Maximal pseudocompact spaces

Jack R. Porter, R.M. Stephenson Jr., R. Grant Woods[1)]

Abstract. Maximal pseudocompact spaces (*i.e.* pseudocompact spaces possessing no strictly stronger pseudocompact topology) are characterized. It is shown that submaximal pseudocompact spaces whose pseudocompact subspaces are closed need not be maximal pseudocompact. Various techniques for constructing maximal pseudocompact spaces are described. Maximal pseudocompactness is compared to maximal feeble compactness.

Keywords: maximal pseudocompact, maximal feebly compact, submaximal topology

Classification: Primary 54A10; Secondary 54C30

§1 Introduction

A topological space X is called *pseudocompact* if every continuous real-valued function with domain X is bounded; it is called *feebly compact*, (or, *lightly compact*) if every locally finite collection of open sets is finite. Both classes of spaces have been extensively studied; see [GJ] or [PW], for example. The following well-known relationships link these classes of spaces:

1.1 Theorem. *A completely regular Hausdorff space is feebly compact if and only if it is pseudocompact.*

1.2 Theorem. *Every feebly compact space is pseudocompact, but there are pseudocompact spaces that are not feebly compact.*

A proof of 1.1 and the first part of 1.2 can be found in 1.11 (d) of [PW]; examples witnessing the second part of 1.2 can be found in [St_1], and one of these appears as problem 1U of [PW].

Let τ and σ be two topologies on a set X. If $\tau \subseteq \sigma$ we say that σ is an *expansion* of τ and that τ is a *compression* of σ. If $\sigma \setminus \tau \neq \emptyset$ then σ is a *proper expansion* of τ and τ is a *proper compression* of σ. If $A \subset X$, the closure of A in (X,τ) will be denoted by $\mathrm{cl}_\tau A$. If only one topology τ on X is under discussion, we write $\mathrm{cl}\, A$ or $\mathrm{cl}_X A$ instead of $\mathrm{cl}_\tau A$. Similar conventions apply to closures in subspaces.

Let $\mathcal{P}$ be a topological property. A space (X,τ) is said to be a *maximal $\mathcal{P}$-space* if (X,τ) has $\mathcal{P}$ and if σ is a proper expansion of τ, then (X,σ) does not have $\mathcal{P}$.

[1)] The research of the third-named author was supported by Grant No. A7592 from the Natural Sciences and Engineering Research Council of Canada.

Maximal $\mathcal{P}$-spaces have already received considerable study; see [R] and the five papers by Douglas Cameron listed in the references. Related to the above is the notion of a strongly $\mathcal{P}$-space; a space (X,τ) with $\mathcal{P}$ is said to be *strongly* $\mathcal{P}$ if there is an expansion σ of τ for which (X,σ) is a maximal $\mathcal{P}$-space (see [Ca$_4$]).

In this paper we study the class of maximal pseudocompact spaces and considerably extend the previously known results about this class. In §2 we characterize maximal pseudocompact spaces, and provide counterexamples to other plausible candidates for characterizations. In §3 we produce examples of maximal pseudocompact spaces and strongly pseudocompact spaces, and describe ways in which maximal pseudocompact spaces can be constructed. In §4 we study the relation between maximal pseudocompactness and maximal feeble compactness. We finish the paper by posing some open questions.

In the remainder of this introduction we present some known concepts and results. We will make considerable use of results appearing in [PSW$_1$] and [PSW$_2$], which are "companion papers" to this.

1.3 Definition. A topological space is *submaximal* if its dense subsets are open.

We collect a miscellany of facts about submaximal spaces below; 1.4 (a) and 1.4 (b) are respectively Theorems 15 and 14 of [R].

1.4 Theorem.

(a) *A maximal pseudocompact space is submaximal.*
(b) *A maximal feebly compact space is submaximal.*
(c) *If S is a dense subspace of the submaximal space X, then $X\setminus S$ is a closed discrete subspace of X.*

If $\mathcal{D}$ denotes the set of dense subsets of a space (X,τ), then by a *dense ultrafilter* on (X,τ) we mean an ultrafilter on the poset $(\mathcal{D},\subseteq)$. By Zorn's lemma any nonempty subfamily of $\mathcal{D}$ that is closed under finite intersection is contained in some dense ultrafilter on (X,τ). If $\mathcal{S}$ is a collection of subsets of (X,τ), then $\tau(\mathcal{S})$ denotes the topology generated on X by $\tau\cup\mathcal{S}$, *i.e.* the smallest topology containing both τ and $\mathcal{S}$. If $\mathcal{S}=\{A\}$ (resp. $\{A,B\}$), we write $\tau(A)$ (resp. $\tau(A,B)$) instead of $\tau(\{A\})$ (resp. $\tau(\{A,B\})$). If S is a subset of X, then $\tau|S$ as usual denotes the restriction of τ to S. Observe that $\tau|S=\tau(S)|S$ and $\tau|X\setminus S=\tau(S)|X\setminus S$.

If (X,τ) is a space then $R(\tau)$ (resp. $\mathrm{RO}\,(\tau)$) will denote the set of regular closed subsets (resp. regular open subsets) of X. The *semiregularization* $s\tau$ of the topology τ is the topology on X for which $\mathrm{RO}\,(\tau)$ is an open base. Observe that $R(s\tau)=R(\tau)$ and $\mathrm{RO}\,(s\tau)=\mathrm{RO}\,(\tau)$. See Chapter 2 of [PW] for proofs of these claims and a detailed discussion of semiregularizations.

The following is 2.23 of [PSW$_1$] (which in turn is drawn from exercises 20 and 22, pages 138 and 139 of [Bo]) and immediate consequences thereof.

1.5 Proposition. *Let $\mathcal{V}$ be a dense ultrafilter on (X,τ). Then:*

(a) $\big(X,\tau(\mathcal{V})\big)$ *is submaximal, and its dense subsets are precisely the members of $\mathcal{V}$.*

(b) $\mathrm{RO}\,(\tau) = \mathrm{RO}\,\bigl(\tau(\mathcal{V})\bigr)$ *and* $s\bigl(\tau(\mathcal{V})\bigr) = s\tau$.
(c) (X, τ) *is feebly compact if and only if* $\bigl(X, \tau(\mathcal{V})\bigr)$ *is feebly compact.*
(d) *If* (X, τ) *is* T_1 *and* $p \in X$ *then* $\{p\} \in \tau$ *if and only if* $\{p\} \in \tau(\mathcal{V})$
(e) *If* σ *is a submaximal expansion of* τ *and if* $\tau = s\sigma$, *then there is a dense ultrafilter* $\mathcal{V}$ *on* (X, τ) *such that* $\sigma = \tau(\mathcal{V})$.

In 2.2 of [PSW_1] maximal feebly compact spaces are characterized as follows.

1.6 Theorem. *A space* (X, τ) *is maximal feebly compact if and only if it is feebly compact, submaximal, and all its feebly compact subspaces are closed.*

If (X, τ) is a space then $C(X, \tau)$ will denote the set of real-valued continuous functions on (X, τ); we write $C(X)$ if it is unnecessary to specify τ. The set of bounded members of $C(X, \tau)$ is denoted $C^*(X, \tau)$ or $C^*(X)$. The weak topology induced on X by $C(X, \tau)$ is denoted by $w\tau$; thus $w\tau$ is the topology on X for which $\mathrm{coz}\,(X, \tau)$ is a base. [Here $\mathrm{coz}\,(X, \tau)$ denotes the set of cozero sets of (X, τ), *i.e.* the set $\{X \setminus f^{\leftarrow}(0) : f \in C(X, \tau)\}$. If no ambiguity can arise about the topology τ, we write $\mathrm{coz}\,X$ instead of $\mathrm{coz}\,(X, \tau)$.] We call (X, τ) completely regular if $\tau = w\tau$. Completely regular spaces need not be T_1; observe that a space is Tychonoff if and only if it is completely regular and T_1, but $(X, w\tau)$ need not be T_1 even if (X, τ) is. However, the proof of 1.1 found, for example, in [PW] can be used essentially unchanged to prove the generalization of 1.1 given in 1.7 (a) below. The proof of 1.7 (b) is straightforward; the inclusion $w\tau \subseteq s\tau$ follows from 2.1 of [PV].

1.7 Proposition. (a) *A completely regular space* (X, τ) *is feebly compact if and only if it is pseudocompact.*
(b) *If* (X, τ) *is any space, then* $s(w\tau) = w\tau \subseteq s\tau \subseteq \tau$.

Following Guthrie and Stone [GS], we will call an expansion σ of a topology τ on X an $\mathbf{R}$*-invariant expansion* if $C(X, \sigma) = C(X, \tau)$. Theorem 1.9 below links this notion to those of 1.5. We precede it with Lemma 1.8; 1.8 (a) appears on page 103 of [Ca_2] and in 1Q of [PW], while 1.8 (b) may be found in 1G(1) of [GJ].

1.8 Lemma. (a) *Continuous images of feebly compact spaces are feebly compact.*
(b) *Continuous images of pseudocompact spaces are pseudocompact.*

1.9 Theorem. (a) *If* σ *is an expansion of a topology* τ *on a set* X, *then* (X, σ) *is feebly compact* (*resp. pseudocompact*) *only if* (X, τ) *is feebly compact* (*resp. pseudocompact*).
(b) *If* (X, τ) *is a space, then* τ *is an* $\mathbf{R}$*-invariant expansion of* $w\tau$ (*and hence of* $s\tau$), *i.e.* $C(X, \tau) = C(X, s\tau) = C(X, w\tau)$; *thus* (X, τ) *is pseudocompact if and only if* $(X, s\tau)$ *is if and only if* $(X, w\tau)$ *is.*

Clearly 1.9 (a) follows from 1.8. The first assertion in 1.9 (b) follows from 1.7 (b) and a special case of a result attributed to Katětov on page 4 of [GS]; a proof of it may be found by combining 2.2(g)(2) of [PW] and 1.7 (b). The second part of 1.9 (b) clearly follows from the first.

Finally, in [Ca_3] it is shown that:

1.10 Proposition. *A maximal pseudocompact space is T_1.*

In what follows, all hypothesized separation axioms will be stated explicitly. Undefined concepts and notation are explained in [GJ] and/or [PW]. In particular, $\mathbf{N}$ denotes the set of positive integers.

§2 A characterization of maximal pseudocompactness

In this section we characterize maximal pseudocompact spaces (Theorem 2.4), and derive some consequences of this characterization (Corollaries 2.5 and 2.6). After introducing the concept of a "Herrlich expansion" of a topology, and deriving some properties of it (2.7 and 2.8), we use such expansions to show in 2.10 that the statement obtained from 1.6 by replacing "feebly compact" by "pseudocompact" throughout is in fact false, and that maximal pseudocompactness and maximal feeble compactness do not have parallel characterizations.

The following definition appears in [M]; Theorem 2.2 combines Theorem 2.1 and Corollary 1 of [M].

2.1 Definition. A subset S of a space X is called *relatively pseudocompact* if for all $f \in C(X)$, $f|S \in C^*(S)$.

2.2 Theorem. *Let X be a space.*

(a) *If $C \in \operatorname{coz} X$ and if $\operatorname{cl} C$ is a relatively pseudocompact subset of X, then $\operatorname{cl} C$ is a pseudocompact subspace of X.*
(b) *If X is pseudocompact and $C \in \operatorname{coz} X$, then $\operatorname{cl} X$ is pseudocompact.*

We can generalize 2.2 as follows. The proof below is modelled after Mandelker's proof of 2.2.

2.3 Theorem. *Let (X, τ) be a space and let V be a nonempty union of cozero-sets of (X, τ). Then:*

(a) *If $\operatorname{cl}_\tau V$ is a relatively pseudocompact subset of X, then it is a pseudocompact subspace of X.*
(b) *If (X, τ) is pseudocompact, then $\operatorname{cl}_\tau V$ is pseudocompact.*

Proof: Suppose that $\operatorname{cl}_\tau V$ is not a pseudocompact subspace of X. We will show that $\operatorname{cl}_\tau V$ is not a relatively pseudocompact subset of (X, τ).

If $f \in C(\operatorname{cl}_\tau V) \setminus C^*(\operatorname{cl}_\tau V)$, there exists $(d_i)_{i \in \mathbf{N}} \subseteq V$ such that $f(d_{i+1}) \geq f(d_i) + 1$. (We may assume without loss of generality that $f \geq \mathbf{0}$). As V is union of cozero-sets of (X, τ), for each $i \in \mathbf{N}$ there exists $C_i \in \operatorname{coz}(X, \tau)$ such that $d_i \in C_i \subseteq V_i$. Let $C = \bigcup\{C_i : i \in \mathbf{N}\}$. Then $C \in \operatorname{coz}(X, \tau)$, $C \subseteq V$, and by 1.19 of [GJ] the set $(d_i)_{i \in \mathbf{N}} = E$ is C-embedded in $\operatorname{cl}_\tau V$ and hence in $\operatorname{cl}_\tau C$. Now $\operatorname{cl}_\tau C \setminus C$ is a zero-set of $\operatorname{cl}_\tau C$, so by 1.18 of [GJ] there exists $g \in C(\operatorname{cl}_\tau V)$ such that $\mathbf{0} \leq g \leq \mathbf{1}$, $g(d_i) = 1$ for each $i \in \mathbf{N}$, and $g[\operatorname{cl}_\tau C \setminus C] = \{0\}$. Define $h \colon X \to \mathbf{R}$ by: $h[X \setminus C] = \{0\}$, $h|\operatorname{cl}_\tau C = fg$. Then $h \in C(X, \tau)$ by 1A(1) of [GJ], and $h[(d_i)_{i \in \mathbf{N}}]$ is unbounded, so $h|\operatorname{cl}_\tau V \in C(\operatorname{cl}_\tau V) \setminus C^*(\operatorname{cl}_\tau V)$. Hence $\operatorname{cl}_\tau V$ is not a relatively pseudocompact subset of X.

(b) Since X is pseudocompact, $\mathrm{cl}_\tau V$ is relatively pseudocompact. $\square$

We now characterize maximal pseudocompact spaces.

2.4 Theorem. *Let (X,τ) be a pseudocompact space. The following are equivalent.*

(a) *(X,τ) is maximal pseudocompact.*

(b) *If $F \subseteq X$ and F is a relatively pseudocompact subspace of $\big(X,\tau(X\setminus F)\big)$, then F is closed in (X,τ).*

Proof: (b) implies (a): Suppose (a) fails. Then there is a proper pseudocompact expansion σ of τ. Suppose $F\subseteq X$ and $X\setminus F\in\sigma\setminus\tau$. Then $\tau\neq\tau(X\setminus F)\subseteq\sigma$, and by 1.9 (a) $\big(X,\tau(X\setminus F)\big)$ is pseudocompact. Hence F is a relatively pseudocompact subset of $\big(X,\tau(X\setminus F)\big)$ but is not closed in (X,τ). Hence (b) fails.

(a) implies (b): Suppose that (a) holds but (b) is false. Then there exists $F\subseteq X$ such that $X\setminus F\notin\tau$ but F is a relatively pseudocompact subset of $\big(X,\tau(X\setminus F)\big)$. Let $p\in\mathrm{cl}_\tau F\setminus F$ and put $H=\mathrm{cl}_\tau F\setminus\{p\}$. Then $X\setminus H\notin\tau$, so by the maximality of τ the space $\big(X,\tau(X\setminus H)\big)$ is not pseudocompact. Hence there exists $f\in C\big(X,\tau(X\setminus H)\big)$ for which $f[X]$ has no upper bound in $\mathbf{R}$.

Clearly $\tau(X\setminus H)|X\setminus\{p\}=\tau|X\setminus\{p\}$. Since $X\setminus\{p\}\in\tau$ as (X,τ) is T_1 by 1.10, it follows that f is τ-continuous, and hence $\tau(X\setminus F)$-continuous, at each point of $X\setminus\{p\}$. Let W be a neighborhood of $f(p)$ in $\mathbf{R}$. Since f is $\tau(X\setminus H)$-continuous at p, there exists $V\in\tau$ such that $p\in V$ and $f[V\setminus H]\subseteq W$. Since (X,τ) is maximal pseudocompact it is submaximal by 1.4 (a) and hence by 1.4 (c) $\mathrm{cl}_\tau F\setminus F$ is a discrete subspace of (X,τ). Hence there exists $T\in\tau$ such that $T\cap(\mathrm{cl}_\tau F\setminus F)=\{p\}$. Thus $T\cap(X\setminus H)=T\cap(X\setminus F)$, and so the set $U=V\cap T\cap(X\setminus F)$ is a $\tau(X\setminus F)$-neighborhood of p for which $f[U]\subseteq W$. Thus f is $\tau(X\setminus F)$-continuous at p, and so $f\in C\big(X,\tau(X\setminus F)\big)$.

As F is a relatively pseudocompact subspace of $\big(X,\tau(X\setminus F)\big)$, there exists $k>0$ such that $f[F]\subseteq(-k,k)$. As $H=\mathrm{cl}_{\tau(X\setminus H)}F$, it follows that $f[H]\subseteq[-k,k]$. Choose $b\in\mathbf{R}$ such that $b>\max\{k,f(p)\}$. Define $g\colon X\to\mathbf{R}$ as follows: $g(x)=\max\big(f(x),b\big)$. Since f is τ-continuous at each point of $X\setminus\{p\}$, so is g; as $f[X]$ is unbounded so is $g[X]$. Since f is $\tau(X\setminus H)$-continuous at p, there exist $J,K\in\tau$ such that $p\in J\cup(K\setminus H)$ and $f[J\cup(K\setminus H)]\subseteq\big(f(p)-1,b\big)$. Clearly $g[J\cup(K\setminus H)]=\{b\}$ by the definition of g, and as $f[H]\subseteq[-k,k]$, it follows that $g[J\cup K]=\{b\}$. As $J\cup K$ is a τ-neighborhood of p, it follows that g is τ-continuous at p. Thus $g\in C(X,\tau)\setminus C^*(X,\tau)$, contradicting the assumption that (X,τ) is pseudocompact. Hence assuming that (a) holds and (b) is false leads to a contradiction, and so (a) implies (b). $\square$

2.5 Corollary. (a) *If (X,τ) is maximal pseudocompact then it is submaximal and its pseudocompact subspaces are closed and maximal pseudocompact.*

(b) *If (X,τ) is a pseudocompact submaximal space whose pseudocompact subspaces are closed, then its pseudocompact subspaces are of the form $A\cup T$, where*

$A \in R(\tau)$ and T is finite. (In particular, this holds for maximal pseudocompact spaces.)

Proof: (a) Submaximality follows from 1.4 (a). Let F be a pseudocompact subspace of the maximal pseudocompact space (X,τ). As $\tau(X \setminus F)|F = \tau|F$ it follows that if $f \in C\big(X, \tau(X \setminus F)\big)$ then $f|F \in C(F, \tau|F)$ and so $f[F]$ is a bounded subset of $\mathbf{R}$ since $(F, \tau|F)$ is pseudocompact. Hence $\big(F, \tau(X \setminus F)|F\big)$ is a relatively pseudocompact subspace of $\big(X, \tau(X \setminus F)\big)$. Thus by 2.4 F is a closed subspace of (X,τ).

To show that $(F, \tau|F)$ is maximal pseudocompact, suppose that $G \subseteq F$ and that G is a relatively pseudocompact subspace of $\big(F, (\tau|F)(F \setminus G)\big)$. Now $(\tau|F)(F \setminus G) = \big(\tau(X \setminus G)\big)|F$, so if $f \in C\big(X, \tau(X \setminus G)\big)$, then $f|F \in C\big(F, (\tau|F)(F \setminus G)\big)$. Consequently $f[G]$ is a bounded subset of $\mathbf{R}$, and so G is a relatively pseudocompact subspace of $\big(X, \tau(X \setminus G)\big)$. By 2.4 and the maximal pseudocompactness of (X,τ) it follows that G is closed in (X,τ) and hence in $(F, \tau|F)$. Thus by 2.4 $(F, \tau|F)$ is maximal pseudocompact.

(b) Let F be a pseudocompact (and hence closed) subspace of the space (X,τ), whose properties are as hypothesized. Then $F = (\mathrm{cl}_\tau \mathrm{int}_\tau F) \cup T$, where T is defined to be $F \setminus \mathrm{cl}_\tau \mathrm{int}_\tau F$. As (X,τ) is submaximal, by 1.4 (c) T is a closed discrete subspace of it. But T is open in F, so T is an open-and-closed set of isolated points of F. If T were infinite it would fail to be pseudocompact, in contradiction to 2.2 (b). Hence T is finite. □

2.6 Corollary. *If X is maximal pseudocompact and if V is a union of cozero-sets of X then $\mathrm{cl}\, V$ is a maximal pseudocompact subspace of X.*

Proof: This follows from 2.2 (a), 2.3 (a) and 2.5. □

In analogy with the characterization of maximal feebly compact spaces given in 1.6, one might conjecture that a pseudocompact space is maximal pseudocompact if and only if it is submaximal and its pseudocompact subspaces are closed. As 2.5 (a) attests, maximal pseudocompactness implies submaximality and that pseudocompact subspaces are closed; however, the converse fails. The remainder of this section is devoted to a discussion of "Herrlich expansions" of topologies, and how the Herrlich expansion of the closed unit interval (for example) witnesses the failure of the above-mentioned converse.

2.7 Definition. A *Herrlich expansion* of a topology τ on a set X is a topology generated by $\tau \cup \{A, B\}$, where $\{A, B, C\}$ is a partition of X into three subsets each of which is dense in (X,τ). As indicated in the introduction, we denote such a topology as $\tau(A, B)$.

Herrlich expansions were considered by Herrlich in [Her], and are also discussed in Stephenson [St$_1$]. Note that $\tau(A, B) = \{S \cup (T \cap A) \cup (U \cap B) : S, T, U \in \tau\}$. We will make use of the following properties of Herrlich expansions. These properties are due to Herrlich, and proofs are at least implicit in [Her], but for the sake of completeness we sketch proofs here.

2.8 Proposition. *Let (X,τ) be a space and $\{A,B,C\}$ a partition of X into three dense subsets of (X,τ). Denote $\tau(A,B)$ by η. Then:*

(a) *$C(X,\tau)=C(X,\eta)$ and hence $w\tau=w\eta$. (See the paragraph following* 1.4 *for notation).*

(b) *$A,B\in \mathrm{RO}\,(\eta)$.*

(c) *If τ is a semiregular topology then so is η.*

(d) *If some point of $A\cup B$ is a regular G_δ-point of (X,τ), then (X,η) is not feebly compact. (A point $p\in X$ is called a regular G_δ-point of (X,τ) if there exist $(V_n)_{n\in\mathbf{N}}\subseteq\tau$ such that $\{p\}=\bigcap\{V_n : n\in\mathbf{N}\}=\bigcap\{\mathrm{cl}_\tau V_n : n\in\mathbf{N}\}$).*

Proof: (Sketch): One can prove that:

(i) If $T\in\tau$ then $\mathrm{cl}_\eta(T\cap A)=(\mathrm{cl}_\tau T)\setminus B$ and $\mathrm{cl}_\eta(T\cap B)=(\mathrm{cl}_\tau T)\setminus A$.

(ii) If $T\in\tau$ then $\mathrm{cl}_\tau T=\mathrm{cl}_\eta T$.

(a) Since $\tau\subseteq\eta$, clearly $C(X,\tau)\subseteq C(X,\eta)$. Conversely, suppose that $f\in C(X,\eta)$. Clearly f is τ-continuous at each point of C. If $x_0\in A$ and $f(x_0)\in V$, where V is open in $\mathbf{R}$, find open subsets U and Y of $\mathbf{R}$ such that $f(x_0)\in U\subseteq \mathrm{cl}_{\mathbf{R}}U\subseteq Y\subseteq \mathrm{cl}_{\mathbf{R}}Y\subseteq V$. As $f\in C(X,\eta)$, there exists $T\in\tau$ such that $x_0\in T$ and $f[T\cap A]\subseteq U$. Thus $f[\mathrm{cl}_\eta(T\cap A)]\subseteq \mathrm{cl}_{\mathbf{R}}f[T\cap A]\subseteq \mathrm{cl}_{\mathbf{R}}U\subseteq Y$. But $T\cap C\subseteq \mathrm{cl}_\eta(T\cap A)$ by (i), so $f[T\cap C]\subseteq Y$. As f is τ-continuous at each point of C, for each $x\in T\cap C$ there exists $W(x)\in\tau$ such that $x\in W(x)$ and $f[W(x)]\subseteq Y$. Let $W=\bigcup[W(x) : x\in T\cap C]$; then $T\cap C\subseteq W$ and $f[W]\subseteq Y$. Now $\mathrm{cl}_\tau T=\mathrm{cl}_\tau(T\cap C)$ as C is τ-dense in X. But $\mathrm{cl}_\tau(T\cap C)\subseteq \mathrm{cl}_\tau W$ and $\mathrm{cl}_\tau W=\mathrm{cl}_\eta W$ by (ii) above; hence $\mathrm{cl}_\tau T\subseteq \mathrm{cl}_\eta W$, and as f is η-continuous, $f[\mathrm{cl}_\tau T]\subseteq f[\mathrm{cl}_\eta W]\subseteq \mathrm{cl}_{\mathbf{R}}f[W]\subseteq \mathrm{cl}_{\mathbf{R}}Y$. Thus $f[T]\subseteq V$, so f is τ-continuous at x_0. Interchange the roles of A and B in the above and conclude that $f\in C(X,\tau)$.

(b) Setting $T=X$ in (i) above, we see that $B=X\setminus \mathrm{cl}_\eta A$ and $A=X\setminus \mathrm{cl}_\eta B$. Thus $A,B\in \mathrm{RO}\,(\eta)$.

(c) By hypothesis $\{S\cup(T\cap A)\cup(W\cap B) : S,T,W\in \mathrm{RO}\,(\tau)\}$ is a base for η. By (i) above,

$$\begin{aligned}\mathrm{int}_\eta\mathrm{cl}_\eta(T\cap A)&=\mathrm{int}_\eta[(\mathrm{cl}_\tau T)\cap(X\setminus B)]\\&=\mathrm{int}_\eta(\mathrm{cl}_\tau T)\cap \mathrm{int}_\eta(X\setminus B).\end{aligned}$$

But $\mathrm{int}_\eta\mathrm{cl}_\tau T=\mathrm{int}_\tau\mathrm{cl}_\tau T$ by (ii) above and $\mathrm{int}_\eta(X\setminus B)=A$ by (i) above. Thus since $T\in \mathrm{RO}\,(\tau)$, we see that $\mathrm{int}_\eta\mathrm{cl}_\eta(T\cap A)=T\cap A$. Similarly $W\cap B\in \mathrm{RO}\,(\eta)$, and $S\in \mathrm{RO}\,(\eta)$ by (ii) above. The result follows.

(d) Suppose $p\in A$ and $\{p\}=\bigcap\{V_n : n\in\mathbf{N}\}=\bigcap\{\mathrm{cl}_\tau V_n : N\in\mathbf{N}\}$, where $(V_n)_{n\in\mathbf{N}}$ is, without loss of generality, a decreasing sequence of members of τ. Then $\{V_n\cap B : n\in\mathbf{N}\}$ is an infinite, locally finite (in (X,η)) subfamily of η. Hence (X,η) is not feebly compact. $\square$

2.9 Lemma. *Let σ and τ be two topologies on a set X. If $s\sigma \subseteq \tau \subseteq \sigma$, then $s\sigma = s\tau$.*

PROOF: Let $U \in \tau$. Then $U \in \sigma$ and $\mathrm{cl}_{s\sigma} U \supseteq \mathrm{cl}_\tau U \supseteq \mathrm{cl}_\sigma U$. As $\mathrm{int}_\sigma \mathrm{cl}_\sigma U \in \tau$, by 2.2 (f) of [PW] it follows that $\mathrm{cl}_\tau U = \mathrm{cl}_\sigma U = \mathrm{cl}_{s\sigma} U$. Thus $X \setminus \mathrm{cl}_\sigma U \in \tau$, and by repeating the above calculation with $X \setminus \mathrm{cl}_\sigma U$ in place of U and then taking complements, we see that $\mathrm{int}_\tau \mathrm{cl}_\tau U = \mathrm{int}_\sigma \mathrm{cl}_\sigma U$. It follows that $s\tau \subseteq s\sigma \subseteq \tau$. Repeating the above (with τ replacing σ and $s\sigma$ replacing τ) we see that $s(s\sigma) \subseteq s\tau$. By 2.2 (f)(6) of [PW] we see that $s(s\sigma) = s\sigma$, and so $s\sigma \subseteq s\tau$. Hence $s\sigma = s\tau$. □

The following technical property of maximal pseudocompact expansions of Herrlich expansions will be key in the construction of examples.

2.10 Lemma. *Let (X, τ) be a semiregular pseudocompact space, and let η be the Herrlich expansion of τ generated by $\tau \cup \{A, B\}$, where $\{A, B, C\}$ is a partition of X into three dense subsets of (X, τ). If σ is a maximal pseudocompact expansion of η, then $s\sigma \setminus \eta \neq \emptyset$. (See remarks preceding 1.5 for notation).*

PROOF: As (X, τ) is semiregular, so is (X, η) (see 2.8 (c)). If the lemma fails then $s\sigma \subseteq \eta \subseteq \sigma$, and hence by 2.9 above it follows that $s\sigma = s\eta = \eta$. Thus as $A \in \mathrm{RO}(\eta) = \mathrm{RO}(\sigma)$, we see that $\mathrm{cl}_\sigma A = \mathrm{cl}_\eta A = A \cup C$ (as noted in (i) in the proof of 2.8). Let $p \in C$ and set $E = B \cup \{p\}$; by the above, $B \cup \{p\} \notin \sigma$, so $\sigma(E)$ is a proper expansion of σ. We will show that $C(X, \sigma) = C\big(X, \sigma(E)\big)$, which will imply that $\big(X, \sigma(E)\big)$ is pseudocompact and thereby contradict the maximality of σ. Let $f \in C\big(X, \sigma(E)\big)$. By 1.10 $X \setminus \{p\} \in \sigma$. Observe that $\sigma | X \setminus \{p\} = \sigma(E) | X \setminus \{p\}$, and hence f is σ-continuous at each point of $X \setminus \{p\}$. We will show that f is σ-continuous at p, thereby showing that $f \in C(X, \sigma)$ and completing our proof.

Let W be an open subset of $\mathbf{R}$ for which $f(p) \in W$. Choose open subsets U and V of $\mathbf{R}$ such that $f(p) \in U \subseteq \mathrm{cl}_{\mathbf{R}} U \subseteq V \subseteq \mathrm{cl}_{\mathbf{R}} V \subseteq W$. By 1.4 (a) and 1.5 (e) there is an ultrafilter $\mathcal{V}$ of dense subsets of (X, η) such that $\sigma = \eta(\mathcal{V})$. Since f is $\sigma(E)$-continuous at p, there exist $D \in \mathcal{V}$ and $T \in \eta$ such that $p \in S = D \cap T \cap (B \cup \{p\})$ and $f[S] \subseteq U$. Now as $p \notin A \cup B$ and $\eta = \tau(A, B)$, we may assume that $T \in \tau$. Because $p \in S$, it follows that $\mathrm{cl}_{\sigma(E)} S = \mathrm{cl}_\sigma S$. Since $\sigma = \eta(\mathcal{V})$, the σ-dense subsets of X are precisely the members of $\mathcal{V}$. Thus D is a dense subset of (X, σ) and so $\mathrm{cl}_\sigma(D \cap T \cap B) = \mathrm{cl}_\sigma(T \cap B) = \mathrm{cl}_{\sigma(s)}(T \cap B) = \mathrm{cl}_\eta(T \cap B) \supseteq T \cap (B \cup C)$. Consequently

(i) $$\begin{aligned} f[T \cap (B \cup C)] &\subseteq f[\mathrm{cl}_\sigma(D \cap T \cap B)] \subseteq f[\mathrm{cl}_\sigma S] = f[\mathrm{cl}_{\sigma(E)} S] \\ &\subseteq \mathrm{cl}_{\mathbf{R}} f[S] \subseteq \mathrm{cl}_{\mathbf{R}} U \subseteq V. \end{aligned}$$

Also note that since $E \cap A = \emptyset$, it follows that if $L \subseteq X$ then

(ii) $$A \cap \mathrm{cl}_{\sigma(E)} L = A \cap \mathrm{cl}_\sigma L.$$

In particular,

(iii) $$A \cap \mathrm{cl}_{\sigma(E)}(T \cap A \cap f^{\leftarrow}[V]) = A \cap \mathrm{cl}_{\sigma}(T \cap A \cap f^{\leftarrow}[V]).$$

We claim that $T \cap A \cap f^{\leftarrow}[V]$ is $\sigma(E)$-dense in $T \cap A$. If not, then by (iii) above there would exist $D \in \mathcal{V}$ and $P \in \tau$ such that $P \cap D \cap T \cap A \neq \emptyset$ but $P \cap D \cap T \cap A \cap f^{\leftarrow}[V] = \emptyset$. Let $P \cap T = G$; then $f[G \cap D \cap A] \subseteq \mathbf{R} \setminus V$, and as f is $\sigma(E)$-continuous it follows that $f[\mathrm{cl}_{\sigma(E)}(G \cap D \cap A)] \subseteq \mathbf{R} \setminus V$. But by (ii) above,

$$\begin{aligned} A \cap \mathrm{cl}_{\sigma(E)}(G \cap D \cap A) &= A \cap \mathrm{cl}_{\sigma}(G \cap D \cap A) \\ &\supseteq G \cap A; \end{aligned}$$

the latter inclusion follows from the fact that $D \in \mathcal{V}$ and hence is dense in (X, σ), while $G \cap A \in \sigma$. Hence $f[G \cap A] \subseteq \mathbf{R} \setminus V$. Now $G \in \tau \setminus \{\emptyset\}$ so $G \cap C$ is infinite. Hence we can choose $q \in (G \cap C) \setminus \{p\}$. By (i) in the proof of 2.8, $q \in \mathrm{cl}_{\eta}(G \cap A)$. But $\mathrm{cl}_{\sigma}(G \cap A) = \mathrm{cl}_{\eta}(G \cap A)$ as $\eta = s\sigma$ and $G \cap A \in \sigma$. Hence $q \in \mathrm{cl}_{\sigma}(G \cap A)$. If $q \in J \cup (K \cap E)$, where $J, K \in \sigma$ then $q \in J$ (as $q \notin E$) so $J \cap G \cap A \neq \emptyset$. Hence basic $\sigma(E)$-neighborhoods of q meet $G \cap A$, and so $q \in \mathrm{cl}_{\sigma(E)}(G \cap A)$. Consequently

$$f(q) \in f[\mathrm{cl}_{\sigma(E)}(G \cap A)] \subseteq \mathrm{cl}_{\mathbf{R}} f[G \cap A] \subseteq \mathbf{R} \setminus V \text{ (see above).}$$

But $q \in G \cap C \subseteq T \cap (B \cup C)$, so by (i) $f(q) \in V$. This contradiction verifies our claim that $T \cap A \cap f^{\leftarrow}[V]$ is $\sigma(E)$-dense in $T \cap A$.

Thus

$$\begin{aligned} f[T \cap A] &\subseteq f\left[\mathrm{cl}_{\sigma(E)}(T \cap A \cap f^{\leftarrow}[V])\right] \\ &\subseteq \mathrm{cl}_{\mathbf{R}} V \subseteq W. \end{aligned}$$

Thus $f[T] = f[T \cap (B \cup C)] \cup f[T \cap A] \subseteq V \cup W \subseteq W$. Consequently f is σ-continuous at p, and so as noted above, $f \in C(X, \sigma)$. Thus $C\big(X, \sigma(E)\big) = C(X, \sigma)$, which as observed above violates the maximality of σ. The lemma follows. $\square$

The following class of examples shows that pseudocompact submaximal spaces whose pseudocompact subspaces are closed (in fact, whose singleton sets are zero-sets) need not be maximal pseudocompact.

2.11 Example. Let (X, τ) be a semiregular pseudocompact space whose pseudocompact subspaces are closed; suppose also τ has a Herrlich expansion η. (These conditions will be satisfied if (X, τ) is a compact metric space without isolated points, for example.) Let $\mathcal{V}$ be a dense ultrafilter on (X, η). As $\eta(\mathcal{V})$ is an expansion of τ, by 1.8 (b) and hypothesis, pseudocompact subspaces of $\big(X, \eta(\mathcal{V})\big)$ are closed. Furthermore, by 1.5 (a) $\big(X, \eta(\mathcal{V})\big)$ is submaximal and by 1.5 (b) and

1.9 (b) it is pseudocompact. But since $s\eta(\mathcal{V}) = s\eta$ by 1.5 (b) and $s\eta = \eta$ by 2.8 (c) it follows that $s\eta(\mathcal{V}) = \eta$, and hence $(X, \eta(\mathcal{V}))$ cannot be maximal pseudocompact by 2.10.

Observe also that both (X, η) and $(X, \eta(\mathcal{V}))$ are pseudocompact spaces that have regular closed subspaces that fail to be pseudocompact spaces. (This contrasts to feebly compact spaces, whose regular closed subspaces are feebly compact.) If $\eta = \tau(A, B)$, where $\{A, B, C\}$ is a partition of X into three dense subspaces of (X, τ), then $\mathrm{cl}_{\eta} A = A \cup C$ (see (i) in proof of 2.8) and as $A \cup C$ fails to be closed in (X, τ), it is not a pseudocompact subspace of (X, τ). Hence $\mathrm{cl}_{\eta} A$ is a regular closed subset of (X, η) which by 1.8 (b) is not a pseudocompact subspace of (X, η). By 1.5 (b) $\mathrm{cl}_{\eta} A$ is a regular closed subspace of $(X, \eta(\mathcal{V}))$ and by 1.8 (b) is not a pseudocompact subspace of it.

§3 Construction and examples of maximal pseudocompact spaces

Having characterized maximal pseudocompact spaces in 2.4, and shown in 2.10 that a plausible alternate candidate for such a characterization fails, we now consider ways of constructing maximal pseudocompact spaces. We begin by investigating maximal pseudocompact expansions of pseudocompact Tychonoff space whose pseudocompact subspaces are closed.

The following definition first appeared in [W].

3.1 Definition. Let $\mathcal{T}(X)$ denote the set of topologies on a nonempty set X. We define $\leq$ on $\mathcal{T}(X)$ as follows. $\tau \leq \sigma$ if $\tau \subseteq \sigma$ and for each $V \in \tau$, $\mathrm{cl}_{\tau} V = \mathrm{cl}_{\sigma} V$.

Clearly $\leq$ is a partial order on $\mathcal{T}(X)$, and $\sigma(s) \leq \sigma$ for each $\sigma \in \mathcal{T}(X)$ by 1.5 (b).

3.2 Lemma. *Let (X, τ) be a pseudocompact Tychonoff space in which every pseudocompact subspace is closed. If σ is a pseudocompact expansion of τ, then $w\sigma = \tau$ and $\tau \leq \sigma$.*

Proof: As $\tau \subseteq \sigma$ and (X, τ) is Tychonoff, it follows that $\tau = w\tau \subseteq w\sigma$. Now $w\sigma$ is a Tychonoff topology as σ is an expansion of a Tychonoff topology. If $A \in R(w\sigma)$, then by 2.3 (b) A is a pseudocompact subspace of $(X, w\sigma)$. By 1.8 (b) A is a pseudocompact subspace of (X, τ) and hence is closed in (X, τ). Hence $s(w\sigma) \subseteq \tau$. Since $w\sigma$ is Tychonoff, it follows that $s(w\sigma) = w\sigma$, and so $\tau = w\sigma$.

If $\tau \leq \sigma$ fails, there exists $V \in \tau$ for which $\mathrm{cl}_{\sigma} V \neq \mathrm{cl}_{\tau} V$. Now $\mathrm{cl}_{\sigma} V \subseteq \mathrm{cl}_{\tau} V$ since $\tau \subseteq \sigma$, so there exists $p \in \mathrm{cl}_{\tau} V \backslash \mathrm{cl}_{\sigma} V$. Let $F = \mathrm{cl}_{\tau} V \backslash \{p\}$; then $X \backslash F \notin \tau$ so as pseudocompact subsets of (X, τ) are closed, F is not a pseudocompact subspace of (X, τ). Hence F is not a feebly compact subspace of (X, τ), and so there is a countably infinite pairwise disjoint family $(M_n)_{n \in \mathbf{N}}$ of nonempty open subsets of $(F, \tau|F)$ that is locally finite in $(F, \tau|F)$. As V is τ-dense in F, it follows that $(M_n \cap V)_{n \in \mathbf{N}}$ is a countably infinite family of nonempty members of τ that is locally finite in $(F, \tau|F)$. Denote $M_n \cap V$ by L_n and let $y_n \in L_n$. As (X, τ) is Tychonoff there exists $g_n \in C(X, \tau)$ such that $\mathbf{0} \leq g_n \leq \mathbf{1}$, $y_n \in \mathrm{int}_{\tau} g_n^{\leftarrow}(1)$

and $g_n[X \setminus L_n] = \{0\}$ (for each $n \in \mathbf{N}$). Let $f = \Sigma\{\mathbf{n}g_n : n \in \mathbf{N}\}$. Clearly f is well-defined as for each $x \in X$, $g_n(x) \neq 0$ for only finitely many $n \in \mathbf{N}$ (recall $(M_n)_{n\in\mathbf{N}}$ is pairwise disjoint). As $(L_n)_{n\in\mathbf{N}}$ is a locally finite subfamily of $(F, \tau|F)$, clearly $f|F \in C(F, \tau|F)$. But f is identically zero on $X \setminus V$, so $f|X \setminus V \in C(X \setminus V, \tau|X \setminus V)$. Consequently $f|X \setminus \{p\} \in C(X \setminus \{p\}, \tau|X \setminus \{p\})$. But as $p \notin \mathrm{cl}_\sigma V$, and $f[X \setminus \mathrm{cl}_\sigma V] = \{0\}$, it follows that f is continuous at p with respect to the topology σ. But as $f|X \setminus \{p\} \in C(X \setminus \{p\}, \tau|X \setminus \{p\}) \subseteq C(X \setminus \{p\}, \sigma|X \setminus \{p\})$, it follows that $f \in C(X, \sigma)$. As (X, σ) is pseudocompact and f is unbounded, this is a contradiction. Consequently $\tau \leq \sigma$ as claimed. □

3.3 Theorem. *Let (X, τ) be a pseudocompact Tychonoff space in which each pseudocompact subspace is closed. Then:*

(a) *If $\mathcal{V}$ is a dense ultrafilter on (X, τ), then $(X, \tau(\mathcal{V}))$ is a maximal pseudocompact space.*

(b) *If σ is a maximal pseudocompact expansion of τ that is feebly compact, then there is a dense ultrafilter $\mathcal{V}$ on (X, τ) such that $\sigma = \tau(\mathcal{V})$ (and hence $\tau = s\sigma$).*

Proof: (a) As (X, τ) is Tychonoff and pseudocompact, by 1.1 it is feebly compact. By 1.5 (c), $\tau(\mathcal{V})$ is also a feebly compact topology on X. Thus by 1.2 $(X, \tau(\mathcal{V}))$ is pseudocompact. Now suppose that σ were a pseudocompact expansion of $\tau(\mathcal{V})$ (and hence of τ). We now can argue as in the proof of Theorem 11 of [GS]; what follows is an elaboration and slight modification of the brief argument presented there, where (X, τ) is also assumed to be first countable. Denote $\tau(\mathcal{V})$ by μ, and suppose that $U \in \sigma \setminus \mu$. By 3.2 $w\sigma = \tau$ and $\tau \leq \sigma$; hence by the equivalence of (1) and (3) in Theorem 6 of [GS], there exist $V \in \tau$ and $D \subseteq X$ such that $U = V \cap D$ and D is τ-dense in X. Without loss of generality we may assume that $X \setminus V \subseteq D$. (Replace D by $D \cup (X \setminus V)$ if necessary.) Now $U \notin \mu$ so $D \notin \mathcal{V}$; consequently by the maximality of $\mathcal{V}$, there exists $E \in \mathcal{V}$ such that $D \cap E$ is not τ-dense in X. Hence there exists $W \in \tau \setminus \{\emptyset\}$ such that $W \cap D \cap E = \emptyset$. Let $T = W \cap V$. Then $T \neq \emptyset$, for if $T = \emptyset$ then $W \subseteq X \setminus V$ and hence $W \subseteq D$. Consequently $\emptyset = W \cap D \cap E = W \cap E$, which is a contradiction as E is τ-dense and $W \in \tau \setminus \{\emptyset\}$. As $T \in \tau$ and $U \in \sigma$, it follows that $T \cap U \in \sigma$. But $T \cap U = (W \cap V) \cap (V \cap D) = W \cap V \cap D$, so $T \cap U \neq \emptyset$ since $W \cap V \in \tau \setminus \{\emptyset\}$. However, $T \cap U \cap E = (W \cap D \cap E) \cap V = \emptyset$. Choose $p \in T \cap U$; then $E \cup \{p\} \in \mathcal{V}$ since $E \in \mathcal{V}$, so $E \cup \{p\} \in \mu \subseteq \sigma$. Consequently $(T \cap U) \cap (E \cup \{p\}) \in \sigma$; however, $(T \cap U) \cap (E \cup \{p\}) = \{p\}$, so $\{p\} \in \sigma$. Thus the characteristic function $\chi_{\{p\}} \in C(X, \sigma)$, but as $\{p\} \notin \tau$ (since E is τ-dense and $p \in X \setminus E$), $\chi_{\{p\}} \notin C(X, \tau)$. But $w\sigma = \tau$ by 3.2, and this implies that $C(X, \sigma) = C(X, \tau)$. Hence we have reached a contradiction, and so $\sigma \setminus \mu = \emptyset$. Hence $\tau(\mathcal{V})$ has no proper pseudocompact expansions, and hence it is a maximal pseudocompact topology on X.

(b) If η were a feebly compact expansion of σ, then by 1.2 η would be a pseudocompact expansion of σ. Consequently $\eta = \sigma$ by the maximality of σ. Thus σ is a maximal feebly compact expansion of τ. In [PSW_2] it is shown that if each feebly compact subspace of (X,τ) is closed, then each maximal feebly compact expansion of (X,τ) is of the form $\tau(\mathcal{V})$, where $\mathcal{V}$ is a dense ultrafilter on (X,τ). Now subspaces of Tychonoff spaces are feebly compact iff they are pseudocompact, so it follows that $\sigma=\tau(\mathcal{V})$ for some dense ultrafilter $\mathcal{V}$ on (X,τ). Then $\tau=s\sigma$ by 1.5 (b). □

The class of examples presented in 2.11 witnesses the fact that the assumption that (X,τ) is Tychonoff cannot be dropped in 3.3 (a). A Herrlich expansion η of the interval topology τ on the closed unit interval is a Urysohn, semiregular pseudocompact topology, and pseudocompact subspaces of $([0,1],\eta)$ are closed, but no topology on $[0,1]$ of the form $\eta(\mathcal{V})$ is maximal pseudocompact (where $\mathcal{V}$ is a dense ultrafilter on $([0,1],\eta)$).

Note that the special case of 3.3 (a) in which the hypothesis "each pseudocompact subspace is closed" is replaced by the stronger hypothesis "(X,τ) is first countable" was proved in Corollary 11 (a) of [GS].

Finally, it is shown in [PSW_2] that the Tychonoff plank (T,τ) does not have a maximal feebly compact expansion. If $\mathcal{V}$ were a dense ultrafilter on (T,τ), then $\big(T,\tau(\mathcal{V})\big)$ would have a dense set of isolated points as (T,τ) does. In 4.2 we will show that a space with a dense set of isolated points is maximal feebly compact if and only if it is maximal pseudocompact. Consequently no expansion of τ of the form $\tau(\mathcal{V})$ can be a maximal pseudocompact topology. This shows that the hypothesis "each pseudocompact subspace is closed" cannot be dropped from 3.3 (a).

We give another sufficient condition for maximal pseudocompactness. Recall that a space (X,τ) is called $\mathbf{R}$-*maximal* if $C(X,\sigma)\setminus C(X,\tau)\neq\emptyset$ whenever σ is a proper expansion of τ. This concept is discussed in [GS] in some detail.

3.4 Theorem. *Suppose that (X,τ) is an $\mathbf{R}$-maximal pseudocompact space such that every pseudocompact subspace of $(X,w\tau)$ is a closed subspace of (X,τ). Then (X,τ) is a maximal pseudocompact space.*

Proof: Suppose that σ is a proper expansion of τ and that (X,σ) is pseudocompact. Then by 1.8 (b) $(X,w\sigma)$ is pseudocompact and hence feebly compact by 1.7 (a). Hence every regular closed subset of $(X,w\sigma)$ is feebly compact by 1.4 (a) of [PSW_1], and hence is pseudocompact by 1.2. Now $\tau\subseteq\sigma$ so $w\tau\subseteq w\sigma$; consequently every regular closed subset of $(X,w\sigma)$ is a pseudocompact subspace of $(X,w\tau)$ by 1.8 (b) and hence by hypothesis is closed in (X,τ). But $(X,w\sigma)$ is semiregular by 1.7 (b), so every closed subset of $(X,w\sigma)$ is closed in (X,τ). Consequently $w\tau\subseteq w\sigma\subseteq\tau$, and hence $C(X,\tau)=C(X,w\sigma)$. But $C(X,w\sigma)=C(X,\sigma)$ and by the $\mathbf{R}$-maximality of (X,τ) it follows that $\tau=\sigma$. Hence (X,τ) is maximal pseudocompact. □

3.5 Corollary. *If (X,τ) is a pseudocompact Tychonoff space whose pseudocompact subspaces are closed, and if σ is a pseudocompact $\mathbf{R}$-maximal expansion of τ, then (X,σ) is a maximal pseudocompact space.*

The next example shows that the assumption in 3.2 that every pseudocompact subset of (X,τ) be closed cannot be dropped, and that (in contrast to 3.3) not every maximal pseudocompact completely Hausdorff (*i.e.* distinct points can be separated by a real-valued continuous function) space (X,τ) has the property that each pseudocompact subspace of $(X,w\tau)$ is a closed subspace of $(X,w\tau)$.

Recall that if $\mathcal{M}$ is maximal almost disjoint (henceforth abbreviated m.a.d.) family of infinite subsets of $\mathbf{N}$, and if $\{p(M) : M \in \mathcal{M}\}$ is a set D, disjoint from $\mathbf{N}$, faithfully indexed by $\mathcal{M}$, then the set $Y = \mathbf{N} \cup D$ can be given a topology $\tau(\mathcal{M})$ as follows:

$$\tau(\mathcal{M}) = \{U \subseteq Y : \text{ if } p(M) \in U \text{ then } M \setminus U \text{ is finite}\}.$$

The resulting space $\big(Y,\tau(\mathcal{M})\big)$ will be denoted by $\psi(\mathcal{M})$. It is a locally compact pseudocompact Hausdorff space; its set of isolated points is $\mathbf{N}$ and is dense, and D is an uncountable closed discrete subset of it. See 5I of [GJ] or 1N of [PW] for details.

Observe that if $U \in \tau(\mathcal{M})$ and $[\mathrm{cl}_{\tau(\mathcal{M})}(U \cap \mathbf{N})] \setminus \mathbf{N}$ is infinite, then it is uncountable; for by 1Q(2) and 1.11 (d)(2) of [PW], $\mathrm{cl}_{\tau(\mathcal{M})}(U\cap\mathbf{N})$ is pseudocompact, but it is not countably compact as $[\mathrm{cl}_{\tau(\mathcal{M})}(U\cap\mathbf{N})]\setminus\mathbf{N}$ is an infinite closed discrete subset of it. Consequently it cannot be countable (see 5F(5) of [PW]).

Also observe that if W is open in $\psi(\mathcal{M})$ and $D \setminus W$ is finite, then $\psi(\mathcal{M}) \setminus W$ is compact. To see this suppose $D \setminus W = \{p(M_i) : i = 1 \text{ to } n\}$, where $\{M_i : i = 1 \text{ to } n\} \subseteq \mathcal{M}$. Then $(\mathbf{N} \setminus \bigcup\{M_i : i = 1 \text{ to } n\}) \setminus W$ is finite; for if not, by the maximality of $\mathcal{M}$ there exists $S \in \mathcal{M}$ such that $S \bigcap [(\mathbf{N}\setminus\bigcup\{M_i : i = 1 \text{ to } n\})\setminus W]$ is infinite. One easily checks that $p(S) \in D \setminus W$ but $p(S) \notin \{p(M_i) : i = 1 \text{ to } n\}$, which is a contradiction. Thus $(\mathbf{N} \setminus \bigcup\{M_i : i = 1 \text{ to } n\}) \setminus W$ is finite, and $\psi(\mathcal{M}) \setminus W \subseteq \bigcup\big\{M_i \cup \{p(M_i)\} : i = 1 \text{ to } n\big\} \cup \big((\mathbf{N} \setminus \bigcup\{M_i : i = 1 \text{ to } n\}) \setminus W\big)$ which is compact. Thus $\psi(\mathcal{M}) \setminus W$ is compact.

In Theorem 1 of [Hec] it is shown that there exists a m.a.d. family $\mathcal{M}$ on $\mathbf{N}$ and a subset B of $D\big(= \psi(\mathcal{M}) \setminus \mathbf{N}\big)$ with this property:

$(*)$ $|B| = \mathbf{c}$ and if $U \in \tau(\mathcal{M})$ and $|U \cap B| > \aleph_0$ then $|(\mathrm{cl}_{\tau(\mathcal{M})}U) \cap (D \setminus B)| = \mathbf{c}$

(Here $\mathbf{c}$ denotes $2^{\aleph_0}$. Observe that $(*)$ implies that $|D \setminus B| = \mathbf{c}$.)

If B is partitioned into two sets A_1 and A_2, each of cardinality $\mathbf{c}$, then clearly $(*)$ remains true if B is replaced by either A_1 or A_2. Henceforth we assume that $\mathcal{M}$ is as in $(*)$, and that A_1 and A_2 are as above. We let $\psi(\mathcal{M}) \setminus \mathbf{N} = D$, and $D \setminus (A_1 \cup A_2) = E$.

3.6 Lemma. *Let $\mathcal{M}$ be as above, let $f \in C\big(\psi(\mathcal{M}),\tau(\mathcal{M})\big)$, and let $r > 0$ and $\varepsilon > 0$ be given. If there is a finite subset G of $E \cup A_1$ such that $f[(E\cup A_1)\setminus G] \subseteq [-r,r]$, then there exists a finite subset F_ε of $E \cup A_2$ such that $f[(E\cup A_2)\setminus F_\varepsilon] \subseteq [-r-\varepsilon, r+\varepsilon]$.*

Proof: Suppose not; then there exists an infinite subset S of $E \cup A_2$ such that $|f(x)| > r + \varepsilon$ whenever $x \in S$. By hypothesis $f(y) \in [-r,r]$ for all but finitely many members y of E, so without loss of generality, we may assume that $S \subseteq A_2$. Let $U = \{x \in \psi(\mathcal{M}) : |f(x)| > r + \frac{\varepsilon}{2}\}$. Clearly $S \subseteq \mathrm{cl}_{\tau(\mathcal{M})}(U \cap \mathbf{N}) \setminus \mathbf{N}$, so $\mathrm{cl}_{\tau(\mathcal{M})}(U \cap \mathbf{N})$ is infinite and hence, as noted above, uncountable. But $\mathrm{cl}_{\tau(\mathcal{M})}(U \cap \mathbf{N}) \setminus \mathbf{N} \subseteq \{x \in \psi(\mathcal{M}) : |f(x)| > r + \frac{\varepsilon}{4}\}$, which is an open set of $\psi(\mathcal{M})$ that we denote by V. Since by hypothesis f takes all but finitely many members of $E \cup A_1$ into $[-r, r]$, clearly $|V \cap A_2| > \aleph_0$. But as $(*)$ holds with B replaced by A_2, it follows that $\left|\left(\mathrm{cl}_{\tau(\mathcal{M})} V\right) \cap (E \cup A_1)\right| = \mathbf{c}$. Thus there are $\mathbf{c}$ elements y of $E \cup A_1$ for which $|f(y)| \geq r + \frac{\varepsilon}{4}$, contradicting the above-mentioned hypothesis. The lemma follows. □

Observe that 3.6 holds if A_1 and A_2 are interchanged. The example (X, σ) constructed below is a modification by Hechler [Hec] of an example due to Stephenson [St3]. We modify it further to construct an example showing that 3.2 cannot be generalized much.

3.7 Example. . Let $\mathcal{M}$, A_1, A_2, D and E be defined as above, and note that as $|E| = |A_1| = |A_2| = \mathbf{c}$, there are bijections $j_i \colon A_i \to E$ $(i = 1, 2)$. Define a partition $\mathcal{P}$ on $\psi(\mathcal{M}) \times \mathbf{N}$.

$$\mathcal{P}_1 = \Big\{\big\{\big(j_1^{\leftarrow}(d), n-1\big), (d, n), \big(j_2^{\leftarrow}(d), n+1\big)\big\} : n \in \mathbf{N}, n > 1, \text{ and } d \in E\Big\}.$$
$$\bigcup \Big\{\big\{(x, 1), \big(j_2(x), 2\big)\big\} : x \in E\Big\}$$
$$\mathcal{P} = \mathcal{P}_1 \bigcup \Big\{\{x\} : x \in \psi(\mathcal{M}) \times \mathbf{N} \text{ and } x \notin \bigcup\{L : L \in \mathcal{P}_1\}\Big\}.$$

Let $q \colon \psi(\mathcal{M}) \times \mathbf{N} \to \mathcal{P}$ map each point of $\psi(\mathcal{M}) \times \mathbf{N}$ to the member of $\mathcal{P}$ to which it belongs, give $\mathcal{P}$ the quotient topology η induced by q, and let $X = \mathcal{P} \cup \{\infty\}$, where $\infty \notin \mathcal{P}$. Topologize X by letting $(\mathcal{P}, \eta)$ be an open subspace of X and letting $\big\{\{\infty\} \cup \bigcup\{T_n : n \geq k\} : k \in \mathbf{N}\big\}$ be a neighborhood base at ∞, where T_n is defined to be $q[\psi(\mathcal{M}) \times \{n\}]$. Denote the resulting topology on X by σ. One verifies easily that (X, σ) is a first countable T_3 space with a countable dense set of isolated points (namely $\mathbf{N} \times \mathbf{N}$). Observe that each T_n is homeomorphic to $\psi(\mathcal{M})$. Note that (X, σ) can be intuitively visualized as being obtained by gluing (for $n > 1$) the nth copy T_n of $\psi(\mathcal{M})$ to the $(n-1)$st and $(n+1)$st copies by gluing E (in the nth copy) to A_1 (in the $(n-1)$st copy) and to A_2 (in the $(n+1)$st copy); the copy of E in T_1 is also glued to the copy of A_2 in T_2. Then the sequence $(T_n)_{n \in \mathbf{N}}$ of glued copies is required to converge to ∞.

Observe that (X, σ) is pseudocompact; for if $f \in C(X, \sigma)$ and $f(\infty) = r$, then there exists $n \in \mathbf{N}$ such that $\bigcup\{T_k : k \geq n\} \subseteq f^{\leftarrow}[(r-1, r+1)]$. But $\bigcup\{T_k : k \leq n\}$ is a finite union of pseudocompact spaces and hence is pseudocompact; thus f is bounded on this union and hence on (X, σ).

It is easily verified that $(X \setminus \{\infty\}, \sigma | X \setminus \{\infty\})$ is a locally compact zero-dimensional Hausdorff space, and its one-point compactification can be viewed as X equipped with a topology β; as β is a Tychonoff topology, it is evident that $\beta \subseteq w\sigma \subseteq \sigma$. We claim that $\beta = w\sigma$; to prove this it clearly suffices to show that if $f \in C(X,\sigma)$, $f(\infty) = 0$, and $r > 0$, then $X \setminus f^{\leftarrow}[(-r,r)]$ is a compact subspace of (X,σ). To verify this, let $U = f^{\leftarrow}[(-\frac{r}{2},\frac{r}{2})]$. As noted above, there exists $k \in \mathbf{N}$ such that $\bigcup\{T_m : m \geq k\} \subseteq U$. Then $q[(E \cup A_1) \times \{k\}] \subseteq f^{\leftarrow}[(-\frac{r}{2},\frac{r}{2})]$. Applying 3.6 (with $\psi(\mathcal{M})$ replaced by its homeomorph T_k, r by $\frac{r}{2}$, ε by $\frac{r}{8k}$, and G by $\emptyset$) we see that there is a finite subset F_k of $E \cup A_2$ such that $f[q[\big((E \cup A_2) \setminus F_k\big) \times \{k\}]] \subseteq [-\frac{r}{2} - \frac{r}{8k}, \frac{r}{2} + \frac{r}{8k}]$. But $q\big[\big((E \cup A_2) \setminus F_k\big) \times \{k\}\big] = q\big[\big((E \cup A_1) \setminus F_k\big) \times \{k-1\}\big]$ by definition of q, so we can apply 3.6 again, this time with $\psi(\mathcal{M})$ replaced by T_{k-1}, $\frac{r}{2}$ by $\frac{r}{2} + \frac{r}{8k}$, and G by F_k. We continue this process "inductively downward" until we reach T_1. We see that at each stage ($j = k$ to 1) $f^{\leftarrow}[(-\frac{r}{2} - \frac{r}{4}, \frac{r}{2} + \frac{r}{4})]$ is an open set containing all points of $q[D \times \{j\}]$ except perhaps for the finite set F_j; hence by the remark before our definition of $(*)$,

$$T_j \setminus f^{\leftarrow}\Big[\Big(-\frac{r}{2} - \frac{r}{4k}, \frac{r}{2} + \frac{r}{4k}\Big)\Big] \text{ is a compact subset } K_j \text{ of } T_j.$$

Thus $X \setminus f^{\leftarrow}[(-r,r)] \subseteq \bigcup\{K_j : j = 1 \text{ to } k\}$, which is compact, and our verification is complete.

Observe that $q[\psi(\mathcal{M}) \times \{1\}]$ is a pseudocompact subspace of $(X, w\sigma)$ that is not closed there. Furthermore, the unique dense ultrafilter on (X,σ) is $\{J \subseteq S : q[\mathbf{N} \times \mathbf{N}] \subseteq J\}$ and by 2.25 of [PSW$_1$], $\big(X, \sigma(\mathcal{V})\big)$ is maximal feebly compact and hence by 4.2 below maximal pseudocompact. Now if $V = \mathbf{N} \times \{1\}$, then $V \in w\sigma$ but one can show that $\infty \in \mathrm{cl}_{w\sigma} V \setminus \mathrm{cl}_{\sigma} V$; consequently it is false that $w\sigma \leq \sigma$. Using $(S, w\sigma)$ in place of (X,τ), this shows that 3.2 can fail if the assumption that pseudocompact subspaces be closed is dropped. Similarly, since $w\big(\sigma(\mathcal{V})\big) = w\sigma$, $\big(X, \sigma(\mathcal{V})\big)$ is maximal pseudocompact but not all pseudocompact subspaces of $\Big(Xw\big(\sigma(\mathcal{V})\big)\Big)$ are closed in $\Big(X, w\big(\sigma(\mathcal{V})\big)\Big)$.

§4 Maximal pseudocompactness vs. maximal feeble compactness

In this section we investigate the relationship between maximal feebly compact spaces and maximal pseudocompact spaces. In 4.1 and 4.2 we present classes of spaces for which the concepts are equivalent. In 4.3 and 4.4 we present sufficient conditions for a maximal feebly compact space to be maximal pseudocompact. Since maximal feeble compactness has an easily applicable characterization (see 1.6), these criteria are useful. In 4.5 we present a "structure theorem" common to maximal feeble compactness and maximal pseudocompactness. Finally, in 4.6 we give an example of a maximal feebly compact space that is not maximal pseudocompact. We have been unable to determine whether every maximal pseudocompact space must be maximal feebly compact.

In [Ca$_2$], Cameron proved that a Tychonoff space is maximal feebly compact if and only if it is maximal pseudocompact. The following result generalizes this. Recall that a subfamily $\mathcal{F}$ of $\tau \setminus \{\emptyset\}$ is a *π-base* for the space (X,τ) if each nonempty open subset has a subset belonging to $\mathcal{F}$. See [Ho] or 2N(5) of [PW] for details.

4.1 Theorem. (a) *If (X,τ) is a maximal feebly compact space, and if* $\operatorname{coz}(X,\tau)\setminus\{\emptyset\}$ *is a π-base for τ, then (X,τ) is maximal pseudocompact.*

(b) *If (X,τ) is a maximal pseudocompact space that is feebly compact, then it is maximal feebly compact.*

(c) *If (X,τ) is a maximal pseudocompact space, and if* $\operatorname{coz}(X,\tau)\setminus\{\emptyset\}$ *is a π-base for $s\tau$, then (X,τ) is maximal feebly compact. [Observe that* $\operatorname{coz}(X,\tau)\subseteq s\tau$.*]*

Proof: (a) If (X,τ) were maximal feebly compact, then by 1.2 it would be pseudocompact. Suppose that σ were a pseudocompact expansion of τ. We will show that (X,σ) is feebly compact; by the maximality of τ, this will imply that $\tau=\sigma$ and hence that τ is a maximal pseudocompact topology.

If (X,σ) were not feebly compact, there would exist a pairwise disjoint family $(V_n)_{n\in\mathbf{N}}\subseteq\sigma\setminus\{\emptyset\}$ with no limit point in (X,σ). Let $A=\{n\in\mathbf{N}:\operatorname{int}_\tau V_n=\emptyset\}$. By 1.4 (b) (X,τ) is submaximal, and hence V_n is a discrete subspace of (X,τ) for each $n\in A$. Thus V_n consists of isolated points of (X,σ) as $V_n\in\sigma$ and $\tau\subseteq\sigma$. Choose $x_n\in V_n$ for each $n\in A$; then $\{x_n\}_{n\in A}$ has no limit point in (X,σ) as $(V_n)_{n\in\mathbf{N}}$ has none. Hence $\{x_n\}_{n\in A}$ is an open-and-closed discrete subset of (X,σ). Define $f\colon X\to\mathbf{R}$ by: $f(x_n)=n$, $f[X\setminus\{x_n:n\in A\}]=\{0\}$; then $f\in C(X,\sigma)$. If A were infinite then f would be unbounded, contradicting the assumption that (X,σ) is pseudocompact. Thus A is finite. Consequently $\operatorname{int}_\tau V_n\neq\emptyset$ for cofinitely many $n\in\mathbf{N}$; without loss of generality assume that $\operatorname{int}_\tau V_n\neq\emptyset$ for all $n\in\mathbf{N}$. By hypothesis there exists $C_n\in\operatorname{coz}(X,\tau)\setminus\{\emptyset\}$ such that $C_n\subseteq V_n$. Now $(C_n)_{n\in\mathbf{N}}$ is locally finite in (X,σ) as $(V_n)_{n\in\mathbf{N}}$ is. Choose $p_n\in C_n$ and $f_n\in C(X,\tau)$ such that $\mathbf{0}\le f_n\le\mathbf{1}$, $f_n[X\setminus C_n]=\{0\}$, and $f_n(p_n)=1$ for each $n\in\mathbf{N}$. By the local finiteness of $(C_n)_{n\in\mathbf{N}}$ the function $\Sigma\{\mathbf{n}f_n: n\in\mathbf{N}\}$ is a well-defined unbounded member of $C(X,\sigma)$, again contradicting the pseudocompactness of (X,σ). Consequently (X,σ) must be feebly compact, and our result follows.

(b) If σ is a feebly compact expansion of τ, then (X,σ) is pseudocompact by 1.2. Hence $\sigma=\tau$ by the maximality of τ. As (X,τ) is feebly compact, it must be maximal feebly compact.

(c) Let (X,τ) be as in the hypothesis of (c); we claim it is feebly compact. If not, there exists $(V_n)_{n\in N}\subseteq\tau\setminus\{\emptyset\}$ with no limit point in (X,τ). Then $(\operatorname{int}_\tau\operatorname{cl}_\tau V_n)_{n\in\mathbf{N}}$ also has no limit point in (X,τ). By hypothesis there exists $C_n\in\operatorname{coz}(X,\tau)\setminus\{\emptyset\}$ such that $C_n\subseteq\operatorname{int}_\tau\operatorname{cl}_\tau V_n$. Arguing as in (a), we construct an unbounded member of $C(X,\tau)$, contradicting the pseudocompactness of (X,τ). Hence (X,τ) is feebly compact, and (c) now follows from (b). □

4.2 Corollary. *If (X,τ) has a dense set I of isolated points, then it is maximal feebly compact if and only if it is maximal pseudocompact.*

Proof: Clearly $\big\{\{x\}:x\in I\big\}\subseteq\operatorname{coz}(X,\tau)\setminus\{\emptyset\}$, and by hypothesis $\big\{\{x\}:x\in I\big\}$ is a π-base for τ (and hence for $s\tau$). Now apply 4.1. □

In 2.16 of [PSW$_1$] we have shown that semiregular maximal feebly compact spaces must have a dense set of isolated points. Clearly they must also be maximal pseudocompact by 4.2.

We now give sufficient conditions for a maximal feebly compact space to be maximal pseudocompact. Theorem 4.3 below complements 3.3 (b).

4.3 Theorem. *Let (X, τ) be a pseudocompact Tychonoff space whose pseudocompact subspaces are closed. If σ is a maximal feebly compact expansion of τ, then σ is maximal pseudocompact.*

Proof: Since each feebly compact subspace of (X, τ) is closed (by 1.1 and hypothesis), it follows as in the proof of 3.3 (b) that each maximal feebly compact expansion σ of τ is of the form $\tau(\mathcal{V})$, where $\mathcal{V}$ is a dense ultrafilter on (X, τ). Our result now follows from 3.3 (a). $\square$

4.4 Corollary. *If (X, τ) is a maximal feebly compact space whose singleton sets are zero-sets, then it is maximal pseudocompact.*

Proof: The weak topology $w\tau$ has $\operatorname{coz}(X, w\tau)$ as a base, and it is T_1 as zero-sets of (X, τ) are closed in $w\tau$. Hence $(X, w\tau)$ is Tychonoff. As its singleton sets are intersections of countably many closed neighborhoods (in (X, τ)), it is easy to prove (or, one can appeal to 2.7 (c) of [PSW$_1$]) that its feebly compact (and hence, by 1.1, its pseudocompact) subspaces are closed. Hence as τ is a maximal feebly compact expansion of $w\tau$, by 4.3 (X, τ) is maximal pseudocompact. $\square$

We now consider a property shared by maximal feebly compact and maximal pseudocompact spaces.

4.5 Proposition. *If (X, τ) is a submaximal space without isolated points, then each discrete subspace is closed.*

Proof: Let D be a discrete subspace of (X, τ). As each point of $\operatorname{int}_\tau D$ would be isolated in (X, τ), we conclude that $\operatorname{int}_\tau D = \emptyset$. By 1.4 (c) D is closed. $\square$

As maximal feebly compact and maximal pseudocompact spaces are submaximal, it immediately follows from 4.5 that if such a space is Hausdorff and has no isolated points, it contains no nontrivial convergent sequence, and in fact no infinite countable compact spaces; for if K were such a space, it would contain an infinite discrete subspace, which by the countable compactness of K would contain a limit point, in contradiction to the above.

We finish this paper by exhibiting a Hausdorff maximal feebly compact space that is not maximal pseudocompact.

4.6 Example. Let (X, τ) denote Bing's example of a countable, first countable connected Hausdorff space (see [Bi]). [The underlying set X is $\{(p, q) \in \mathbf{Q} \times \mathbf{Q} : q \geq 0\}$, where $\mathbf{Q}$ denotes the rationals. A neighborhood base at $(p, 0)$ is $\big\{\{(x, 0) \in X : |p - x| < \frac{1}{n}\} : n \in \mathbf{N}\big\}$. If $(p, q) \in X$ and $q > 0$ let $(p_1, 0)$ and $(p_2, 0)$ be the other two corners of the equilateral triangle in $\mathbf{R}^2$ whose vertex is (p, q) and whose base lies on the X-axis. Then a neighborhood base at (p, q) is $\big\{\{(p, q)\} \cup \{(x, 0) : |p_1 - x| < \frac{1}{n} \text{ or } |p_2 - x| < \frac{1}{n}\} : n \in \mathbf{N}\big\}$. Connectedness follows from the fact that if $V, W \in \tau \setminus \{\emptyset\}$ then $\mathrm{cl}_\tau V \cap \mathrm{cl}_\tau W \neq \emptyset$. It is easy to check that the semiregularization $(X, s\tau)$ is also a countable, first countable connected Hausdorff space. Real-valued continuous functions on connected countable spaces must be constant, so $(X, s\tau)$ is pseudocompact. One can check that $\big\{\mathrm{int}_\tau \mathrm{cl}_\tau \{(x,0) : n - \frac{1}{4} < x < n + \frac{1}{4}\} : n \in \mathbf{N}\big\}$ is an infinite pairwise disjoint subfamily of $s\tau \setminus \{\emptyset\}$ that is locally finite in $(X, s\tau)$, so $(X, s\tau)$ is not feebly compact.

By 5.7 and 5.9 of [St$_2$], there exists a first countable feebly compact Hausdorff extension (T, σ) of $(X, s\tau)$. Thus X is a proper dense subset of (T, σ). Let $\mathcal{V}$ be a dense ultrafilter on (T, σ) such that $X \in \mathcal{V}$. Then by 2.25 of [PSW$_1$], $\big(T, \sigma(\mathcal{V})\big)$ is a maximal feebly compact space. Now by 2.2(i)(2) of [PW], $s\big(\sigma(\mathcal{V})|X\big) = \Big(s\big(\sigma(\mathcal{V})\big)\Big)|X$. But $s\big(\sigma(\mathcal{V})\big)|X = \sigma|X = s\tau$ (see 1.5 (b)), so $(X, s\tau)$ is the semiregularization of $\big(X, \sigma(\mathcal{V})|X\big)$. By 1.9 (b) $\big(X, \sigma(\mathcal{V})|X\big)$ is pseudocompact. As $X \in \mathcal{V}$, $\big(X, \sigma(\mathcal{V})|X\big)$ is a dense subspace of $\big(T, \sigma(\mathcal{V})\big)$, and so not all pseudocompact subspaces of $\big(T, \sigma(\mathcal{V})\big)$ are closed. Hence by 2.5 (a) $\big(T, \sigma(\mathcal{V})\big)$ is not maximal pseudocompact, and hence is an example of a Hausdorff maximal feebly compact space that is not maximal pseudocompact.

References

[Bi] Bing R.H., *A connected countable Hausdorff space*, Proc. Amer. Math. Soc. **4** (1953), 474.

[Bo] Bourbaki N., *Elements of Mathematics, General Topology, Part I*, Addison-Wesley, Reading, 1966.

[Ca$_1$] Douglas Cameron, *Maximal and minimal topologies*, Trans. Amer. Math. Soc. **160** (1971), 229–248.

[Ca$_2$] ———, *A class of maximal topologies*, Pacific J. Math **70** (1977), 101–104.

[Ca$_3$] ———, *A note on maximal pseudocompactness*, Bull. Malaysian Math. Soc. (2) **2** (1979), No. 1, 45–46.

[Ca$_4$] ———, *A survey of maximal topological spaces*, Topology Proceedings **2** (1977), 11–60.

[Ca$_5$] ———, *Δ-maximal topologies for some cardinal functions*, Gen. Top. Applic. **9** (1978), 59–70.

[E] Engelking R., *General Topology*, Polish Scientific Publishers, Warsaw, 1977.

[GJ] Gillman L., Jerison M., *Rings of continuous functions*, Van Nostrand, Princeton, 1960.

[GS] Guthrie J.A., Stone H.E., *Pseudocompactness and invariance of continuity*, Gen. Top. Applic. **7** (1977), 1–13.

[Hec] Hechler S.H., *Two* **R***-closed spaces revisited*, Proc. Amer. Math. Soc. **56** (1976), 303–309.

[Her] Herrlich H., *T_υ-Abgeschlossenheit and T_υ-Minimalität*, Math. Zeit. **88** (1965), 285–294.

[Ho] Hodel R., *Cardinal functions I, Handbook of Set-theoretic Topology* (Kunen, Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 1–61.

[M] Mandelker M., *Supports of continuous functions*, Trans. Amer. Math. Soc. **156** (1971), 73–83.

[PSW$_1$] Porter J.R., Stephenson R.M., Jr., and Woods R.G., *Maximal feebly compact spaces*, Topology Applic. (to appear).

[PSW$_2$] ———, *Spaces with maximal feebly compact expansions*, preprint.

[PV] Porter J.R., Vermeer J., *Spaces with coarser minimal Hausdorff topologies*, Trans. Amer. Math. Soc. **289** (1985), 59–71.

[PW] Porter J.R., Woods R.G., *Extensions and Absolutes of Hausdorff spaces*, Springer-Verlag, New York, 1988.

[R] Raha A.B., *Maximal topologies*, J. Austral. Math. Soc. **15** (1973), 279–290.
[St$_1$] Stephenson R.M., Jr., *Pseudocompact spaces*, Trans. Amer. Math. Soc. **134** (1968), 437–448.
[St$_2$] ———, *Minimal first countable topologies*, Trans. Amer. Math. Soc. **138** (1969), 115–127.
[St$_4$] ———, *Two* **R**-*closed spaces*, Canad. J. Math. **24** (1972), 286–292.
[W] Weston J.D., *On the comparison of topologies*, J. Lond. Math. Soc. **37** (1957), 342–354.

The University of Kansas, Lawrence, Kansas 66045, U.S.A.

The University of South Carolina, Columbia, South Carolina 29208, U.S.A.

The University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada

(*Received* January 18, 1993)