Boundedness and pointwise differentiability of weak solutions to quasi-linear elliptic differential equations and variational inequalities

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Abstract. The local boundedness of weak solutions to variational inequalities (obstacle problem) with the linear growth condition is obtained. Consequently, an analogue of a theorem by Reshetnyak about a.e. differentiability of weak solutions to elliptic divergence type differential equations is proved for variational inequalities.

Keywords: quasi-linear elliptic equations and inequalities, weak solution, local boundedness, pointwise differentiability, difference quotient

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1. Introduction

In this paper we are interested in local boundedness and a.e. differentiability of weak solutions to the quasi-linear differential equation

$$\operatorname{div} \mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u)$$

and to the variational inequality

$$\int\limits_{\Omega} \mathcal{A}(x,u,\nabla u)\nabla(u-w) + \int\limits_{\Omega} \mathcal{B}(x,u,\nabla u)(u-w) \le 0 \qquad \text{for all } w \in K,$$

where $K = \{u \in W_0^{1,2}(\Omega) : u \ge \psi \text{ in } \Omega\}$.

We will show that a theorem by Serrin about local boundedness of weak solutions (and thus their a.e. differentiability, see [4]) can be proved not only for elliptic differential equations with linear growth conditions on the coefficients but also for variational inequalities of the same type.

We also extend the result about a.e. differentiability to equations and inequalities with coefficients satisfying a quadratic growth condition.

In the following, Ω will be an open subset of $\mathbf{R}^n$, $n \ge 3$. $B_r(x)$ will denote the ball with center at x and radius r, for simplicity we will write B_r instead of

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$B_r(0)$ unless otherwise stated. By $⨍_N f$ we will denote the integral mean value $|N|^{-1}\int_N f$, where $|N|$ is the n-dimensional Lebesgue measure of $N \subset \mathbf{R}^n$.

Since we will be concerned with values of Sobolev functions at a given point, we will, for clarity, consider the representative of a Sobolev function, say u, which satisfies

$$u(x) = \limsup_{r\to 0} ⨍_{B_r(x)} u(y)\,dy\,.$$

Let us first consider the following quasi-linear equation

$$\operatorname{div} \mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u), \tag{1.1}$$

where $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $\mathcal{A} : \Omega\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$ and $\mathcal{B} : \Omega\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ are Carathéodory functions.

We will moreover assume that the function $\mathcal{A}$ satisfies the following ellipticity condition, namely that

$$\begin{aligned} |\mathcal{A}(x,u,q)| &\le a|q| + b(x)|u| + e(x),\\ q\cdot\mathcal{A}(x,u,q) &\ge |q|^2 - d(x)|u|^2 - g(x)\end{aligned} \tag{1.2}$$

hold for all $x\in\Omega$, $u\in R$ and $q\in R^n$. Here $a\ge 1$ is a constant, $b,e\in L^n_{\mathrm{loc}}(\Omega)$ and $d,g\in L^{\frac{n}{2-\varepsilon}}_{\mathrm{loc}}(\Omega)$ for some $0<\varepsilon<1$.

It was shown by Reshetnyak in [9] that if the function $\mathcal{B}$ satisfies the linear growth condition

$$|\mathcal{B}(x,u,q)| \le c(x)|q| + d(x)|u| + f(x), \tag{1.3}$$

where $c\in L^{\frac{n}{1-\varepsilon}}_{\mathrm{loc}}(\Omega)$ and $d,f\in L^{\frac{n}{2-\varepsilon}}_{\mathrm{loc}}(\Omega)$ for some $0<\varepsilon<1$, then the a.e. differentiability of weak solutions to (1.1) is an easy consequence of their Hölder continuity. In the case of the linear equation $\operatorname{div}\big(a(x)\nabla u\big)=0$, the a.e. differentiability of weak solutions was proved independently by Bojarski, see [1]. Hajłasz and Strzelecki showed in [4] that using Bojarski's method one can under the conditions (1.2) and (1.3) simplify Reshetnyak's proof. The idea of the method is as follows:

Definition 1.1. Let $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $x_0\in\Omega$. For $0<h<\frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)$ and $X\in B_2$, we define the difference quotient v_h of u at the point x_0 by

$$v_h(X) = \frac{u(x_0+hX)-u(x_0)}{h}.$$

Theorem 1.2 (Reshetnyak, see Theorem 1 in [8])**.** *Let $u\in W^{k,p}(\Omega)$. Then for a.a. $x\in\Omega$*

$$\lim_{h\to 0}\left\|\frac{1}{h^k}\left[u(x+hX) - \sum_{0\le|\alpha|\le k}\frac{D^\alpha u(x)}{\alpha!}h^{|\alpha|}X^\alpha\right]\right\|_{W^{k,p}(B_2)} = 0.$$

Remark. It is also possible to use a standard result concerning the L^p-derivatives of Sobolev functions, see e.g. Theorem 3.4.2 in [12], instead.

Theorem 1.3 (Stepanov, see [11] or Theorem 3.1.9 in [2]). *For $u : \Omega \to \mathbf{R}^m$ put*

$$A = \left\{ a \in \Omega : \limsup_{x \to a} \frac{|u(x) - u(a)|}{|x - a|} < \infty \right\}.$$

Then A is Lebesgue measurable and u is differentiable a.e. in A.

It is shown that v_h solves an equation similar to (1.1) and this together with a theorem by Serrin about local boundedness of weak solutions to such equations (see Theorem 1 in [10]) is used to obtain the estimate

$$\operatorname{ess\,sup}_{X \in B_1} |v_h| \le Q_h, \tag{1.4}$$

where the constant Q_h depends only on the parameters of the equation (1.1) and on $\|v_h\|_{L^2(B_2)}$. Reshetnyak's theorem (for $k = 1$) implies that

$$\|v_h\|_{L^2(B_2)} \to \left\| \sum_{i=1}^{n} u_{x_i}(x_0) X_i \right\|_{L^2(B_2)}, \qquad \text{as } h \to 0$$

and thus $\|v_h\|_{L^2(B_2)} \le 2|\nabla u(x_0)| + 1$ for small h. It follows that there exists a constant $Q < \infty$ such that $Q_h \le Q$ for sufficiently small h. Hence

$$\limsup_{h \to 0} \frac{|u(x_0 + hX) - u(x_0)|}{h} < \infty$$

for a.a. $x_0 \in \Omega$ and by Stepanov's theorem, the weak solution u is totally differentiable a.e. in Ω.

2. Quadratic growth condition

We will show that with some modifications the above method can be used to prove the almost everywhere differentiability of weak solutions of the equation (1.1) even in the case when the function $\mathcal{B}$ satisfies a (more natural) limited quadratic growth condition

$$|\mathcal{B}(x, u, q)| \le c(x)|q|^2 + d(x)|u|^2 + f(x), \tag{2.1}$$

where $d, f \in L_{\mathrm{loc}}^{\frac{n}{2-\varepsilon}}(\Omega)$ for some $0 < \varepsilon < 1$, $c \in L_{\mathrm{loc}}^{\infty}(\Omega)$ and for a.a. $x_0 \in \Omega$ there exist $0 < \rho < \frac{1}{2} \operatorname{dist}(x_0, \partial\Omega)$ and $\xi > 0$ such that

$$2M \operatorname{ess\,sup}_{x \in B_{2\rho}(x_0)} |c(x)| < 1 - \xi, \tag{2.2}$$

where $M = \operatorname{ess\,sup}_{x \in B_{2\rho}(x_0)} |u(x)|$.

A function $u \in W_{\mathrm{loc}}^{1,2}(\Omega) \cap L^\infty(\Omega)$ is called a weak solution of the equation (1.1), if

$$\int_\Omega \mathcal{A}(x, u, \nabla u) \nabla\varphi \, dx + \int_\Omega \mathcal{B}(x, u, \nabla u)\varphi \, dx = 0$$

is satisfied for all $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

We will need the following simple lemma (for the proof see Lemma 2 in [10]).

Lemma 2.1. *Let α be a positive exponent and let a_i and β_i, $i = 1, 2, \dots, N$ be two sets of real numbers such that $0 < a_i < \infty$ and $0 \le \beta_i < \alpha$. Suppose that z is a positive number satisfying*

$$z^\alpha \le \sum_{i=1}^{N} a_i z^{\beta_i}.$$

Then

$$z \le C \sum_{i=1}^{N} a_i^{\gamma_i},$$

where $\gamma_i = (\alpha - \beta_i)^{-1}$ and the constant C depends only on N, α and β_i.

The following theorem generalizes Serrin's theorem in such a way that it combines Serrin's method with that of Hajłasz and Strzelecki and applies it directly to the difference quotient v_h. This makes it possible to handle the quadratic growth in the calculations and obtain the required estimate (1.4).

Theorem 2.2. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to the equation* (1.1) *and suppose that the conditions* (1.2), (2.1) *and* (2.2) *are satisfied.*

Then for a.a. $x_0 \in \Omega$ there exists $0 < \delta < \rho$ and a constant C depending only on n, ε, ξ, a, M, δ, $u(x_0)$, $b(x_0)$, $d(x_0)$, $e(x_0)$, $f(x_0)$ and $g(x_0)$, such that for $0 < h < \delta$, the difference quotient v_h of the solution u at the point x_0 satisfies the a priori estimate

$$\|v_h\|_{L^\infty(B_1)} \le C \left(\|v_h\|_{L^2(B_2)} + 1 \right).$$

Proof: Step 1: Let $x_0 \in \Omega$ be an L^p-Lebesgue point of the functions b, d, e, f and g (p is taken for each function according to (1.2) and (2.1)), which also satisfies (2.2). It is clear that a.a. $x_0 \in \Omega$ have the above properties. Put $u_0 = u(x_0)$.

Using the change of variables $x = x_0 + hX$ and the definition of a weak solution to the equation (1.1) it is easy to show that for $0 < h < \delta < \rho$, the difference quotient v_h of u is a weak solution to the equation

$$\operatorname{div} \mathcal{A}_h(X, v_h, \nabla v_h) = \mathcal{B}_h(X, v_h, \nabla v_h),$$

where

$$\begin{aligned} \mathcal{A}_h(X, v, q) &= \mathcal{A}(x_0 + hX, u_0 + hv, q), \\ \mathcal{B}_h(X, v, q) &= h\mathcal{B}(x_0 + hX, u_0 + hv, q) \end{aligned} \tag{2.3}$$

for $X \in B_2$, $v \in \mathbf{R}$ and $q \in \mathbf{R}^n$. Since $u \in L^\infty(\Omega)$, we may assume that $b = 0$ and $d = 0$, for if we define

$$\begin{aligned}\bar{e}(x) &= Mb(x)\chi_{B_{2\rho}(x_0)}(x) + e(x),\\ \bar{f}(x) &= M^2 d(x)\chi_{B_{2\rho}(x_0)}(x) + f(x),\\ \bar{g}(x) &= M^2 d(x)\chi_{B_{2\rho}(x_0)}(x) + g(x),\end{aligned}$$

then $\bar{e} \in L^n(\Omega)$, $\bar{f}, \bar{g} \in L^{\frac{n}{2-\varepsilon}}$ and for $x \in B_{2\rho}(x_0)$, $u \in \mathbf{R}$, $|u| \le M$ and $q \in \mathbf{R}^n$, the following simplified conditions hold:

$$\begin{aligned}|\mathcal{A}(x,u,q)| &\le a|q| + \bar{e}(x),\\ |\mathcal{B}(x,u,q)| &\le c(x)|q|^2 + \bar{f}(x),\\ q \cdot \mathcal{A}(x,u,q) &\ge |q|^2 - \bar{g}(x).\end{aligned}$$

It is now straightforward that the functions $\mathcal{A}_h$ and $\mathcal{B}_h$ satisfy

$$\begin{aligned}|\mathcal{A}_h(X,v,q)| &\le a_h|q| + e_h(X),\\ |\mathcal{B}_h(X,v,q)| &\le c_h(X)|q|^2 + f_h(X),\\ q \cdot \mathcal{A}_h(X,v,q) &\ge |q|^2 - g_h(X),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}a_h &= a,\\ c_h(X) &= hc(x_0 + hX),\\ e_h(X) &= \bar{e}(x_0 + hX),\\ f_h(X) &= h\bar{f}(x_0 + hX),\\ g_h(X) &= \bar{g}(x_0 + hX).\end{aligned}$$

An easy calculation (using the fact that x_0 is an L^p-Lebesgue point) shows that by making δ sufficiently small, one can ensure that for $0 < h < \delta$,

$$\begin{aligned}\|e_h\|_{L^n(B_2)} &< 2\alpha(n)^{1/n}\big|e(x_0) + Mb(x_0)\big| + 1\\ \|f_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} &< 1,\\ \|g_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} &< 2^{2-\varepsilon}\alpha(n)^{\frac{2-\varepsilon}{n}}\big|g(x_0) + M^2 d(x_0)\big| + 1,\end{aligned}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbf{R}^n$. For example

$$\begin{aligned}\|f_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} &= h\Big(\int_{B_2} \big|\bar{f}(x_0 + hX)\big|^{\frac{n}{2-\varepsilon}}\, dX\Big)^{\frac{2-\varepsilon}{n}}\\ &= h\Big(2^n\alpha(n) \fint_{B_{2h}(x_0)} \big|f(x) + Md(x)\big|^{\frac{n}{2-\varepsilon}}\, dx\Big)^{\frac{2-\varepsilon}{n}}\\ &\to 0, \quad \text{as } h \to 0.\end{aligned} \tag{2.5}$$

Step 2: We continue by Moser's iteration method (see also [6] and [7]). The calculations are similar to those in the proof of Serrin's theorem, see [10]. Put $\bar{v} = |v_h| + 1$, then clearly

$$1 \le \bar{v} \le \frac{2M}{h} + 1. \tag{2.6}$$

Define for fixed $k \ge 1$

$$\begin{aligned} F(\bar{v}) &= \bar{v}^k, \\ G(v_h) &= F(\bar{v})F'(\bar{v}) \operatorname{sgn} v_h, \\ \phi(X) &= \eta(X)^2 G(v_h), \end{aligned}$$

where η is a nonnegative C^∞ function with compact support in B_2. It then follows from (2.4) that

$$\begin{aligned} &\mathcal{A}_h(X, v_h, \nabla v_h)\nabla\phi(X) + \mathcal{B}_h(X, v_h, \nabla v_h)\phi(X) \\ &\quad \ge \Big((\eta F')^2 - \eta^2 c_h(X)|G|\Big)\,|\nabla v_h|^2 - 2a\eta|\nabla\eta|\,|G|\,|\nabla v_h| \\ &\qquad - 2e_h(X)\eta|\nabla\eta|\,|G| - f_h(X)\eta^2|G| - 2g_h(X)(\eta F')^2. \end{aligned}$$

Using $|G| = \bar{v}(F')^2/k$ and $|F'| \le k|F|$ together with $1 \le \bar{v}$, the last inequality can be simplified by setting $w = w(X) = F(\bar{v})$

$$\begin{aligned} &\mathcal{A}_h(X, v_h, \nabla v_h)\nabla\phi(X) + \mathcal{B}_h(X, v_h, \nabla v_h)\phi(X) \\ &\quad \ge \big(1 - c_h(X)\bar{v}\big)|\eta\nabla w|^2 - 2a|\eta\nabla w|\,|w\nabla\eta| \\ &\qquad - 2ke_h(X)|\eta w|\,|w\nabla\eta| - k^2\widehat{f}(X)|\eta w|^2, \end{aligned}$$

where $\widehat{f} = 2g_h + f_h$.

Using the estimates (2.2) and (2.6) together with the definition of c_h it follows that for $0 < h < \delta < 2M\xi$ and $\widehat{\xi} = \xi^2$,

$$c_h(X)\bar{v} < c(x_0 + hX)(2M + h) < \frac{1-\xi}{2M}(2M + 2M\xi) = 1 - \widehat{\xi}. \tag{2.7}$$

Thus the integration over B_2 together with the definition of a weak solution leads to

$$\begin{aligned} \widehat{\xi}\|\eta\nabla w\|^2_{L^2(B_2)} &\le 2a\int_{B_2} |\eta\nabla w|\,|w\nabla\eta|\,dX \; + 2k\int_{B_2} e_h(X)|\eta w|\,|w\nabla\eta|\,dX \\ &\quad + k^2\int_{B_2} \widehat{f}(X)|\eta w|^2\,dX\,. \end{aligned} \tag{2.8}$$

The terms on the right-hand side can be estimated by means of the Hölder, Sobolev and Minkowski inequalities as follows (see also pages 257 and 258 in [10])

$$
\begin{aligned}
\int_{B_2} |\eta\nabla w|\,|w\nabla\eta|\,dX &\le \|\eta\nabla w\|_{L^2(B_2)}\|w\nabla\eta\|_{L^2(B_2)},\\
\int_{B_2} e_h(X)|\eta w|\,|w\nabla\eta|\,dX &\le \|e_h\|_{L^n(B_2)}\|w\nabla\eta\|_{L^2(B_2)}\|\eta w\|_{L^{2^*}(B_2)}\\
&\le c_1(n)\|e_h\|_{L^n(B_2)}\|w\nabla\eta\|_{L^2(B_2)}\\
&\qquad\cdot\Big(\|w\nabla\eta\|_{L^2(B_2)}+\|\eta\nabla w\|_{L^2(B_2)}\Big),\\
\int_{B_2}\widehat{f}(X)|\eta w|^2\,dX &= \int_{B_2}\widehat{f}(X)|\eta w|^{\varepsilon}|\eta w|^{2-\varepsilon}\,dX\\
&\le c_1(n)\|\widehat{f}\|_{L^{\frac{n}{2-\varepsilon}}(B_2)}\|\eta w\|^{\varepsilon}_{L^2(B_2)}\\
&\qquad\cdot\Big(\|w\nabla\eta\|^{2-\varepsilon}_{L^2(B_2)}+\|\eta\nabla w\|^{2-\varepsilon}_{L^2(B_2)}\Big),
\end{aligned}
$$

where $2^*=2n/(n-2)$ is the Sobolev exponent and $c_1(n)$ is the absolute constant from the Sobolev inequality. Putting $z=\|\eta\nabla w\|/\|w\nabla\eta\|$, $s=\|\eta w\|/\|w\nabla\eta\|$ and inserting the above estimates in (2.8) yields

$$z^2\le\widehat{\xi}^{-1}\Big(2az+2c_1(n)k\|e_h\|(1+z)+c_1(n)k^2\|\widehat{f}\|(s^{\varepsilon}+s^{\varepsilon}z^{2-\varepsilon})\Big).$$

It now follows from Lemma 2.1 that $z\le C_1k^{2/\varepsilon}(1+s)$, or rather

$$\|\eta\nabla w\|_{L^2(B_2)}\le C_1k^{2/\varepsilon}\Big(\|\eta w\|_{L^2(B_2)}+\|w\nabla\eta\|_{L^2(B_2)}\Big),$$

where the constant C_1 depends only on n, ε, a, ξ and on the norms of e_h and $\widehat{f}$. Another use of the Sobolev inequality gives

$$\|\eta w\|_{L^{2^*}(B_2)}\le C_2k^{2/\varepsilon}\Big(\|\eta w\|_{L^2(B_2)}+\|w\nabla\eta\|_{L^2(B_2)}\Big),$$

where $C_2=c_1(n)(C_1+1)$.

Let r and r' be real numbers satisfying $1\le r'<r\le 2$ and let the function $\eta\in C_0^\infty(B_2)$ be chosen so that $0\le\eta\le 1$, $\eta=1$ in $B_{r'}$, $\eta=0$ outside B_r and $|\nabla\eta|\le 2(r-r')^{-1}$. Inserting η to the last estimate yields immediately

$$\|\bar{v}^k\|_{L^{2^*}(B_{r'})}\le 3C_2k^{2/\varepsilon}(r-r')^{-1}\|\bar{v}^k\|_{L^2(B_r)}$$

and by putting $p = 2k$ and $\kappa = n/(n-2)$ it becomes

$$\|\bar{v}\|_{L^{p\kappa}(B_{r'})} \le \left[3C_2(p/2)^{2/\varepsilon}(r-r')^{-1}\right]^{2/p} \|\bar{v}\|_{L^p(B_r)}.$$

Iterating this inequality (with $p_j = 2\kappa^j$, $r_j = 1 + 2^{-j}$ and $r_j' = r_{j+1}$, see also page 259 in [10]) we finally get

$$\|\bar{v}\|_{L^{p_{j+1}}(B_{r_{j+1}})} \le C_3^{\Sigma_1} K^{\Sigma_2} \|\bar{v}\|_{L^2(B_2)},$$

where $K = 2\kappa^{2/\varepsilon}$, $C_3 = 6C_2$ and

$$\Sigma_1 = \sum_{j=0}^{\infty} \kappa^{-j} = \frac{\kappa}{\kappa-1}, \quad \Sigma_2 = \sum_{j=0}^{\infty} j\kappa^{-j} = \frac{\kappa}{(\kappa-1)^2}.$$

By taking a limit for $j \to \infty$, it follows from the definition of $\bar{v}$ that

$$\|v_h\|_{L^\infty(B_1)} \le C\left(\|v_h\|_{L^2(B_2)} + 1\right).$$

It is clear that the constant C depends only on n, δ, a, ξ, M, $u(x_0)$ and on the values of the functions b, d, e, f and g at the point x_0. □

3. Variational inequalities

In this section, we will deal with variational inequalities and will show that a method similar to that described above can be applied to prove that their weak solutions satisfy the a priori estimate (1.4) (and are thus differentiable a.e.).

Let $u \in K = \{u \in W_0^{1,2}(\Omega) : u \ge \psi \text{ in } \Omega\}$, $\psi \le 0$ on $\partial\Omega$, be a weak solution to the variational inequality

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\nabla(u-w)\,dx + \int_\Omega \mathcal{B}(x,u,\nabla u)(u-w)\,dx \le 0 \quad \text{for all } w \in K, \tag{3.1}$$

where $\mathcal{A} : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ and $\mathcal{B} : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ are Carathéodory functions.

To prove the main results of this section, namely Theorems 3.4, 3.5 and 3.6, we will need the following three lemmas.

Lemma 3.1. *Let $x_0 \in \Omega$ and $\delta > 0$ be such that $B_{2\delta}(x_0) \subset \Omega$. Let u be a weak solution to the inequality* (3.1) *and put $u_0 = u(x_0)$. Then the difference quotient v_h (see Definition* 1.1) *satisfies, for $0 < h < 2\delta$ and for all $w_h \in K_h$, the variational inequality*

$$\int_{\Omega_{h,x_0}} \mathcal{A}_h(X,v_h,\nabla v_h)\nabla(v_h - w_h)\,dX + \int_{\Omega_{h,x_0}} \mathcal{B}_h(X,v_h,\nabla v_h)(v_h - w_h)\,dX \le 0, \tag{3.2}$$

where the functions $\mathcal{A}_h$ and $\mathcal{B}_h$ are defined as in (2.3) *and*

$$\Omega_{h,x_0} = \{X \in \mathbf{R}^n : x_0 + hX \in \Omega\},$$

$$\psi_h(X) = (\psi(x_0+hX) - u_0)/h,$$

$$K_h = \{u = v - u_0/h : v \in W_0^{1,2}(\Omega_{h,x_0}),\ u \ge \psi_h \ \text{ in } \Omega_{h,x_0}\}.$$

PROOF: Clearly $v_h \in K_h$. Let $w_h \in K_h$ and put, for $0 < h < 2\delta$ and $X \in \Omega_{h,x_0}$,

$$w(x_0 + hX) = u_0 + hw_h(X).$$

Then $w \in K$ and inserting w into (3.1) and using the definition of the difference quotient v_h we obtain the required result. □

Lemma 3.2 (Lemma 3.1 in Chapter V in [3]). *Let $f(t)$ be a nonnegative function defined on $[r_1, r_2]$, where $r_1 \ge 0$. Suppose that for all $r_1 \le t < s \le r_2$*

$$f(t) \le \theta f(s) + [(s-t)^{-\alpha} A + B],$$

where A, B, α and θ are nonnegative constants and $\theta < 1$. Then for all $r_1 \le r < R \le r_2$

$$f(r) \le C[(R-r)^{-\alpha} A + B],$$

where C is a constant depending only on α and θ.

Lemma 3.3 (Theorem 5.3 in Chapter II in [5]). *Let $u \in W^{1,2}(\Omega)$ and $x_0 \in \Omega$. Suppose that for all $k \ge k_0 > 0$ and $T/2 \le t < s \le T < \operatorname{dist}(x_0, \partial\Omega)$*

$$\int\limits_{A_{k,t}} |\nabla u|^2 \, dx \le \gamma \left[\frac{1}{(s-t)^2} \int\limits_{A_{k,s}} \omega_k^2 \, dx + k^2 |A_{k,s}|^{1-\frac{2-\varepsilon}{n}} \right],$$

where $0 < \varepsilon \le 1$, $\omega_k = \max(u-k, 0)$ and $A_{k,s} = \{x \in B_s(x_0) : u(x) > k\}$.

Then there exists $k' \ge k_0$ depending only on γ, ε, k_0, T and on $\int_{A_{k_0,T}} \omega_{k_0}^2 \, dx$, such that

$$\operatorname{ess\,sup}_{B_{T/2}(x_0)} u(x) \le 2k'.$$

In the following, we will show that the local boundedness of weak solutions can be proved also for variational inequalities, cf. Theorem 1 in [10].

Theorem 3.4. *Let $u \in K$ be a weak solution to the variational inequality* (3.1) *with $K = \{u = v - S : v \in W_0^{1,2}(\Omega),\ u \ge \psi \text{ in } \Omega\}$, where $\psi \le -S$ on $\partial\Omega$ and $S \in \mathbf{R}$. Let $x_0 \in \Omega$, $0 < T < \operatorname{dist}(x_0, \partial\Omega)$ and suppose that*

$$\operatorname{ess\,sup}_{B_T(x_0)} \psi(x) < \infty.$$

We further assume that for all $x \in B_T(x_0)$ and all $u \in \mathbf{R}$, $q \in \mathbf{R}^n$, the following conditions are satisfied:

$$\begin{aligned} |\mathcal{A}(x,u,q)| &\le a|q| + b(x)|u| + e(x), \\ |\mathcal{B}(x,u,q)| &\le c(x)|q| + d(x)|u| + f(x), \\ q \cdot \mathcal{A}(x,u,q) &\ge |q|^2 - d(x)|u|^2 - g(x), \end{aligned} \tag{3.3}$$

where $a \geq 1$ is a real constant, $b, c, e \in L^{\frac{n}{1-\varepsilon}}(B_T(x_0))$ and $d, f, g \in L^{\frac{n}{2-\varepsilon}}(B_T(x_0))$ for some $0 < \varepsilon < 1$.

Then there exists $T' \leq T$ (T' depending on a, ε, n and on the L^p-norms of b, c, d, e, f and g) and $Q \in \mathbf{R}$ (depending only on a, ε, n, x_0, $\|u\|_{L^2(B_{T'}(x_0))}$, $\operatorname{ess\,sup}_{B_T(x_0)} \psi(x)$ and on the L^p-norms of b, c, d, e, f and g) such that

$$\operatorname{ess\,sup}_{B_{T'/2}(x_0)} u(x) \leq Q.$$

Proof: Step 1: Let us, for simplicity, write $B_T = B_T(x_0)$ and $B_{T'} = B_{T'}(x_0)$. First we will show that without loss of generality it can be assumed that $e = 0$, $f = 0$ and $g = 0$. Put

$$m = \|e\|_{L^{\frac{n}{1-\varepsilon}}(B_T)} + \|f\|_{L^{\frac{n}{2-\varepsilon}}(B_T)} + \|g\|^{1/2}_{L^{\frac{n}{2-\varepsilon}}(B_T)}$$

and $\bar{u} = |u| + m$. Then the functions $\mathcal{A}$ and $\mathcal{B}$ obviously satisfy

$$\begin{aligned} |\mathcal{A}(x,u,q)| &\leq a|q| + \bar{b}(x)|\bar{u}|, \\ |\mathcal{B}(x,u,q)| &\leq c(x)|q| + \bar{d}(x)|\bar{u}|, \\ q \cdot \mathcal{A}(x,u,q) &\geq |q|^2 - \bar{d}(x)|\bar{u}|^2, \end{aligned} \tag{3.4}$$

where $\bar{b}(x) = b(x) + e(x)/m$ and $\bar{d}(x) = d(x) + f(x)/m + g(x)/m^2$.

Step 2: For $0 < s \leq T$ and $k \geq \max(\operatorname{ess\,sup}_{B_T} \psi(x), m)$ put

$$\omega(x) = \max(u(x) - k, 0)$$

and define $A_{k,s}$ as in Lemma 3.3. Choose, for $0 < t < s$, a function $\eta \in C^\infty(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_t(x_0)$, $\eta = 0$ outside $B_s(x_0)$ and $|\nabla \eta| \leq 2/(s-t)$.

Put $w = u - \eta\omega$. It is easy to check that $w \in K$ and w is admissible as a test function in (3.1). Since $w = u$ outside $A_{k,s}$, we can integrate over $A_{k,s}$ in (3.1) and the inequality will still remain true. Hence

$$\int\limits_{A_{k,s}} \mathcal{A}(x,u,\nabla u)\nabla(\eta\omega)\,dx + \int\limits_{A_{k,s}} \mathcal{B}(x,u,\nabla u)\eta\omega\,dx \leq 0$$

and using $\nabla u = \nabla \omega$ on $A_{k,s}$ together with (3.4) it follows that

$$\begin{aligned} 0 \geq \int\limits_{A_{k,s}} |\nabla u|^2\,dx - \int\limits_{A_{k,s}} (1-\eta)|\nabla u|^2\,dx - \int\limits_{A_{k,s}} \bar{d}(x)|\bar{u}|^2\,dx \\ - \int\limits_{A_{k,s}} \Big(a|\nabla\omega| + \bar{b}(x)|\bar{u}|\Big)\omega|\nabla\eta|\,dx - \int\limits_{A_{k,s}} \Big(c(x)|\nabla u| + \bar{d}(x)|\bar{u}|\Big)\omega\,dx\,. \end{aligned}$$

Since $0 < \omega \le u \le \bar{u}$ in $A_{k,s}$, we obtain

$$\begin{aligned}\int\limits_{A_{k,s}} |\nabla u|^2\,dx \le \int\limits_{A_{k,s}} (1-\eta)|\nabla u|^2\,dx \\ + 2\int\limits_{A_{k,s}} \bar{d}(x)|\bar{u}|^2\,dx + \int\limits_{A_{k,s}} a\omega|\nabla\eta||\nabla\omega|\,dx \\ + \int\limits_{A_{k,s}} \bar{b}(x)\omega|\nabla\eta|\,|\bar{u}|\,dx + \int\limits_{A_{k,s}} c(x)\omega|\nabla u|\,dx\,.\end{aligned} \tag{3.5}$$

The terms on the right-hand side are estimated by means of the Hölder and Poincaré inequalities. Assuming $|B_s(x_0)| \le 1$, $|\operatorname{spt}\omega| \le \frac{1}{2}|B_s(x_0)|$ and using $|\bar{u}| \le 2k + w$ in $A_{k,s}$ we get

$$\int\limits_{A_{k,s}} a\omega|\nabla\eta||\nabla\omega|\,dx \le \frac{a^2}{2}\int\limits_{A_{k,s}} |\nabla\eta|^2\omega^2\,dx + \frac{1}{2}\int\limits_{A_{k,s}} |\nabla\omega|^2\,dx\,, \tag{3.6}$$

$$\begin{aligned}\int\limits_{A_{k,s}} \bar{b}(x)\omega|\nabla\eta||\bar{u}|\,dx &\le \frac{1}{2}\int\limits_{A_{k,s}} |\nabla\eta|^2\omega^2\,dx + 4k^2\int\limits_{A_{k,s}} \bar{b}(x)^2\,dx + \int\limits_{A_{k,s}} \bar{b}(x)^2\omega^2\,dx \\ &\le \frac{1}{2}\int\limits_{A_{k,s}} |\nabla\eta|^2\omega^2\,dx + 4k^2\|\bar{b}\|^2_{L^{\frac{n}{1-\varepsilon}}(B_T)}|A_{k,s}|^{1-\frac{2(1-\varepsilon)}{n}} \\ &\quad + \|\bar{b}\|^2_{L^{\frac{n}{1-\varepsilon}}(B_T)}\Big(\int\limits_{A_{k,s}} \omega^{2^*}\,dx\Big)^{2/2^*}|A_{k,s}|^{2\varepsilon/n} \\ &\le \frac{1}{2}\int\limits_{A_{k,s}} |\nabla\eta|^2\omega^2\,dx + 4k^2\|\bar{b}\|^2_{L^{\frac{n}{1-\varepsilon}}(B_T)}|A_{k,s}|^{1-\frac{2-\varepsilon}{n}} \\ &\quad + c_1(n)\|\bar{b}\|^2_{L^{\frac{n}{1-\varepsilon}}(B_T)}|A_{k,s}|^{\varepsilon/n}\int\limits_{A_{k,s}} |\nabla\omega|^2\,dx\,,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\int\limits_{A_{k,s}} c(x)\omega|\nabla\omega|\,dx &\le \|c\|_{L^{\frac{n}{1-\varepsilon}}(B)}\Big(\int\limits_{A_{k,s}} |\nabla\omega|^2\,dx\Big)^{1/2} \\ &\quad \cdot\Big(\int\limits_{A_{k,s}} \omega^{2^*}\,dx\Big)^{1/2^*}|A_{k,s}|^{\varepsilon/n}\end{aligned} \tag{3.8}$$

$$\le c_1(n)\|c\|_{L^{\frac{n}{1-\varepsilon}}(B_T)}|A_{k,s}|^{\varepsilon/n}\int\limits_{A_{k,s}}|\nabla\omega|^2\,dx\,,$$

(3.9)

$$\begin{aligned}\int\limits_{A_{k,s}}\bar d(x)|\bar u|^2\,dx &\le 2\int\limits_{A_{k,s}}\bar d(x)\omega^2\,dx + 8k^2\int\limits_{A_{k,s}}\bar d(x)\,dx\\ &\le 2c_1(n)\|\bar d\|_{L^{\frac{n}{2-\varepsilon}}(B_T)}|A_{k,s}|^{\varepsilon/n}\int\limits_{A_{k,s}}|\nabla\omega|^2\,dx\\ &\quad + 8k^2\|\bar d\|_{L^{\frac{n}{2-\varepsilon}}(B_T)}|A_{k,s}|^{1-\frac{2-\varepsilon}{n}},\end{aligned}$$

where $c_1(n)$ is the constant from the Poincaré inequality. We find $T' \le T$ small enough to ensure $|B_{T'}| \le 1$ and

$$c_1(n)\Big(\|\bar b\|^2_{L^{\frac{n}{1-\varepsilon}}(B_T)} + \|c\|_{L^{\frac{n}{1-\varepsilon}}(B_T)} + 4\|\bar d\|_{L^{\frac{n}{2-\varepsilon}}(B_T)}\Big)|B_{T'}|^{\varepsilon/n} \le \frac14.$$

By putting $C = 4\|\bar b\|^2_{L^{\frac{n}{1-\varepsilon}}(B_T)} + 16\|\bar d\|_{L^{\frac{n}{2-\varepsilon}}(B_T)}$, the inequality (3.5) can be for $s \le T'$ rewritten as

$$\text{(3.10)}\quad \begin{aligned}\frac14\int\limits_{A_{k,s}}|\nabla u|^2\,dx &\le \int\limits_{A_{k,s}}(1-\eta)|\nabla u|^2\,dx\\ &\quad + \frac{1+a^2}{2}\int\limits_{A_{k,s}}\omega^2|\nabla\eta|^2\,dx + Ck^2|A_{k,s}|^{1-\frac{2-\varepsilon}{n}}.\end{aligned}$$

Notice that T' and the constant C depend only on a, ε, n and on the norms of $\bar b$, c and $\bar d$. To ensure the assumption $|\operatorname{spt}\omega| \le \frac12|B_s(x_0)|$, we first notice that for all $s \le T'$

$$k^2|A_{k,s}| \le \int\limits_{B_{T'}}|u|^2\,dx$$

and thus there exists $k_0 \ge \max(\operatorname{ess\,sup}_{B_T}\psi, m)$ such that for all $k \ge k_0$, it is

$$|A_{k,s}| \le k^{-2}\|u\|^2_{L^2(B_{T'})} \le \frac12|B_{T'/2}(x_0)|.$$

For such k and for $T'/2 \le s \le T'$, then $|\operatorname{spt}\omega| \le \frac12|B_{T'/2}(x_0)| \le \frac12|B_s(x_0)|$ and the estimates (3.6) to (3.9) hold.

Again, k_0 can be chosen in such a way so that its value depends only on T', $\|u\|_{L^2(B_{T'})}$, x_0, m and $\operatorname{ess\,sup}_{B_T} \psi(x)$. Using $\eta = 1$ in $B_t(x_0)$, it follows from (3.10) that

$$\int\limits_{A_{k,t}} |\nabla u|^2\,dx \le \gamma\Big[\int\limits_{A_{k,s}\setminus A_{k,t}} |\nabla u|^2\,dx + \int\limits_{A_{k,s}} \omega^2|\nabla\eta|^2\,dx + k^2|A_{k,s}|^{1-\frac{2-\varepsilon}{n}}\Big],$$

where $\gamma = 4\max(C,(1+a^2)/2)$.

We will continue by "hole-filling"—add γ-times the left-hand side to both sides of the inequality and using $|\nabla\eta| \le 2/(s-t)$ conclude that for all $k \ge k_0$ and $T'/2 \le t < s \le s_1$ (s_1 is an arbitrary number not exceeding T'),

$$\int\limits_{A_{k,t}} |\nabla u|^2\,dx \le \frac{\gamma}{\gamma+1}\Big[\int\limits_{A_{k,s}} |\nabla u|^2\,dx + \frac{4}{(s-t)^2}\int\limits_{A_{k,s_1}} \omega^2\,dx + k^2|A_{k,s_1}|^{1-\frac{2-\varepsilon}{n}}\Big].$$

Lemma 3.2 implies

$$\int\limits_{A_{k,t}} |\nabla u|^2\,dx \le \widetilde{\gamma}\Big[\frac{1}{(s_1-t)^2}\int\limits_{A_{k,s_1}} \omega^2\,dx + k^2|A_{k,s_1}|^{1-\frac{2-\varepsilon}{n}}\Big],$$

where $\widetilde{\gamma}$ depends only on γ and thus on C and a. By Lemma 3.3, we conclude that

$$\operatorname{ess\,sup}_{B_{T'/2}(x_0)} u(x) \le Q,$$

where Q depends only on $\widetilde{\gamma}$, ε, k_0, T' and on $\int_{A_{k_0,T'}} w_{k_0}^2\,dx \le \int_{B_{T'}} |u|^2\,dx$. The special choice of the constants $\widetilde{\gamma}$ and k_0 above completes the proof. $\square$

Remark. If we put

$$\begin{aligned}\widetilde{\mathcal{A}}(x,u,q) &= -\mathcal{A}(x,-u,-q),\\ \widetilde{\mathcal{B}}(x,u,q) &= -\mathcal{B}(x,-u,-q),\\ \widetilde{K} = -K &= \big\{u = v + S : v \in W_0^{1,2}(\Omega), u \le -\psi \text{ in } \Omega\big\},\end{aligned}$$

then the functions $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ satisfy the conditions (3.3) and $\widetilde{u} = -u$ is a weak solution to the inequality

$$\int\limits_\Omega \widetilde{\mathcal{A}}(x,\widetilde{u},\nabla\widetilde{u})\nabla(\widetilde{u}-\widetilde{w}) + \int\limits_\Omega \widetilde{\mathcal{B}}(x,\widetilde{u},\nabla\widetilde{u})(\widetilde{u}-\widetilde{w}) \le 0 \quad \text{for all } \widetilde{w} \in \widetilde{K}.$$

Assume that the constant Q from Theorem 3.4 satisfies

$$Q < -\operatorname{ess\,sup}_{B_T(x_0)} \psi.$$

Using the notation $\widetilde{A}_{k,s} = \{x \in B_s(x_0) : \widetilde{u}(x) > k\}$, $\widetilde{\omega}(x) = \max(\widetilde{u}(x) - k, 0)$ and $\widetilde{w} = \widetilde{u} - \eta\widetilde{\omega}$, for $s \le T$, $m \le k \le Q$ one easily verifies that $\widetilde{w} \in \widetilde{K}$ and in the same way as in the proof of Theorem 3.4 (with u, w, ω, $\mathcal{A}$ and $\mathcal{B}$ replaced by $\widetilde{u}$, $\widetilde{w}$, $\widetilde{\omega}$, $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$) it can be shown that

$$\operatorname{ess\,sup}_{B_{T'/2}(x_0)}(-u(x)) = \operatorname{ess\,sup}_{B_{T'/2}(x_0)} \widetilde{u}(x) \le Q$$

holds for all $0 < h < \delta$.

The following theorem provides us with the required estimate (1.4).

Theorem 3.5. *Let $u \in K$ be a weak solution to the inequality* (3.1) *with*

$$K = \{u \in W_0^{1,2}(\Omega) : \; u \ge \psi \text{ in } \Omega\}$$

and let $\psi : \mathbf{R}^n \to \mathbf{R}$ be a continuous function differentiable almost everywhere, $\psi \le 0$ on $\partial\Omega$. Assume that the functions $\mathcal{A}$ and $\mathcal{B}$ satisfy, for all $x \in \Omega$ and all $u \in \mathbf{R}$, $q \in \mathbf{R}^n$, the ellipticity condition

$$\begin{aligned} |\mathcal{A}(x,u,q)| &\le a|q| + b(x)|u| + e(x), \\ q \cdot \mathcal{A}(x,u,q) &\ge |q|^2 - d(x)|u|^2 - g(x), \end{aligned} \tag{3.11}$$

where $a \ge 1$ is a real constant, $b, e \in L_{\text{loc}}^{\frac{n}{1-\varepsilon}}(\Omega)$ and $d, g \in L_{\text{loc}}^{\frac{n}{2-\varepsilon}}(\Omega)$ for some $0 < \varepsilon < 1$, and the linear growth condition (1.3).

Then for a.a. $x_0 \in \Omega$ there exists $\delta > 0$ such that the difference quotient v_h of u at x_0 (see Definition 1.1*) satisfies for $0 < h < \delta$ the a priori estimate*

$$\operatorname{ess\,sup}_{X \in B_1} |v_h(X)| \le Q_h.$$

Here the constant Q_h depends only on δ, u, x_0, on the parameters of the variational inequality and on $\|v_h\|_{L^2(B_2)}$.

Proof: Step 1: Let $x_0 \in \Omega$ be an L^p-Lebesgue point of b, c, d, e, f and g (p taken for each function in accordance with (1.3) and (3.11)) and let ψ be totally differentiable at x_0. Clearly a.a. $x_0 \in \Omega$ have the above property. Put $u_0 = u(x_0)$.

By Lemma 3.1 there exists $0 < \delta < \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$ such that the difference quotient v_h of u at x_0 satisfies, for $0 < h < \delta$ and for all $w_h \in K_h$, the inequality

$$\int\limits_{\Omega_{h,x_0}} \mathcal{A}_h(X, v_h, \nabla v_h)\nabla(v_h - w_h)\,dX + \int\limits_{\Omega_{h,x_0}} \mathcal{B}_h(X, v_h, \nabla v_h)(v_h - w_h)\,dX \le 0,$$

with K_h defined as in Lemma 3.1. An easy calculation yields that the functions $\mathcal{A}_h$ and $\mathcal{B}_h$ satisfy

$$\begin{aligned} |\mathcal{A}_h(X,v,q)| &\le a_h|q| + b_h(X)|v| + e_h(X), \\ |\mathcal{B}_h(X,v,q)| &\le c_h(X)|q| + d_h(X)|v| + f_h(X), \\ q \cdot \mathcal{A}_h(X,v,q) &\ge |q|^2 - d_h(X)|v|^2 - g_h(X), \end{aligned}$$

where

$$
\begin{aligned}
a_h &= a, \\
b_h(X) &= hb(x_0 + hX), \\
c_h(X) &= hc(x_0 + hX), \\
d_h(X) &= 2h^2 d(x_0 + hX), \\
e_h(X) &= e(x_0 + hX) + |u_0|\, b(x_0 + hX), \\
f_h(X) &= hf(x_0 + hX) + h|u_0|\, d(x_0 + hX), \\
g_h(X) &= g(x_0 + hX) + 2|u_0|^2 d(x_0 + hX).
\end{aligned}
$$

Another calculation similar to that of (2.5) shows that by making δ small enough one obtains for $0 < h < \delta$

$$
\begin{aligned}
&\|b_h\|_{L^{\frac{n}{1-\varepsilon}}(B_2)} < 1, \\
&\|c_h\|_{L^{\frac{n}{1-\varepsilon}}(B_2)} < 1, \\
&\|d_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} < 1, \\
&\|f_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} < 1, \\
&\|e_h\|_{L^{\frac{n}{1-\varepsilon}}(B_2)} < 2^{1-\varepsilon}\alpha(n)^{\frac{1-\varepsilon}{n}} \left|e(x_0) + |u_0| b(x_0)\right| + 1, \\
&\|g_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} < 2^{2-\varepsilon}\alpha(n)^{\frac{2-\varepsilon}{n}} \left|g(x_0) + 2|u_0|^2 d(x_0)\right| + 1.
\end{aligned}
$$

Step 2: We will distinguish between two cases:

(i) $u_0 = \psi(x_0)$: Then $\lim_{h \to 0} \psi_h(X) = \partial_X \psi(x_0) = \nabla\psi(x_0)X$, since ψ is differentiable at x_0. Further diminishing of δ gives

$$\operatorname{ess\,sup}_{B_2} |\psi_h(X)| \le 2|\nabla\psi(x_0)| + 1.$$

(ii) $u_0 > \psi(x_0)$: Then since ψ is continuous, there exist $\zeta > 0$ and $\delta > 0$ such that $u_0 > \psi(x_0 + hX) + \zeta$ for $0 < h < \delta$ and $X \in B_2$, and thus $\psi_h(X) < -\zeta/\delta < 0$.

In both cases $\operatorname{ess\,sup}_{B_2} \psi_h(X) < \infty$ and thus the assumptions of Theorem 3.4 with $T = 2$ and $x_0 = 0$ are satisfied. Hence there exists $0 < T \le 2$ such that for all $0 < h < \delta$

$$\operatorname{ess\,sup}_{B_{T/2}} v_h(X) \le Q_h. \tag{3.12}$$

Here Q_h depends only on n, ε, a, u, δ, $\|v_h\|_{L^2(B_T)}$ and on the values of b, c, d, e, f and g at the point x_0. To finish the proof we need to estimate $\operatorname{ess\,sup}_{B_{T/2}} (-v_h(X))$.

For **(i)**, it is straightforward that for small enough h

$$\operatorname{ess\,sup}_{B_{T/2}}(-v_h(X)) \le \operatorname{ess\,sup}_{B_2}(-\psi_h(X)) \le 2|\nabla\psi(x_0)| + 1.$$

For **(ii)**, it first follows from Reshetnyak's theorem that δ can be made smaller so that

$$\|v_h\|_{L^2(B_T)} \le L < \infty$$

and thus $Q_h < Q$ holds for $0 < h < \delta$ and some $Q < \infty$. Further diminishing of δ (so that $Q < \zeta/\delta$) yields (for $0 < h < \delta$ and $X \in B_T$) $Q_h < \zeta/h < -\psi_h(X)$, and using the remark after Theorem 3.4 we conclude that

$$\operatorname{ess\,sup}_{B_{T/2}}(-v_h(X)) \le Q_h.$$

Putting $\widehat{\delta} = \delta T/2$ finishes the proof. $\square$

Combining the methods used in the proofs of Theorems 2.2, 3.4 and 3.5, it is possible to prove the a priori estimate (1.4) also for variational inequalities with the limited quadratic growth condition. We will just sketch the main idea of the proof.

Theorem 3.6. *Let $u \in K$ be a weak solution to the variational inequality* (3.1) *with $K = \{u \in W_0^{1,2} \cap L^\infty(\Omega) : u \ge \psi \text{ in } \Omega\}$, where $\psi : \mathbf{R}^n \to \mathbf{R}$ is a continuous function differentiable almost everywhere and $\psi \le 0$ on $\partial\Omega$. Assume that the conditions* (2.1), (2.2) *and* (3.11) *hold.*

Then for a.a. $x_0 \in \Omega$ there exists $\delta > 0$ and constants $Q_h \in \mathbf{R}$, which depend only on δ, u, x_0, on the parameters of the inequality and on $\|v_h\|_{L^2(B_2)}$, such that for $0 < h < \delta$

$$\operatorname{ess\,sup}_{X \in B_1} |v_h(X)| \le Q_h.$$

Proof: Step 1: As in Theorem 3.5, the difference quotient v_h is a solution to the inequality (3.2) with

$$K_h = \{u = v - u_0/h : v \ \in W_0^{1,2}(\Omega_{h,x_0}) \cap L^\infty(\Omega_{h,x_0}),\ u \ge \psi_h \ \text{ in } \Omega_{h,x_0}\}.$$

As in the proof of Theorem 2.2, it can be assumed that $b = 0$ and $d = 0$ and the same calculation yields that for small enough δ and $0 < h < \delta$

$$\begin{aligned}
&\|e_h\|_{L^{\frac{n}{1-\varepsilon}}(B_2)} < 2^{1-\varepsilon}\alpha(n)^{\frac{1-\varepsilon}{n}}\Big|e(x_0) + Mb(x_0)\Big| + 1,\\
&\|f_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} < 1,\\
&\|g_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} < 2^{2-\varepsilon}\alpha(n)^{\frac{2-\varepsilon}{n}}\Big|g(x_0) + M^2 d(x_0)\Big| + 1.
\end{aligned}$$

Step 2: The method used in Step 2 of the proof of Theorem 3.4 is applied to obtain the estimate (3.12). It goes as follows:
We assume that ψ_h is bounded from above on B_2 and put for $s \le 2$ and $k \ge \max(\operatorname{ess\,sup}_{B_2} \psi_h(X), 1)$

$$\begin{aligned} A_{k,s} &= \{x \in B_s : v_h(x) > k\}, \\ \omega = \max(v_h - k, 0), &\quad w_h = v_h - \eta\omega, \end{aligned}$$

where the function η is chosen as in the proof of Theorem 3.4. Then $w_h \in K_h$ and inserting w_h in (3.2) we get using (2.7)

$$\begin{aligned} \widehat{\xi} \int\limits_{A_{k,s}} |\nabla v_h|^2 \, dX \le & \int\limits_{A_{k,s}} (1-\eta)|\nabla v_h|^2 \, dX + \int\limits_{A_{k,s}} a_h \omega |\nabla \eta| |\nabla \omega| \, dX \\ & + \int\limits_{A_{k,s}} e_h(X)\, \omega |\nabla \eta| \, dX + \int\limits_{A_{k,s}} f_h(X) \omega \, dX \\ & + \int\limits_{A_{k,s}} g_h(X) \, dX \, . \end{aligned} \tag{3.13}$$

The terms on the right-hand side are estimated as in the proof of Theorem 3.4. We choose $0 < T \le 2$ such that $\frac{1}{2} c_1(n) \, \|f_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} |B_T|^{\varepsilon/n} \le \widehat{\xi}/4$ and $|B_T| \le 1$ hold and find $k_0 \ge \max(\operatorname{ess\,sup}_{B_T} \psi_h(X), 1)$ such that for $k \ge k_0$, the assumption $|\operatorname{spt} \omega| \le \frac{1}{2} |B_{T/2}|$ holds. The Hölder and Poincaré inequalities then yield

$$\begin{aligned} \int\limits_{A_{k,s}} a_h \omega |\nabla \eta| \, |\nabla \omega| \, dX &\le \frac{a_h^2}{2\widehat{\xi}} \int\limits_{A_{k,s}} |\nabla \eta|^2 \omega^2 \, dX + \frac{\widehat{\xi}}{2} \int\limits_{A_{k,s}} |\nabla \omega|^2 \, dX \, , \\ \int\limits_{A_{k,s}} e_h(X) \omega |\nabla \eta| \, dX &\le \frac{1}{2} \int\limits_{A_{k,s}} |\nabla \eta|^2 \omega^2 \, dX + \frac{1}{2} \|e_h\|^2_{L^{\frac{n}{1-\varepsilon}}(B_2)} |A_{k,s}|^{1-\frac{2-\varepsilon}{n}}, \\ \int\limits_{A_{k,s}} f_h(X) \omega \, dX &\le \frac{1}{2} c_1(n) \|f_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} \int\limits_{A_{k,s}} |\nabla \omega|^2 \, dX \, |A_{k,s}|^{\varepsilon/n} \\ & \quad + \frac{1}{2} \|f_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} |A_{k,s}|^{1-\frac{2-\varepsilon}{n}}, \\ \int\limits_{A_{k,s}} g_h(X) \, dX &\le \|g_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} |A_{k,s}|^{1-\frac{2-\varepsilon}{n}} \end{aligned}$$

and (3.13) can be rewritten as

$$\frac{\widehat{\xi}}{4}\int\limits_{A_{k,s}} |\nabla v_h|^2\,dX \le \int\limits_{A_{k,s}} (1-\eta)|\nabla v_h|^2\,dX + \frac{\widehat{\xi}+a_h^2}{2\widehat{\xi}}\int\limits_{A_{k,s}} \omega^2|\nabla\eta|^2\,dX + Ck^2|A_{k,s}|^{1-\frac{2-\varepsilon}{n}},$$

where $C = \frac{1}{2}\|e_h\|_{L^{\frac{n}{1-\varepsilon}}(B_2)} + \frac{1}{2}\|f_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)} + \|g_h\|_{L^{\frac{n}{2-\varepsilon}}(B_2)}$. The rest of Step 2 goes as in the proof of Theorem 3.4.

Step 3: It is first shown that the function ψ_h is bounded from above on B_2. This is done in the same way as in Step 2 of Theorem 3.5. The estimate (3.12) follows. Finally the trick of the remark after Theorem 3.4 is used on the inequality (3.2) and this gives the required estimate for $\widetilde{v}_h = -v_h$. $\square$

References

[1] Bojarski B., *Pointwise differentiability of weak solutions of elliptic divergence type equations*, Bull. Acad. Polon. Sci. **33** (1985), 1–6.

[2] Federer H., *Geometric Measure Theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[3] Giaquinta M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, Princeton, New Jersey, 1983.

[4] Hajłasz P., Strzelecki P., *A new proof of Reshetnyak's theorem concerning the pointwise differentiability of solution of quasilinear equations*, Preprint, Institute of Mathematics, Warsaw University, PKIN IXp., 00-901 Warsaw.

[5] Ladyzhenskaya O.A., Ural'tseva N.N., *Linear and Quasilinear Elliptic Equations*, 2nd ed., Nauka Press, Moscow, 1973, English translation Academic Press, New York, 1968.

[6] Moser J., *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **XIII** (1960), 457–468.

[7] ———, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **XIV** (1961), 577–591.

[8] Reshetnyak Yu.G., *Generalized derivatives and differentiability almost everywhere*, Mat. Sb. **75** (**117**) (1968), 323–334 (in Russian); Math. USSR–Sb. **4** (1968), 293–302 (English translation).

[9] ———, *O differentsiruemosti pochti vsyudu resheniĭ ellipticheskikh uravneniĭ*, Sibirsk. Mat. Zh. **XXVIII** (1987), 193–195.

[10] Serrin J., *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302.

[11] Stepanoff M.W., *Sur les conditions de l'existence de la différentielle totale*, Matematiceskij Sbornik, Rec. Math. Soc. Math. Moscou **XXXII** (1925), 511–527.

[12] Ziemer W.P., *Weakly Differentiable Functions*, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

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