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Non-idempotent left symmetric left distributive groupoids

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Abstract. Subdirectly irreducible non-idempotent groupoids satisfying $x \cdot xy = y$ and $x \cdot yz = xy \cdot xz$ are studied.

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Selfdistributive groupoids appear in various situations both algebraic and non-algebraic. In the latter case they usually turn out to be idempotent and (left) symmetric. Two-sided distributive groupoids are very special subdirect products of idempotent distributive groupoids and semigroups nilpotent of class at most 3, and so many structural problems are easily reduced to the idempotent case. Nothing like that is true for general one-sided distributive groupoids; even the structure of one-generated left distributive groupoids is complicated enough (see e.g. [1] and [2]). The purpose of this short note is to initiate the study of non-idempotent left symmetric left distributive groupoids and to make a few first steps toward the question how far from the idempotent case these groupoids are.

1. Introduction

By an LSLD-groupoid we shall mean a left symmetric left distributive groupoid, i.e. a groupoid satisfying the identities $x \cdot xy = y$ and $x \cdot yz = xy \cdot xz$. By an LSLDI-groupoid we shall mean an idempotent LSLD-groupoid.

Let G be an LSLD-groupoid. Define a relation p_G on G by $(a, b) \in p_G$ iff $ax = bx$ for every $x \in G$. Then p_G is a congruence of G and G/p_G is idempotent. Moreover, $ip_G \subseteq p_G$, where ip_G is defined by $(a, b) \in ip_G$ iff either $a = b$ or $a = bb$. Clearly, ip_G is the smallest congruence of G such that the corresponding factor is idempotent. If A is a non-trivial block of ip_G, then A is a two-element subgroupoid of G and A is isomorphic to the following groupoid T:

T	0	1
0	1	0
1	1	0

1.1 Proposition. *Let G be an LSLD-groupoid. Then T is a homomorphic image of G iff G is isomorphic to the cartesian product $T \times H$, $H = G/ip_G$ (H is an LSLDI-groupoid).*

Proof: Let r be a congruence of G such that $G/r \cong T$. Then $r \cap ip_G = id_G$ and $f : x \to (g(x), h(x))$ is an isomorphism of G onto $G/r \times H$, where $g : G \to G/r$ and $h : G \to H$ are the natural projections.

Let G be an LSLD-groupoid. Define a relation u_G on G by $(a, b) \in u_G$ iff $a = c_1(\dots(c_n b))$ for some $n \geq 1$ and $c_1, \dots, c_n \in G$. Then u_G is just the smallest congruence of G such that the corresponding factor is a semigroup of right zeros.

Let G be an LSLD-groupoid. Then the set $Id(G) = \{a \in G; a = aa\}$ is either empty or a left ideal of G. Moreover, the transformation $o_G : x \to x^2$ is an automorphism of G, $o_G^2 = id_G$ and $(x, o_G(x)) \in p_G$ for every $x \in G$. □

2. Examples

2.1. Let f be a transformation of a non-empty set G such that $f^2 = id_G$. Define a multiplication on G by $xy = f(y)$. Then G becomes an LSLD-groupoid and $Id(G) = \{a \in G; f(a) = a\}$.

2.2. Let G be a groupoid such that $G = A \cup B$, where A is a subgroupoid of G and an LSLD-groupoid and every element from B is left neutral and right absorbing in G. Then G is an LSLD-groupoid.

2.3. Let f be an automorphism of a group G such that $f^2 = id_G$ and let $a \in G$ be such that $a^2 = 1$ and $f(a) = a$. Put $x * y = xf(x^{-1}y)a$ for all $x, y \in G$. Then $G(*)$ is an LSLD-groupoid and $Id(G(*)) = \emptyset$ for $a \neq 1$.

2.4. Let f be an automorphism of a group G such that $f^2 = id_G$ and $xf(x) \in Z(G)$ for every $x \in G$ and let $a \in G$ be such that $f(a) = a^{-1}$ and $a^{-1}x^{-1}ax \in Z(G)$ for every $x \in G$. Put $x * y = xf(y)af(x^{-1})$ for all $x, y \in G$. Then $G(*)$ is an LSLD-groupoid and $Id(G(*)) = \emptyset$ for $a \neq 1$.

2.5. Let f be an automorphism of a group G such that $f^2 = id_G$ and $f(x^2)x^2 = 1$, $f(x)x \in Z(G)$ for every $x \in G$. Put $x * y = xf(y)x$ for all $x, y \in G$. Then $G(*)$ is an LSLD-groupoid and $Id(G(*)) = \{a \in G; f(a) = a^{-1}\}$.

2.6. Let G be an LSLD-groupoid and f an automorphism of G such that $f^2 = id_G$ and $(x, f(x)) \in p_G$ for every $x \in G$. Let e be an element not belonging to G, $K = G \cup \{e\}$, and define a multiplication on K as follows: G is a subgroupoid of K; $xe = e$ and $ey = f(y)$ for all $x \in K$ and $y \in G$. We obtain a new groupoid $K = G[e, f]$ and it is easy to check that K is an LSLD-groupoid.

2.7. Let f be an automorphism of an LSLD-groupoid G such that $f^2 = id_G$ and $(x, f(x)) \in p_G$ for every $x \in G$. Put $x * y = f(xy)$ for all $x, y \in G$. Then $G(*)$ is an LSLD-groupoid and $G(*)$ is idempotent if $f = o_G$.

2.8. The following two groupoids are the only two-elements LSLD-groupoids (up to isomorphism):

T	0	1
0	1	0
1	1	0

S	0	1
0	0	1
1	0	1

2.9. The following five groupoids are the only three-element LSLD-groupoids:

R_1	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

R_2	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

R_3	0	1	2
0	0	2	1
1	0	1	2
2	0	1	2

R_4	0	1	2
0	0	1	2
1	0	2	1
2	0	2	1

R_5	0	1	2
0	0	2	1
1	0	2	1
2	0	2	1

2.10. Consider the following four-element groupoid Q:

Q	0	1	2	3
0	1	0	2	3
1	1	0	2	3
2	1	0	2	3
3	0	1	2	3

Then Q is a subdirectly irreducible LSLD-groupoid.

3. Subdirectly irreducible non-idempotent left symmetric left distributive groupoids

3.1 Proposition. *Let G be a non-idempotent subdirectly irreducible LSLD-groupoid, $I = Id(G)$ and $K = G - I$. Then:*

(i) *K is a block of u_G and $ip_G = ip_K \cup id_G$ is just the smallest non-trivial congruence of G.*

(ii) *If $I = \emptyset$, then G is left-ideal-free (i.e. G possesses no proper left ideals).*

(iii) *$p_G \mid I = id_I$.*

Proof: (i) Let s denote the smallest non-trivial congruence of G. Further, let A be a block of u_G such that $A \subseteq K$. Then A is a left ideal of both G and K and $r = ip_A \cup id_G$ is a congruence of G. Consequently, $s \subseteq r \subseteq ip_G$.

Now, let $a, b \in G$, $a \neq b$, $(a, b) \in s$. Then $a, b \in A$ and this shows that A is the only block of u_G contained in K. In other words, $K = A$ is a block of u_G. Further, if $c \in K$, then $c = d_1(\dots(d_n a))$, $(c, d) \in s$, where $d = d_1(\dots(d_n b))$, and $c = dd$, since $a = bb$. This shows that $s = ip_G$.

(ii) and (iii) These assertions are clear. $\square$

3.2 Corollary. *A non-idempotent LSLD-groupoid G is subdirectly irreducible iff every proper factorgroupoid of G is idempotent.*

3.3 Proposition. *Let f be an automorphism of an LSLD-groupoid K such that $f^2 = id_K$ and $(x, f(x)) \in p_K$ for every $x \in K$. Put $G = K[0, f]$, where $0 \neq K$* (*see* 2.6). *The following conditions are equivalent:*

(i) *G is subdirectly irreducible and $Id(K) = \emptyset$.*
(ii) *G is subdirectly irreducible, non-idempotent and K is a block of u_G.*
(iii) *K is subdirectly irreducible and $Id(K) = \emptyset$.*

If these conditions are satisfied, then K is left-ideal-free, $\{0\}$, K and G are the only left ideals of G and either $f = id_K$ or *$f = o_K$.*

Proof: (i) implies (ii) This follows immediately from 3.1 (i).

(ii) implies (iii) Since G is not idempotent, we have $K \neq Id(K)$ and K is non-trivial. Further, $Id(K) \subseteq Id(G)$, $Id(G)$ is a left ideal of G and K is a block of u_G. This implies that $Id(K)$ is empty. G is subdirectly irreducible and ip_G is the smallest non-trivial congruence of G by 3.1 (i). Assume, for a moment, that $f \neq id_K$. Then $s \neq id_G$, where s is the congruence of G defined by $(a, b) \in s$ iff either $a = b$ or $a, b \in K$ and $a = f(b)$. Consequently, $ip_G \subseteq s$ and we see that $f = o_K$.

Now, let $r \neq id_K$ be a congruence of K. Then $r \cup id_G$ is a congruence of G, $ip_G \subseteq r \cup id_G$ and $ip_K \subseteq r$. We have proved K is subdirectly irreducible. Finally, K is left-ideal-free by 3.1 (ii).

(iii) implies (i) Let r denote the smallest non-trivial congruence of K. Define a relation q on K by $(a, b) \in q$ iff $(f(a), f(b)) \in r$. Then $q \neq id_K$ is a congruence of K and we conclude that r is invariant under f. However, then $s = r \cup id_G$ is a congruence of G.

Now, let $t \neq id_G$ be a congruence of G. If $t \mid K \neq id_K$, then $s \subseteq t$. Hence, assume that $t \mid K = id_K$. Since $t \neq id_G$, the set $J = \{a \in K; (a, 0) \in t\}$ is non-empty. But then J is a left ideal of K, $J = K$, $t = G \times G$ and $t \mid K \neq id_K$, a contradiction. □

3.4 Theorem. **(i)** *Let K be a subdirectly irreducible LSLD-groupoid without idempotent elements, let $0 \notin K$ and $G_1 = K[0, id_K]$, $G_2 = K[0, o_K]$. Then both G_1 and G_2 are subdirectly irreducible LSLD-groupoids, $Id(G_1) = \{0\} = Id(G_2)$ and G_1, G_2 are not isomorphic.*

(ii) *Let G be a subdirectly irreducible LSLD-groupoid such that $Id(G) = \{0\}$ is a one-element set. Then $K = G - \{0\}$ is a subdirectly irreducible LSLD-groupoid without idempotents and either $G = K[0, id_K]$ or $G = K[0, o_K]$.*

Proof: (i) See 3.3.

(ii) Clearly, K is non-trivial. Since K is a left ideal of G, $f = L_{0,G} \mid K$ is an automorphism of K; $L_{0,G}$ is the left translation by 0 and we have $f^2 = id_K$. On the other hand, $Id(G) = \{0\}$ is a left ideal of G and hence $x0 = 0$ for every

$x \in K$. Further, if $y \in K$, then $0x \cdot y = 0(x \cdot 0y) = 0(x0 \cdot xy) = 0(0 \cdot xy) = xy$, i.e. $(x, f(x)) \in p_K$. Finally, $G = K[0, f]$ and it remains to use 3.3. □

3.5. T and S are (up to isomorphism) LSLD-groupoids; the only two-element S is idempotent and T is without idempotent elements. Clearly, both T and S are simple, and hence subdirectly irreducible.

3.6. R_2, R_3, R_4 and R_5 are the only subdirectly irreducible three-element LSLD-groupoids; R_2 is simple and idempotent, R_3 is idempotent, not left-ideal-free and $R_3 = S[0, f]$, where f is the unique non-identical automorphism of S; R_4 and R_5 contain each just one idempotent element, $R_4 = T[0, id]$ and $R_5 = T[0, o]$.

3.7. Let G be a four-element subdirectly irreducible LSLD-groupoid. We show that either G is idempotent or G is isomorphic to the groupoid Q from 2.10 (and then G contains just two idempotent elements).

Put $I = Id(G)$ and $K = G - I$. First, let $I \neq \emptyset$, $G = \{a, b, c, d\}$, $b = aa$, $d = cc$. By 3.1, $ip_G = \{(a, b), (b, a), (c, d), (d, c)\} \cup id_G$ is the smallest non-trivial congruence of G and G is left-ideal-free. On the other hand, $cc = d$, $cd = c \cdot cc = cc \cdot cc = c$, and so $ca, cb \in \{a, b\}$. Similarly, $da, db \in \{a, b\}$ and we see that $\{a, b\}$ is a left ideal, a contradiction.

If $\operatorname{card}(I) = 1$, then, by 3.4 (ii), K is a subdirectly irreducible LSLD-groupoid without idempotent elements and K contains three elements. However, by 3.6, such a groupoid does not exist.

If $\operatorname{card}(I) = 2$, then $I \cong S$ and $p_G \mid I = id_I$. Moreover, $K \cong T$. We can assume that $G = \{a, b, c, d\}$, $I = \{c, d\}$, $K\{a, b\}$ and $ca = b$, $cb = a$. Then $da = a$, $db = b$. From this, $a = ab = a \cdot ca = ac \cdot aa = ac \cdot b$, and so $ac = c$, $ad = d$, $bc = c$, $bd = d$ and $G \cong Q$.

If $\operatorname{card}(I) \geq 3$, then $\operatorname{card}(K) \leq 1$, and hence $K = \emptyset$ and G is idempotent.

3.8. Let G be a five-element subdirectly irreducible LSLD-groupoid. We show that G is idempotent.

Again, put $I = Id(G)$ and $K = G - I$. Since G contains an odd number of elements, the involution o_G has a fixed point and this means that $\operatorname{card}(I) \geq 1$. Further, with respect to 3.7 and 3.4 (ii), $\operatorname{card}(I) \geq 2$. If $\operatorname{card}(I) = 2$, then K is a three-element LSLD-groupoid without idempotents, but such a groupoid does not exist. Consequently, $\operatorname{card}(I) \geq 3$. If $\operatorname{card}(I) \geq 4$, then $K = \emptyset$ and G is idempotent. Assume finally that $\operatorname{card}(I) = 3$.

Let $G = \{a, b, c, d\}$, $I = \{c, d, e\}$, $K = \{a, b\}$; clearly, $K \cong T$, $aa = b = ba$, $bb = a = ab$. Define a mapping $q : I \to S$ by $q(x) = 0$ if $L_x \mid K = id_K$ and $q(x) = 1$ if $L_x \mid K = o_K$. Then q is a homomorphism of I into S. Further, I is isomorphic to one of the groupoids R_1, R_2, R_3. If $I \cong R_1$, then $(u, v) \in p_G$, where $u, v \in I$ are such that $u \neq v$ and $q(u) = q(v)$, and this is a contradiction with 3.1 (iii). Similarly if $I \cong R_2$. Finally, if $I \cong R_3$, then $\ker(q) = I \times I$ and it is easy to see that $r = \ker(q) \cup id_G$ is a congruence of G such that $r \neq id_G$ and $r \cap ip_G = id_G$, a contradiction with the subdirect irreducibility of G.

4. Varieties of non-idempotent left symmetric left distributive groupoids

4.1 Proposition. (i) *T is a free LSLD-groupoid of rank* 1.

(ii) *Let F be a free LSLD-groupoid of rank $\alpha \geq 1$. Then $E = F/ip_F$ is a free LSLDI-groupoid of rank α and F is isomorphic to the product $T \times E$.*

Proof: (i) Every LSLD-groupoid satisfies $x = xx \cdot xx$.

(ii) By (i), T is a homomorphic image of F and the rest is clear from 1.1. □

4.2 Corollary. *No free LSLD-groupoid is right cancellative.*

4.3. Let $\mathcal{K}, \mathcal{I}, \mathcal{D}$ and $\mathcal{R}$ denote the varieties of LSLD-groupoids, LSLDI-groupoids, left symmetric left unars ($\mathcal{D}$ is determined by $xz = yz$ and $x \cdot xy = y$) and RZ-semigroups, respectively.

(i) The only proper non-trivial subvariety of $\mathcal{D}$ is $\mathcal{R}$.

(ii) Let $\mathcal{V}$ be a variety of LSLD-groupoids such that $\mathcal{V} \nsubseteq \mathcal{I}$. If $F \in \mathcal{V}$ is free, then $F \cong T \times F/ip_F$ and this implies that $\mathcal{V}$ is generated by $(\mathcal{V} \cap \mathcal{I}) \cup \mathcal{D}$.

(iii) Let $\mathcal{U}$ be a subvariety of $\mathcal{I}$ such that $\mathcal{R} \subseteq \mathcal{U}$ and let $\mathcal{V}$ be the variety generated by $\mathcal{U} \cup \mathcal{D}$. Then $\mathcal{V} \cap \mathcal{I} = \mathcal{U}$.

(iv) Let $C_2 = \{0, 1\}$ denote a two-element chain and let $\mathcal{L}(\mathcal{K})$ be "the lattice of subvarieties" of LSLD-groupoids. Further, let $\mathcal{M}$ designate the collection of ordered pairs $(i, \mathcal{U})$, where $\mathcal{U}$ is a subvariety of $\mathcal{I}$ and either $i = 0$ or $i = 1$ and $\mathcal{R} \subseteq \mathcal{U}$; $\mathcal{M}$ is ordered by $(i, \mathcal{U}) \leq (j, \mathcal{W})$ iff $\mathcal{U} \subseteq \mathcal{W}$ and $i \leq j$. Then $\mathcal{L}(\mathcal{K})$ is isomorphic to $\mathcal{M}$. The isomorphism is given by $\mathcal{V} \to (0, \mathcal{V})$ if $\mathcal{V} \subseteq \mathcal{I}$ and $\mathcal{V} \to (1, \mathcal{V} \cap \mathcal{I})$ if $\mathcal{V} \nsubseteq \mathcal{I}$.

4.4. The variety of LSLD-groupoids is equivalent to the variety of LSLDI-groupoids supplied with one unary operation f satisfying $f(x)f(y) = f(xy)$, $f^2(x) = x$ and $xy = f(x)y$. The equivalence is given by $G \leftrightarrow (G(*), o_G)$, $x * y = xx \cdot yy$ and $xy = o_G(x * y)$.

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