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# Alternative set theory with elementary classes

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*Abstract.* In this paper we sketch the development and give a model of the formal version of a generalization of the Alternative Set Theory.

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## Introduction

In this paper we want to introduce a theory called ASTEC (Alternative Set Theory with Elementary Classes) in which some class can be an element of another class. This theory can be considered as a subtheory of the theory FAST (Fuzzy Alternative Set Theory) sketched in [Pr1] (see also [Pr2]): by this no proof in ASTEC will be given here. The theory can also be considered as a generalization of the formal version of the Alternative Set Theory (AST) of [Vo] as it is given in [So1]. Indeed, in ASTEC, we try to formalize more closely the intuitive ideas of [Vo]. For example in ASTEC, without using the $\eta$-membership of [So1], we have the singleton of a class $X$ of the extended Universe, $\{X\}$, i.e. we have in ASTEC the "true singletons" of the classes of the Extended Universe. By this we have, as in [Vo], more general objects and we can write down explicitly the axiom of Extensional coding (even if it will not be done for practical reasons).

If we want to have the "true singleton" of a class (of the Extended Universe or not), we cannot define sets as usual, then we assume a unary predicate symbol for sets in the language: $Set(A)$ that is read "$A$ is a set". That is, we use a more general notion of set than in [So1] or other set theories. Afterwards we will obtain the sets of the Universe of Sets by suitable definitions from the class of sets.

The axioms of constructions of sets, of induction and of prolongation are generalized to this more general notion of set while other axioms are not. In the Alternative environment, sets can be interpreted as objects with clear (unvague) boundaries and classes as objects with vague boundaries. But, since a set can have a class as an element, it may contain some vagueness, too. Thus it does not seem suitable to generalize to sets all the axioms of the sets of the Universe of Sets that are (as we imagine and define them) the sets containing no vagueness.

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## 1. The theory

**1. Metadefinition.** (a) The language of ASTEC is $\mathcal{L} = (\in, =, Set)$, where $=$ is equality and $Set$ is a unary predicate. Variables are denoted by capital Latin letters and denote "classes".

**2. Definition.** We say that $A$ is an element and write $El(A)$ if $(\exists B)\ A \in B$.

Elements are for our theory collectionable objects, that is, what sets are for Kelley-Morse theory and for AST of [So1].

**3. Axiom ASTEC 1** (Extensionality)**.**

$$(\forall A, B) \quad (A = B \longleftrightarrow (\forall C)\ (C \in A \longleftrightarrow C \in B)).$$

**4. Axiom ASTEC 2** (Comprehension)**.** Given a formula $\varphi$, the following formula is an axiom:

$$(\exists B)\ (\forall A) \quad (A \in B \longleftrightarrow (El(A) \wedge \varphi(A)))$$

where $B$ is not present in $\varphi$.

We write $\{A/El(A) \wedge \varphi(A)\}$ for the class obtained by application of the preceding axiom to the formula $\varphi$; this class is obviously unique by Axiom 1. With this we have usual constructions carried on in classical set theory: in particular, $A \cup \{B\} = \{C/El(C) \wedge (C \in A \vee C = B)\}$. Naturally we denote by $\emptyset$ the empty class, i.e. $\{A/El(A) \wedge A \neq A\}$.

**5. Axiom ASTEC 3** (Existence of sets)**.**

(a) $Set(\emptyset)$,
(b) $(\forall A)\ (Set(A) \longrightarrow El(A))$,
(c) $(\forall A, B)\ ((Set(A) \wedge El(B)) \longrightarrow Set(A \cup \{B\}))$.

This is the first axiom on sets, and it is a formalization of the intuitive description of the ideas of [Vo] about "lists". By this axiom we have that if $A$ and $B$ are elements (not necessarily sets) then $\{A\}$, $\{A, B\}$, $\langle A, B\rangle$ (ordered pair defined as usual) are sets.

**Convention.** From now on capital letters from the end of the Latin alphabet $(X, Y, \dots)$ denote variables relativized to elements; lowercase letters denote variables relativized to sets. So we write shortly $\{X/\varphi(X)\}$ for $\{A/El(A) \wedge \varphi(A)\}$.

Since we have two kinds of collectionable objects, i.e. sets and elements, we must distinguish in many cases what we want to collect. This is particularly important for some operations as "power" and "union".

**6. Definition.** (a) $A \subseteq B$ stands for $(\forall X)\ (X \in A \longrightarrow X \in B)$.

(b) $\mathbb{P}(A) = \{X/X \subseteq A\}$ is the power of $A$.
(c) $\mathbb{P}_s(A) = \{x/x \subseteq A\}$ is the set-power of $A$.
(d) $\bigcup_s(A) = \{X/(\exists y)\ (y \in A \wedge X \in Y)\}$ is the set-union of $A$.
(e) $\bigcup(A) = \{X/(\exists Y)\ (Y \in A \wedge X \in Y)\}$ is the union of $A$.
(f) $El = \{X/El(X)\}$.
(h) $W = \{X/Set(X)\}$.

Now we arrive to the notion of set-formula.

**7. Definition.** Define recursively:

(I) (1) $X \in x$,
(2) $y \in x$,
(3) $y = x$,
(4) $X = x$,
(5) $X = Y$,

are set-formulas of $s$-height 0;

(II) if $\varphi(X)$, $\psi(X)$ and $\nu(X)$ are set-formulas of $s$-height, respectively $p, q, r$, with $p, q, r$ such that $r = p + q = m - 1$, then $\neg\nu(X)$, $\varphi(X) \wedge \psi(X)$, $(\exists\, X)\ \nu(X)$, $(\exists\, x)\ \nu(x)$ are set-formulas of $s$-height $m$.

Set-formulas are formulas speaking only of elements of sets. For example, $x \subseteq W$ is (the abbreviation of) the set-formula $(\forall\, X)\ (X \in x \longrightarrow (\exists\, y)\ y = X)$. Note: $A \subseteq x$, $X \subseteq x$, $x \subseteq A$ and so on are not set-formulas.

If sets are clear objects (i.e. objects with clear boundaries or again collections made in a clear way) then set-formulas are the formulas speaking only of clear objects and then can be understood as clear formulas.

We suppose that on clear objects (sets) we can establish clear statements.

**8. Axiom ASTEC 4** (Induction)**.** If $\varphi(X)$ is a set-formula then the following formula is an axiom:

$$(\varphi(\emptyset) \wedge (\forall\, x)\ (\forall\, X) \quad (\varphi(x) \longrightarrow \varphi(x \cup \{X\}))) \longrightarrow (\forall\, x)\ \varphi(x).$$

We recall that this axiom was given in AST (in [Vo] and [So1]) only for sets of the Universe of Sets. With trivial changes by this axiom we can prove for our notion of sets all the theorems on sets of the Universe of Sets of Chapter 1 of [Vo] and so on.

Now we can define (as in KM or AST) that $A$ is a relation, $\mathrm{Rel}\,(A)$, when every element of $A$ is an ordered pair, and (analogously) when $A$ is an ordering $\mathrm{Or}\,(A)$ and when $A$ is a function, $\mathrm{Fnc}\,(A)$. If $R$ is a relation, we write $R(X, Y)$ instead of $\langle X, Y\rangle \in R$. Note that by Axiom 3: $\mathrm{Rel}\,(R) \longrightarrow R \subseteq W$. Then we define natural numbers as regular ordinals (see below) and we obtain recursion on natural numbers.

**9. Definition.** (a) $\underline{\mathrm{E}} = \{X/(\exists\, Y, Z)\ X = \langle Y, Z\rangle \wedge (Y \in Z \vee Y = z)\}$.

(b) $\mathrm{OL}\,(R, B)$ stands for $\mathrm{Rel}\,(R) \wedge \mathrm{Or}\,(R) \wedge (\forall\, Y, Z)\ ((Y \in B \wedge Z \in B) \longrightarrow (R(Y, Z) \vee Y = Z \vee R(Z, Y)))$.

(c) $\mathrm{TR}\,(A)$ stands for $(\forall\, Y, Z)\ ((Z \in Y \wedge Y \in A) \longrightarrow Z \in A)$.

(d) If $x$ is a set, $\mathrm{reg}\,(x)$ stands for $(\forall\, A)\ [x \in A \longrightarrow (\exists\, y \in A)\ y \cap A = \emptyset]$.

(e) $\mathbf{N} = \{x/\,\mathrm{TR}\,(x) \wedge \mathrm{OL}\,(\underline{\mathrm{E}}, x) \wedge x \subseteq W \wedge \mathrm{reg}\,(x)\}$, if $x \in \mathbf{N}$ we say that $x$ is a natural number.

We must assume that a natural number is regular because there is no form of regularity on $W$. For the same reason we must use the notion of regularity (corresponding to the well-foundedness) given as in (d) above. For this notion of regularity see [Bo1], and [Bo2] for other discussions about the theory of Zermelo-Fraenkel without the axiom of foundation and so on. The given definition obviously implies also: $(x \in \mathbf{N} \longrightarrow (\{x_1, \ldots, x_n\} \subseteq x \longrightarrow \neg(x_1 \in x_2 \wedge \cdots \wedge x_{n-1} \in x_n \wedge x_n \in x_1)))$. By the definition usual properties of natural numbers can be proved (see [Vo]). In particular, we can prove the following theorem (where we use the notation $F \restriction A$ for the restriction of $F$ to $A$).

**10. Theorem** (Recursion). *If $G$ is a set-definable function and* $\mathrm{Dom}\,(G) = W$, *then there exists a unique set-definable function $F$ such that* $\mathrm{Dom}\,(F) = \mathbf{N} \wedge (\forall\, x \in \mathbf{N})\ F(x) = G(F \restriction x)$.

As in [Vo] we introduce the notions of semiset and finite classes.

**11. Definition.** (a) $\mathrm{Sem}\,(A)$ stands for $(\exists\, y)\ A \subseteq y$ and we read this as "$A$ is a semiset"; a semiset is proper if it is not a set.

(b) $A$ is finite, $FIN(A)$, iff $(\forall\, B)\ (B \subseteq A \longrightarrow Set(B))$.

(c) Given $A$, if there exists $R$ such that:

(1) $\mathrm{OL}\,(R, A)$,

(2) $\neg FIN(A)$,

(3) $(\forall\, Z \in A)\ FIN(\{Y/Y \in A \wedge R(Y, Z)\})$,

we say that the pair $A, R$ is an ordering of type $\omega$, $O\omega(R, A)$, and that $A$ is countable, $COUNT(A)$.

(d) $UNCOUNT(A)$ stands for $\neg FIN(A) \wedge \neg COUNT(A)$ and we say that $A$ is "uncountable".

(e) $A \cong B$ stands for $(\exists\, F)$ ("$F$ is an injection" $\wedge\, \mathrm{Dom}\,(F) = A \wedge \mathrm{Rng}\,(F) = B$).

Usual properties of finite sets can be proved (see [Vo]).

**12. Axiom ASTEC 5** (Prolongation).

$$(\forall\, F) \quad [(COUNT(F) \wedge \mathrm{Fnc}\,(F)) \longrightarrow (\exists\, f) \quad (\mathrm{Fnc}\,(f) \wedge F \subseteq f)].$$

Also this axiom is given on sets and not only on sets of the Universe of Sets. By prolongation, we can prove that $\mathbf{FN} = \{x/x \in \mathbf{N} \wedge FIN(x)\}$ is countable, is a proper semiset, and so on. We can prove also that if $\varphi$ is a set-formula and every subset $x$ of a countable class $X$ satisfies $\varphi$, then there is a set $y$ satisfying $\varphi$ such that $X \subseteq y$. A particular care must be paid here to prolongation: for example, the class $A = \{\mathbf{FN} - n/n \in \mathbf{FN}\}$ is countable and then it can be prolonged but using only set-formulas. Note in particular that (as a counter-example) for every $n \in \mathbf{FN}$, $\mathbf{FN} - n$ is not a set then if $n \leq m$ "$\mathbf{FN} - m \subseteq \mathbf{FN} - n$" is not a set-formula.

At last, by recursion we can define the Universe of Sets, $V$. Intuitively $V$ is $\bigcup_{\alpha \in \mathbf{N}} \mathbb{P}_s^{\alpha}(\emptyset)$ or, equivalently, $V$ is the collection of sets that are hereditarily sets;

note that it will be proved that $V$ is regular. In our interpretation of sets, $V$ is the collection of the objects that are clear and (hereditarily) made of clear objects. That is, objects containing no vagueness while all the other sets may contain some vague object in the sense that one of their elements (or an element of an element, $\ldots$ ) can be a class.

**13. Definition.** (a) $\mathbf{P}$ is the unique set-definable function with $\mathrm{Dom}(\mathbf{P}) = \mathbf{N}$, obtained by recursion from $K = \{\langle x, y\rangle / y = \mathbb{P}_s(x)\}$.

(b) $V = \bigcup(\mathrm{Rng}(\mathbf{P}))$ is the Universe of Sets.

(c) $(\forall a)\ a \in V$, $\mathrm{rk}(a)$ (the rank of $a$) stands for the minimum of $\{x/x \in \mathbf{N} \wedge a \in \mathbf{P}(x)\}$.

(d) A $V$-formula is a formula in which every variable (free or bounded) is restricted to $V$.

Using the function $\mathbf{P}$, we can see that $\mathbf{N} \subseteq V \subseteq W$, $\mathrm{TR}(V)$, and also $(\forall A)$ $(\mathit{Set}(A) \longrightarrow (A \subseteq V \longrightarrow A \in V))$. Obviously a $V$-formula is a set-formula. By this we can prove that $V$ satisfies the axioms on sets of [So1], in particular for example:

Regularity on $V$. If $\varphi(x)$ is a $V$-formula:

$$(\exists x) \quad \varphi(x) \longrightarrow (\exists x) \quad (\varphi(x) \wedge (\forall y \in x)\ \neg\varphi(y)).$$

Note that on $W$ there exists no form of regularity while $V$ (which is obtained from $W$) is regular.

The last two axioms that are characteristic of the Alternative Set Theory are given only on $V$.

**14. Axiom ASTEC 6** (Two cardinalities)**.**

$$(\forall A, B \subseteq V) \quad ((\mathit{UNCOUNT}(A) \wedge \mathit{UNCOUNT}(B)) \longrightarrow A \cong B).$$

**15. Axiom ASTEC 7** (Choice)**.** $V$ can be well ordered.

The following is the axiom that makes the theory ASTEC really different from AST of [So1].

**16. Axiom ASTEC 8** (Elementarity of the classes of the extended universe)**.**

$$(\forall A) \quad (A \subseteq V \longrightarrow \mathit{El}(A)).$$

By Axiom 8 we define

**17. Definition.** The Extended Universe is the class $\mathcal{U} = \{X/X \subseteq V\}$.

As in AST (see [So1]), the well ordering of $V$ implies the axiom of Extensional Coding. Even if the axiom of Extensional Coding can be written here explicitly as in [Vo] (we leave this easy task to the reader), we prefer to assume this form of the axiom for the construction of the following model.

**18. Metadefinition.** ASTEC is the theory collecting axioms from ASTEC 1 to ASTEC 8.

## 2. The model

It can be proved then that the theory ASTEC$^-$ = ASTEC – ASTEC 8 is consistent with ZF and that the theory AST of [So1] has an interpretation in it (see [Pr2]): indeed, a model of the theory AST of [So1] is a model of ASTEC$^-$ (see [So2]). Now we want to produce a model of ASTEC.

We work in ZF+GCH for simplicity. First we build a model $\mathcal{M}$ of "ASTEC 1+ ASTEC 2+ASTEC 3+Urelements" where $|\,Ur\,| = \aleph_2$.

Note that "ASTEC 1+ASTEC 2+ASTEC 3+Urelements" is equivalent to $\mathrm{ZFU}_{\mathrm{Fin}}$, that is "ZF+Urelements" where the axiom of infinity is replaced by $((\forall\, x)$ "$x$ is finite"): this equivalence can be proved as in [So2].

The model $\mathcal{M}$ is obtained as follows: we choose a $m \in \omega$ such that $m \geq 4$ and a set $\mathcal{A}$ such that $(|\mathcal{A}| = \aleph_2 \wedge (\forall\, x \in \mathcal{A})\ \operatorname{rank}(x) = \omega + m)$. The reasons for the choice of $m$ are technical: here we note only that $(\mathcal{A} \cap \mathcal{V}_\omega = \emptyset)$, define

$$\mathcal{R}(0) = \mathcal{A},$$
$$\mathcal{R}(1) = \{x/x \subseteq \mathcal{R}(0) \wedge |x| \in \omega\} \cup \mathcal{R}(0);$$

given $\mathcal{R}(n)$,

$$\mathcal{R}(n+1) = \{x/x \subseteq \mathcal{R}(n) \wedge |x| \in \omega\} \cup \mathcal{R}(n),$$
$$\mathcal{R} = \bigcup_{n\in\omega} \mathcal{R}(n), \quad \text{and} \quad \mathcal{W} = \mathcal{R} - \mathcal{A}.$$

We have that $\mathcal{V}_\omega \subseteq \mathcal{R}$.

Denoted by $\mathbf{E} = \{\langle x, y\rangle / x \in y\}$, the model is

$$\mathcal{M} = \langle \mathcal{R}, \mathbf{E} \cap (\mathcal{R} \times \mathcal{W}), =, \mathcal{W}\rangle,$$

where $\mathcal{W}$ is the interpretation of $W$ (note $\emptyset \in \mathcal{W}$).

Obviously the class of urelements, $Ur$, and $El$ are interpreted in $\mathcal{A}$, $\mathcal{R}$, respectively due to the assumptions on $\mathcal{A}$. The only non-trivial proof is that $\mathcal{M}$ satisfies induction: let $\varphi$ be a set-formula such that $\mathcal{M} \models (\varphi(\emptyset) \wedge (\forall\, x, Y)\ (\varphi(x) \longrightarrow \varphi(x \cup \{Y\})))$, that is

$$(\hat{\varphi}\,(\emptyset) \wedge (\forall\, x, y) \quad ((x \in \mathcal{W} \wedge y \in \mathcal{R}) \longrightarrow (\hat{\varphi}\,(x) \longrightarrow \hat{\varphi}\,(x \cup \{y\})))),$$

where $\hat{\varphi}$ is the interpretation of $\varphi$ in $\mathcal{M}$; then take $x \in \mathcal{W}$ such that $(\neg\hat{\varphi}\,(x) \wedge |x| = \min\{n/(\exists\, y \in \mathcal{R})\ \neg\hat{\varphi}\,(y) \wedge |y| = n\})$; note $x \neq \emptyset$. Then if $t \in x$, $x - \{t\} \in \mathcal{W}$ and is such that $\hat{\varphi}\,(x - \{t\})$; by the assumption on $\hat{\varphi}$ we have $\hat{\varphi}\,(x)$, absurd.

It follows easily that $\mathbf{N}$, $\mathbf{FN}$, $V$ are interpreted respectively in $\omega$, $\omega$, $\mathcal{V}_\omega$: indeed, $\mathcal{M}$ is not a model of ASTEC 5 and there exists no infinite set (both in classical and alternative sense).

Now we want to consider the ultrapower of the preceding model: then let $D$ be a countably incomplete ultrafilter on $\omega$ (for simplicity). We use definitions and notations of [C-K], so we write $\prod_D X$ for the ultrapower of $X$. If $Y \subseteq X$, by $\prod_D Y$ we denote the substructure of $\prod_D X$ "naturally isomorphic" to $\prod_D Y$.

**19. Definition.** (a) $\mathbf{R} = \prod_D \mathcal{R}$; $\mathbf{A} = \prod_D \mathcal{A}$.
(b) $\mathbf{W} = \prod_D \mathcal{W}$.
(c) $\Re = \langle \mathbf{R}, \in_D, =_D, \mathbf{W} \rangle$, where $\in_D$ and $=_D$ are the ultrapowers of $\in$ and $=$.
(d) $\mathbf{Q} = \{x \subseteq \mathbf{R} / \neg(\exists\, z \in \mathbf{R}) \quad x = \{y/y \in \mathbf{R} \wedge \Re \models y \in z\}\}$.
(e) $\mathbf{M}' = \mathbf{R} \cup \mathbf{Q}$.
(f) For every $x$ and $y$ in $\mathbf{M}'$:

$$x \,\mathsf{E}\, y \longleftrightarrow ((x \in \mathbf{R} \wedge y \in \mathbf{R} \wedge \Re \models x \in y) \vee (x \in \mathbf{R} \wedge y \in \mathbf{Q} \wedge x \in y)),$$

and

$$x \doteq y \longleftrightarrow ((x \in \mathbf{R} \wedge y \in \mathbf{R} \wedge \Re \models x = y) \vee (x \in \mathbf{Q} \wedge y \in \mathbf{Q} \wedge x = y)).$$

(g) $\mathcal{M}' = \langle \mathbf{M}', \mathsf{E}, \doteq, \mathbf{W} \rangle$ such that
$x \,\mathsf{E}\, y$ is the interpretation of $x \in y$,
$x \doteq y$ is the interpretation of $x = y$,
$x \in \mathbf{W}$ is the interpretation of $Set(x)$.

As in [So2], using properties of the ultrapower it can be shown that $\mathcal{M}'$ is a model of ASTEC$^-$.

We have that (up to isomorphisms): $\mathbf{V} = \prod_D \mathcal{V}_\omega \subseteq \mathbf{R}$, $\mathbf{A} \subseteq \mathbf{R}$ and elements of $\mathbf{A}$ are urelements for $\mathcal{M}'$. Denote by $\mathbf{U} = \{x \in \mathbf{Q}/x \subseteq \mathbf{V} \wedge x \notin \mathbf{V}\}$. By the assumption on $\mathcal{M}$, $D$ and $m$, we have that:

**20. Proposition.** (a) $|\mathbf{A}| = \aleph_2 = |\mathbf{U}|$.
(b) $\mathcal{A} \cap \mathbf{R} = \emptyset$.
(c) $\mathbf{Q} \cap \mathbf{R} = \emptyset$.

**Convention.** We assume that a bijection $\mathcal{K} : \mathbf{U} \longrightarrow \mathbf{A}$ has been given.

In a situation like this one, a general method is given in [Ob1] (and [Ob2]) for showing that certain kinds of formulas are true in a new model $\mathcal{M}^*$ if and only if they are true in $\mathcal{M}'$. Here we use a similar but simplified method because we do not need the complete one.

Now we shall identify $x \in \mathbf{U}$ with $\mathcal{K}(x)$ leaving the other objects untouched as in the following definition. In the definition of the model, an object $x(\in \mathbf{U})$ behaves as $x$ itself (from $\mathcal{M}'$) in formulas of the kind $Y \in x$ and as $\mathcal{K}(x)$ (from $\mathcal{M}'$) in formulas of the kind $x \in A$.

**21. Definition.** (a) $\mathbf{H} = \mathcal{K}[\mathbf{U}]$.
(b) $\mathbf{M}'' = \mathbf{M}' - (\mathbf{H} \cup \mathbf{U})$.
(c) $\mathcal{G} = \mathcal{K} \cup \mathrm{Id}_{\mathbf{M}''}$.
(d) $\mathbf{M}^* = \mathbf{M}'' \cup \mathbf{U} = \mathbf{M}' - \mathbf{H}$.
(e) $\mathcal{M}^* = \langle \mathcal{M}^*, \varepsilon, \doteq, \mathbf{W} \rangle$, where

$$x \,\varepsilon\, y \longleftrightarrow \mathcal{G}(x) \,\mathsf{E}\, y,$$

and

$x \varepsilon y$ is the interpretation of $x \in y$,
$x \doteq y$ is the interpretation of $x = y$,
$x \in \mathbf{W}$ is the interpretation of $Set(x)$.

The most parts of the points in the following lemma are an explanation of the definition of $\mathcal{M}''$.

**22. Lemma.** (a) (i) $(\forall x \in \mathbf{M}^*)$ $(\mathcal{M}^* \models Set(x) \longleftrightarrow \mathcal{M}' \models Set(x) \longleftrightarrow x \in \mathbf{W})$;

(ii) $(\forall x \in \mathbf{M}^*)$ $(\mathcal{M}^* \models Set(x) \longrightarrow (\mathcal{G}(x) = x = \mathcal{G}^{-1}(x)))$;

(b) (i) $(\forall x \in \mathbf{M}^*)$ $(\mathcal{M}^* \models El(x) \longleftrightarrow \mathcal{M}' \models El(\mathcal{G}(x)))$;

(ii) $(\forall x \in \mathbf{U})$ $\mathcal{M}^* \models El(x)$;

(c) $(\forall x, y \in \mathbf{M}^*)$ $(\mathcal{M}^* \models x = y \longleftrightarrow \mathcal{M}' \models x = y \longleftrightarrow \mathcal{M}' \models \mathcal{G}(x) = \mathcal{G}(y))$;

(d) (i) $(\forall y \in \mathbf{M}^* - \mathbf{U})$ $[\mathcal{G}(y) = y \wedge (\forall x \in \mathbf{M}^*)$ $(\mathcal{M}^* \models x \in y \longleftrightarrow \mathcal{M}' \models \mathcal{G}(x) \in y \longleftrightarrow \mathcal{M}' \models \mathcal{G}(x) \in \mathcal{G}(y))]$;

(ii) $(\forall y \in \mathbf{U})$ $[\mathcal{G}(y) \neq y \wedge (\forall x \in \mathbf{M}^*)$ $(\mathcal{M}^* \models x \in y \longleftrightarrow (x \in \mathbf{V} \wedge x = \mathcal{G}(x) \wedge \mathcal{M}' \models \mathcal{G}(x) \in y))]$;

(e) (i) $(\forall x \in \mathbf{M}^* - \mathbf{U})$ $[\mathcal{G}(x) = x \wedge (\forall y \in \mathbf{M}^*)$ $(\mathcal{M}^* \models x \in y \longleftrightarrow \mathcal{M}' \models \mathcal{G}(x) \in y \longleftrightarrow \mathcal{M}' \models x \in y)]$;

(ii) $(\forall x \in \mathbf{U})$ $[\mathcal{G}(x) \neq x \wedge (\forall y \in \mathbf{M}^*)$ $(\mathcal{M}^* \models x \in y \longleftrightarrow \mathcal{M}' \models \mathcal{G}(x) \in y)]$;

(f) $(\forall x, y \in \mathbf{M}^*)$ $(\mathcal{M}^* \models x \subseteq y \longleftrightarrow \mathcal{M}' \models x \subseteq y)$.

Proof: (a) and (e) by the definition of $\mathcal{M}^*$.

(b) (i): the implication $\longrightarrow$ is trivial by the definition of $\mathbf{M}^*$. Vice versa, if $\mathcal{M}' \models \mathcal{G}(x) \in y$, then $y \notin \mathbf{A}$, by which $y \in \mathbf{M}^*$. (ii) follows by (i).

(c) by definition and the fact that $\mathcal{G}$ is a bijection.

(d) (i) by definition. (ii) Remember that if $y \in \mathbf{U}$ then $\mathcal{M}' \models ((\forall x \in y)\ x \in V)$, i.e. (by definition of $\mathcal{M}'$ and $\mathbf{U}$) $(\forall x \in y)$ $x \in \mathbf{V}(\subseteq \mathbf{M}^* - \mathbf{U})$.

(f) Assume $\mathcal{M}^* \models x \subseteq y$, then $(\forall z \in \mathbf{M}^*)$ $\mathcal{M}^* \models (z \in x \longrightarrow z \in y)$, that is by definition, $(\forall z \in \mathbf{M}^*)$ $\mathcal{M}' \models (\mathcal{G}(z) \in x \longrightarrow \mathcal{G}(z) \in y)$. But $\mathcal{K}$ is onto $\mathbf{H}$ and $(\forall z \in \mathbf{U})$ $(\forall t \in \mathbf{M}')$ $\mathcal{M}' \models z \notin t$. By this, it follows $(\forall q \in \mathbf{M}')$ $\mathcal{M}' \models (q \in x \longrightarrow q \in y)$ and then a part of the thesis is obtained. The converse is analogous. □

Now we can prove that $\mathcal{M}^*$ is a model of ASTEC using the fact that $\mathcal{M}'$ is model of ASTEC – ASTEC 8. By the preceding lemma we obtain:

**23. Theorem.** *$\mathcal{M}^*$ is a model of* ASTEC 1 *and* ASTEC 2.

Proof: ASTEC 1: Using the preceding lemma, we have that $\mathcal{M}^* \models x = y \longleftrightarrow \mathcal{M}' \models x = y$, see Lemma 22 (c). The thesis follows by the fact that $\mathcal{M}'$ is a model of "**Ext**+Urelements" and if $x, y \in \mathbf{M}^*(\subseteq \mathbf{M}')$ then $x$ and $y$ are not urelements in $\mathcal{M}'$.

ASTEC 2 follows by the fact $\mathcal{M}'$ is model of ASTEC 2 and we still have all the subclasses of the model. □

We will use in nearly all the following proofs the fact that $\mathcal{M}^*$ is a model of ASTEC 1.

**24. Lemma.** (a) $\mathcal{M}^* \models x = \{y_1, \dots, y_n\} \longleftrightarrow (x \in \mathbf{W} \wedge x = \mathcal{G}(x) \wedge \mathcal{M}' \models x = \{\mathcal{G}(y_1), \dots, \mathcal{G}(y_n)\})$.

(b) $\mathcal{M}^* \models (Set(y) \wedge El(z) \wedge x = y \cup \{z\}) \longleftrightarrow \mathcal{M}' \models (Set(y) \wedge El(\mathcal{G}(z)) \wedge x = y \cup \{\mathcal{G}(z)\})$.

Proof: (a) Assume $\mathcal{M}^* \models x = \{y\}$: this means that $(\forall z \in \mathbf{M}^*)\ \mathcal{M}^* \models (z \in x \longleftrightarrow z = y)$. By Lemma 22 this is equivalent to $(\forall z \in \mathbf{M}^*)\ \mathcal{M}' \models (\mathcal{G}(z) \in x \longleftrightarrow \mathcal{G}(z) = \mathcal{G}(y))$. But $\mathcal{K}$ is onto $\mathbf{H}$ and every element of $\mathbf{U}$ cannot be an element in $\mathcal{M}'$, then $(\forall t \in \mathbf{M}')\ \mathcal{M}' \models (t \in x \longleftrightarrow t = \mathcal{G}(y))$, by which it follows $\mathcal{M}' \models x = \{\mathcal{G}(y)\}$. It also follows that $\mathcal{M}' \models Set(x)$ and then $x \in \mathbf{W}$ and $\mathcal{G}(x) = x$. Analogously the vice versa and the case $x = \{y_1, \dots, y_n\}$.

(b) Assume $\mathcal{M}^* \models (Set(y) \wedge El(z) \wedge x = y \cup \{z\})$: by Lemma 22 we have $\mathcal{M}' \models (Set(y) \wedge El(\mathcal{G}(z)))$. $\mathcal{M}^* \models x = y \cup \{z\}$ stands for $(\forall t \in \mathbf{M}^*)\ \mathcal{M}^* \models (t \in x \longleftrightarrow (t \in y \vee t = z))$. By Lemma 22, this is equivalent to $(\forall t \in \mathbf{M}^*)\ \mathcal{M}' \models (\mathcal{G}(t) \in x \longleftrightarrow (\mathcal{G}(t) \in y \vee \mathcal{G}(t) = \mathcal{G}(z)))$. As in the preceding point we have then $\mathcal{M}' \models x = y \cup \{\mathcal{G}(z)\}$. Analogously the vice versa. □

**25. Theorem.** *$\mathcal{M}^*$ is a model of* ASTEC 3.

Proof: It easily follows by the preceding lemma. □

**26. Lemma.** *For every set-formula $\varphi(X_1, \dots, X_n)$, then*

$$\mathcal{M}^* \models \varphi(X_1, \dots, X_n)\ [z_1, \dots, z_n] \longleftrightarrow \mathcal{M}' \models \varphi(X_1, \dots, X_n)\ [\mathcal{G}(z_1), \dots, \mathcal{G}(z_n)].$$

Proof: For set-formulas of $s$-height 0, the proof follows by Lemma 22. Then we continue by an easy induction on the $s$-height of the set-formulas using Lemmas 22, 24 and Theorem 25. □

By this theorem, we have

**27. Theorem.** (a) *$\mathcal{M}^*$ is a model of* ASTEC 4.

(b) $\mathcal{M}^* \models x = V \longleftrightarrow \mathcal{M}' \models x = V \longleftrightarrow x = \mathbf{V}$.

Proof: (a) by the preceding lemma.

(b) By Lemma 22 $\mathcal{M}^* \models x = V \longleftrightarrow \mathcal{M}' \models x = V$. By the construction of $\mathbf{M}'$, it follows $x = \mathbf{V}$. □

Indeed, $\mathbf{V}$ and $\mathbf{W}$ are contained in $\mathbf{M}''$ and their structure is not changed by the identification we have made. We can easily prove also that:

**28. Lemma.** *For every $x, y, z$ in $\mathbf{M}^*$:*

(a) $\mathcal{M}^* \models x = \langle y, z\rangle \longleftrightarrow (x \in \mathbf{W} \wedge x = \mathcal{G}(x) \wedge \mathcal{M}' \models x = \langle \mathcal{G}(y), \mathcal{G}(z)\rangle)$.

(b) $\mathcal{M}^* \models \mathrm{Rel}\,(x) \longleftrightarrow \mathcal{M}' \models \mathrm{Rel}\,(x)$*; the same for* $\mathrm{Fnc}\,(x)$ *and* $\mathrm{In}\,(x)$.

(c) $\mathcal{M}^* \models x = \mathrm{Dom}\,(y) \longleftrightarrow \mathcal{M}' \models x = \mathrm{Dom}\,(y)$*, and the same for* $\mathrm{Rng}\,(y)$.

(d) $\mathcal{M}^* \models \mathit{FIN}(x) \longleftrightarrow \mathcal{M}' \models \mathit{FIN}(x)$*; and the same for* $\mathit{COUNT}(x)$, $\mathit{UNCOUNT}(x)$.

Proof: Every point follows by Lemmas 22, 24 and the preceding points. □

By which:

**29. Theorem.** $\mathcal{M}^*$ *is a model of* ASTEC 5, ASTEC 6.

Proof: ASTEC 5: By the preceding lemma and Lemma 22.

ASTEC 6: If $x$ and $y$ are uncountable in $\mathcal{M}^*$, then they are uncountable in $\mathcal{M}'$ and then there is a bijection $F(\in \mathbf{M}')$ from $x$ to $y$. But if $F$ is a bijection between two uncountable classes in $\mathcal{M}'$, then $F \neq \emptyset$ and then $F \notin \mathbf{A}$. By the preceding lemma, $F$ will be a bijection from $x$ to $y$ also in $\mathcal{M}^*$. □

But also:

**30. Theorem.** $\mathcal{M}^*$ *is a model of* ASTEC 7, ASTEC 8.

Proof: ASTEC 7: Since $V$ is not touched by the identification, we have that the well order $R$ of $V$ in $\mathcal{M}'$ is a class in $\mathcal{M}'$ and then $R \in \mathbf{M}^*$. By the preceding lemma, it is also a well order of $V$ in $\mathcal{M}^*$.

ASTEC 8: By Theorem 27, $V$ is the same in $\mathcal{M}^*$ and $\mathcal{M}'$ and then $x$ is a class of the extended universe in $\mathcal{M}^*$ if and only if it is a class of the extended universe in $\mathcal{M}'$, i.e. $x \in \mathbf{U}$. Then $\mathcal{G}(x)$ is an element (by Lemma 22). □

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