then we have $\mathbf{P}_s(u) = 0$ for all s if and only if $\partial G_u = 0$ in $\Omega \times \mathcal{Y}$.

Remark 4. In the special case that Ω is the unit ball of $\mathbf{R}^3$, $\mathcal{Y} = S^2$, and $\ell = 2$, there is only one generator of $H^2_{DR}(\mathcal{Y}, \mathbf{R}) \simeq \mathbf{R}$ which is represented by the volume form ω_{S^2} of S^2. In this case

$$\mathbf{D}_1(u)(\alpha) \ = \ \int < \alpha, D(u) > \, dx \qquad \forall\ \alpha \in \mathcal{D}^1(B^3)$$

where $D(u)$ is the vector field

$$D(u) \ := \ (u \cdot u_{x^2} \times u_{x^3}, u \cdot u_{x^3} \times u_{x^1}, u \cdot u_{x^1} \times u_{x^2})\,.$$

Moreover, for $2 < p < 3$ we have

$$RW^{1,p}(\Omega, S^2) \ = \ \{\, u \in W^{1,p}(\Omega, S^2) \ \mid\ \operatorname{div} D(u) = 0\,\}\,.$$

Theorem 1. *Suppose that p is not an integer. Then $RW^{1,p}(\Omega, \mathcal{Y})$ is sequentially weakly closed in $W^{1,p}(\Omega, \mathcal{Y})$.*

Proof: Let $\{u_k\}$ be a weakly converging sequence in $W^{1,p}(\Omega, \mathcal{Y})$. Then $\{Du_k\}$ is equibounded in L^p and $\{M(Du_k)\}$ is equibounded in $L^{p/\ell}$, $p/\ell > 1$. Thus passing to a subsequence

$$\begin{aligned} M(Du_k) &\rightharpoonup M(Du) &\qquad& \text{weakly in } L^{p/\ell} \\ u_k(x) &\to u(x) && \text{for a.e. } x\,. \end{aligned}$$

As the $\psi^{(s)}_\beta$ are bounded in L^∞, we therefore can pass to the limit in

$$G_{u_k}(\pi^{\#} d\varrho \wedge \hat{\pi}^{\#}\sigma_s) = 0\,,$$

compare (9), getting also

$$G_u(\pi^{\#} d\varrho \wedge \hat{\pi}^{\#}\sigma_s) = 0\,.$$

$\square$

Remark 5. Notice that the proof above shows also that $\mathbf{P}_s(u_k) \rightharpoonup \mathbf{P}_s(u)$ provided $M(Du_k) \rightharpoonup M(Du)$ weakly in L^1.

We shall now prove that $RW^{1,p}(\Omega,\mathcal{Y})$ is a proper subspace of $W^{1,p}(\Omega,\mathcal{Y})$ whenever the homology group $H_\ell(\mathcal{Y},\mathbf{Z})$ is not trivial in the sense specified below. Denote by $H_{\ell(tf)}(\mathcal{Y},\mathbf{Z})$ the torsion free part of the singular homology group with integer coefficients $H_\ell(\mathcal{Y},\mathbf{Z})$, $\ell = [p]$. It is well-known that $H_{\ell(tf)}(\mathcal{Y},\mathbf{Z})$ is finitely generated and that it can be represented by choosing a finite set of integer rectifiable cycles $\gamma_1,\dots,\gamma_{\overline{s}}$, $\overline{s}$ being the dimension of $H^\ell_{DR}(\mathcal{Y},\mathbf{R})$, as

$$H_{\ell(tf)}(\mathcal{Y},\mathbf{Z}) \;=\; \Big\{ \sum_{s=1}^{\overline{s}} k_s\,[\gamma_s]_{\mathbf{Z}} \;\Big|\; k_s \in \mathbf{Z} \Big\}\,.$$

We now say that a homology class $[\gamma] \in H_\ell(\mathcal{Y},\mathbf{Z})$ is of the type S^ℓ if $[\gamma]$ contains an S^ℓ-cycle, i.e. there exists a smooth map $\phi : S^\ell \to \mathcal{Y}$, $\phi \in C^1(S^\ell,\mathcal{Y})$, such that the image by ϕ of the current $[\![S^\ell]\!]$ is the homology class of γ. The subgroup of $H_\ell(\mathcal{Y},\mathbf{Z})$ of all homology classes $[\gamma]$ of the type S^ℓ will be denoted by $H_\ell^{(sph)}(\mathcal{Y},\mathbf{Z})$.

Our main assumption on $\mathcal{Y}$ is

(I) *The subgroup*

$$H^{(sph)}_{\ell(tf)}(\mathcal{Y},\mathbf{Z}) \;:=\; H_{\ell(tf)}(\mathcal{Y},\mathbf{Z}) \cap H_\ell^{(sph)}(\mathcal{Y},\mathbf{Z})$$

of $H_\ell(\mathcal{Y},\mathbf{Z})$ is not trivial.

This is clearly equivalent to

(I′) *There exists a map $\phi \in C^1(S^\ell,\mathcal{Y})$ such that, apart from the zero multiple, no integer multiple of the image of $[\![S^\ell]\!]$ by ϕ is homologous to zero*

or to

(I″) *There exists a map $\phi \in C^1(S^\ell,\mathcal{Y})$ and a closed form $\sigma_1 \in \mathcal{D}^\ell(\mathcal{Y})$ such that $\phi_\#[\![S^\ell]\!](\sigma_1) \neq 0$.*

Finally, for the sake of simplicity we shall assume that Ω is bilipschitz homeomorphic to the unit ball of $\mathbf{R}^n$. We then have

Theorem 2. *Suppose p is not an integer and let $\ell = [p]$. If $\mathcal{Y}$ satisfies* (I), *then*

$$RW^{1,p}(\Omega,\mathcal{Y}) \underset{\neq}{\subset} W^{1,p}(\Omega,\mathcal{Y})\,.$$

Proof: It suffices to construct a map $u : B^\ell \times B^{n-\ell} \to \mathcal{Y}$, $u \in W^{1,p}(\Omega,\mathcal{Y})$, such that $u \notin RW^{1,p}(\Omega,\mathcal{Y})$, B^ℓ being the unit ball in $\mathbf{R}^\ell$. We may think of the map $\phi : S^\ell \to \mathcal{Y}$ in (I″) as a map $\psi : B^\ell \to \mathcal{Y}$ which is constant, $\psi = c_0 \in \mathcal{Y}$, on ∂B^ℓ. We now extend ψ to be c_0 on $\mathbf{R}^\ell \setminus B^\ell$. Clearly ψ is a Lipschitz map from $\mathbf{R}^\ell$ into $\mathcal{Y}$.

Next, for $(w,t) \in \mathbf{R}^\ell \times (-1,1)$ we define

$$v(w,t) \;:=\; \begin{cases} \psi(\frac{w}{t}) & t > 0 \\ c_0 & t \leq 0 \end{cases} .$$

Clearly $v \in \operatorname{Lip}(\mathbf{R}^\ell \times (-1,1) \setminus \{0,0\})$ and an easy computation shows that

$$\int\limits_{\mathbf{R}^\ell \times (-1,1)} |Dv|^{\ell+\delta}\, dw\, dt \;<\; \infty\,, \qquad 0<\delta<1,$$

so that $v \in W^{1,p}(B^\ell(0,2) \times B^{n-\ell}(0,2), \mathcal{Y})$.

Finally, set $\Omega = B^\ell(0,2) \times B^{n-\ell}(0,2)$ and consider the map $u\,:\,\Omega \to \mathcal{Y}$ defined by

$$u(w,z) \;:=\; v(w, |z|-1)\,.$$

Clearly, $u \in W^{1,p}(\Omega, \mathcal{Y})$ and $u \in \operatorname{Lip}(\Omega \setminus \Sigma, \mathcal{Y})$ where

$$\Sigma \;:=\; \{\,(w,z) \;\mid\; w=0,\ |z|=1\,\} \;=\; \{0\} \times S^{n-\ell-1}\,.$$

We shall now prove that $u \notin RW^{1,p}(\Omega, \mathcal{Y})$.

As we can assume by (I$''$) that $\psi_{\#}[\![\, B^\ell \,]\!](\sigma_1) = \int_{B^\ell} \psi^{\#}\sigma_1 \neq 0$, it suffices to show that there exists r such that

$$\mathbf{P}_1(u) \;=\; r\delta_0 \times [\![\, S^{n-\ell-1} \,]\!]$$

and, moreover,

$$r \;=\; u_{\#}([\![\, S_{x,\varepsilon} \,]\!])(\sigma_1)$$

where $S_{x,\varepsilon}$ is a small ℓ-sphere centered at a point $x \in \Sigma$ in the $(\ell+1)$-plane orthogonal to Σ at x. Choosing $x_0 = (w_0, z_0),\ z_0 := (1,0,\dots,0)$ we have

$$S_{x_0,\varepsilon} = \{\,(w,z) \;\mid\; |w|^2 + (z_1-1)^2 = \varepsilon^2,\ z_2 = \ldots = z_{n-\ell} = 0\,\}\,,$$

therefore by the definition of u

$$u_{\#}[\![\, S_{x_0,\varepsilon} \,]\!] = u_{\#}[\![\, \mathbf{R}^\ell \times \{(2,0,\dots,0)\} \,]\!] = \psi_{\#}[\![\, \mathbf{R}^\ell \,]\!] = \phi_{\#}[\![\, B^\ell \,]\!]\,.$$

Notice that by a homotopy argument we have

$$u_{\#}[\![\, S_{x,\varepsilon} \,]\!] \;\in\; [\phi_{\#}[\![\, S^\ell \,]\!]]$$

for any $x \in \Sigma_1$ and $\varepsilon < 1$. Therefore we infer

$$r \;=\; u_{\#}[\![\, S_{x,\varepsilon} \,]\!](\sigma_1) = \phi_{\#}[\![\, B^\ell \,]\!](\sigma_1) \neq 0,$$

i.e. $\mathbf{P}_1(u) \neq 0$. □

As we have already mentioned, Bethuel in [1] showed that the sequential weak closure $H^{1,p}(\Omega, \mathcal{Y})$ of $C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y})$ in $W^{1,p}(\Omega, \mathcal{Y})$ agrees with the strong closure of $C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y})$ in $W^{1,p}(\Omega, \mathcal{Y})$, provided p is not an integer. In particular we see that

$$H^{1,p}(\Omega, \mathcal{Y}) \subset RW^{1,p}(\Omega, \mathcal{Y})$$

as trivially $C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y}) \subset RW^{1,p}(\Omega, \mathcal{Y})$ by the Stokes theorem. Under the quite restrictive assumption that $\mathcal{Y}$ is $([p]-1)$-connected, that is, all homotopy groups of order $\leq [p]-1$ of $\mathcal{Y}$ are trivial, it has been proved in [2] that the strong closure of smooth maps in $W^{1,p}$ agrees with $RW^{1,p}(\Omega, \mathcal{Y})$, so that

$$RW^{1,p}(\Omega, \mathcal{Y}) = H^{1,p}(\Omega, \mathcal{Y}).$$

However, the general case seems to be largely open.

Finally, we notice that if p is an integer and $\mathcal{Y}$ has a non trivial geometry as in Theorem 2, then $RW^{1,p}(\Omega, \mathcal{Y})$ is *not* sequentially weakly closed. In order to see that, it suffices to approximate by smooth maps the map u in the proof of Theorem 2 as in [9], [3].

References

[1] Bethuel F., *The approximation problem for Sobolev maps between two manifolds*, Acta Math. **167** (1992), 153–206.

[2] Bethuel F., Coron J., Demengel F., Helein F., *A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds*, C.R.A.S., 1990.

[3] Brezis H., Coron J., Lieb E., *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), 649–705.

[4] Giaquinta M., Modica G., Souček J., *Calculus of Variations and Cartesian Currents*, in preparation.

[5] ———, *Cartesian currents and variational problems for mappings into spheres*, Ann. Sc. Norm. Sup. Pisa **16** (1989), 393–485.

[6] ———, *Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal. **106** (1989), 97–159; *Erratum and addendum*, Arch. Rat. Mech. Anal. **109** (1990, 385–392).

[7] ———, *The Dirichlet energy of mappings with values into the sphere*, Manuscripta Math. **65** (1989), 489–507.

[8] ———, *The Dirichlet integral for mappings between manifolds: Cartesian currents and homology*, Math. Ann. **294** (1992), 325–386.

[9] ———, *The gap phenomenon for variational integrals in Sobolev spaces*, Proc. Roy. Soc. Edinburg **120** (1992), 93–98.

[10] Hardt R., Lin F., *A remark on H^1 mappings*, Manuscripta Math. **56** (1986), p. 1010.

[11] Malý J., *L^p-approximation of Jacobians*, Comment. Math. Univ. Carolinae **32** (1991), 659–666.

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