A measure-theoretic characterization of Boolean algebras among orthomodular lattices

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Abstract. We investigate subadditive measures on orthomodular lattices. We show as the main result that an orthomodular lattice has to be distributive (=Boolean) if it possesses a unital set of subadditive probability measures. This result may find an application in the foundation of quantum theories, mathematical logic, or elsewhere.

Keywords: orthomodular lattice, subadditive probability measure

Classification: Primary 06C15; Secondary 28E15, 81P10

Notions and results

Let us first recall some basic notions as we shall use them in the sequel (see e.g. [1], [10], etc.).

Definition 1. An orthomodular lattice (abbr. OML) is a lattice L endowed with an orthocomplementation relation, $' : L \to L$, such that the following conditions are satisfied ($a, b \in L$):

- (i) if $a \leq b$, then $b' \leq a'$,
- (ii) $(a')' = a$,
- (iii) $a \vee a' = 1$,
- (iv) if $a \leq b$, then $b = a \vee (b \wedge a')$ (the orthomodular law).

Let us reserve the letter L for OMLs in that what follows. As known, a typical example of an OML is the lattice of all closed subspaces in a Hilbert space or a Boolean algebra. Obviously, an OML is Boolean (i.e. it is a Boolean algebra) exactly if it is distributive.

We shall deal with (probability) measures on OMLs. Let us call them states to simplify the notation and, also, indicate the link with quantum theories (see e.g. [5]).

Definition 2. Let L be an OML. A mapping $s : L \to \langle 0, 1 \rangle$ is called a **state** if it fulfils the following two requirements: (i) $s(1) = 1$, (ii) if $a \leq b'$, then $s(a \vee b) = s(a) + s(b)$. Further, a state s on L is called **subadditive** if, in addition, it fulfils the following condition: For any $a, b \in L$, we have $s(a \vee b) \leq s(a) + s(b)$. Finally, a state s on L is called a **valuation** if $s(a \vee b) = s(a) + s(b)$ provided $a \wedge b = 0$.

Obviously, every valuation is a subadditive state. If L is a Boolean algebra, then every state is a valuation. A general OML may have a very poor set of valuations. For instance, the lattice of all closed subspaces of a finite dimensional Hilbert space has only one valuation - a normalized dimension function. Thus, the latter example possesses many states that are not valuations.

We shall employ the following notion (see also [5]).

Definition 3. An OML, L, is called **unital with respect to subadditive states** if for any non-zero $a \in L$ there exists a subadditive state on L such that $s(a) = 1$.

It should be noted that there are (finite) OMLs which do not have any state at all (see [4]). Thus, Definition 3 imposes quite a strong condition on L. In fact, we have the following theorem. (It should be noted that this result generalizes Theorem 4 of [7] and, also, gives another proof to this latter Theorem 4.)

Theorem 1. *An OML is a Boolean algebra if and only if it is unital with respect to subadditive states.*

PROOF: 1. If L is a Boolean algebra and if $(\Omega, \bar{L})$ is its set-representation, where $\bar{L}$ $(\subset \exp \Omega)$ is isomorphic to L, then the collection of states which are concentrated in points of Ω obviously makes $\bar{L}$ a unital OML with respect to subadditive states. Thus, the sufficiency is obvious.

2. In order to establish the necessity, we have to prove that if L is unital with respect to subadditive states, then it is Boolean. We shall do so in a series of propositions. (Let us refer to [10] for technical details and orthomodular folklore.) □

Proposition 1. *Suppose that $a, b \in L$ and suppose also that $a \wedge b = 0$. Then*

$$a \vee b = (a' \wedge (a \vee b)) \vee (b' \wedge (a \vee b))$$

PROOF: See [3]. □

Proposition 2 (see also [7], [11] and [12])**.** *Every subadditive state on L is a valuation.*

PROOF: Suppose that s is a subadditive state on L and suppose that $a, b \in L$ with $a \wedge b = 0$. We have to derive the inequality $s(a \vee b) \geq s(a) + s(b)$. Making use of the orthomodular law, we see that $s(a \vee b) = s(a) + s(a' \wedge (a \vee b))$ and, also, $s(a \vee b) = s(b) + s(b' \wedge (a \vee b))$. Thus, we have $2 \cdot s(a \vee b) = s(a) + s(b) + s(a' \wedge (a \vee b)) + s(b' \wedge (a \vee b))$. By Proposition 1, we infer that $s(a \vee b) \leq s(a' \wedge (a \vee b)) + s(b' \wedge (a \vee b))$. It follows that $2 \cdot s(a \vee b) \geq s(a) + s(b) + s(a \vee b)$, which completes the proof. □

Remark. In view of the above proposition, we can rewrite a classical result of the lattice theory (see [2] and [13]) as follows: If L admits a strictly positive subadditive state, then L is modular and every subset of L consisting of nonzero mutually orthogonal elements is at most countable.

Proposition 3. *Let s be a valuation on L. Then, for any $a, b \in L$, we have $s(a) + s(b) = s(a \vee b) + s(a \wedge b)$.*

Proof: See [2]. □

Proposition 4 (see also [6]). *Suppose that L enjoys the following property: If $a \wedge b = 0$, then $a \leq b'$. Then L is a Boolean algebra.*

Proof: We are to show that if the assumption of Proposition 4 is satisfied, then every couple $a, b \in L$ is compatible in L. Since $(a \wedge (a \wedge b)') \wedge (b \wedge (a \wedge b)') = (a \wedge b) \wedge (a \wedge b)' = 0$, our assumption implies that the elements $a \wedge (a \wedge b)'$ and $b \wedge (a \wedge b)'$ are orthogonal. We therefore have $a = (a \wedge b) \vee (a \wedge (a \wedge b)')$, $b = (a \wedge b) \vee (b \wedge (a \wedge b)')$ and all the elements on the right-hand sides of the equalities are mutually orthogonal. This shows that the couple a, b is compatible in L and Proposition 4 is proved. □

Proposition 5. *Suppose that for any nonzero element $a \in L$ there is such a valuation s on L that $s(a) = 1$. Then L is a Boolean algebra.*

Proof: We shall make use of Proposition 4. Consider two elements $a, b \in L$, such that $a \wedge b = 0$. We have to show that $a \leq b'$. Suppose that this is not the case and look for a contradiction. If $a \not\leq b'$, then $a \wedge b' < a$. It means that $(a \wedge b')' \wedge a \neq 0$. Equivalently, $(a' \vee b) \wedge a \neq 0$. Let s be such a valuation that $s((a' \vee b) \wedge a) = 1$. It follows that $s(a' \vee b) + s(a) - s((a' \vee b) \vee a) = 1$. Consequently, $s(a' \vee b) + s(a) = 2$ and therefore $s(a' \vee b) = 1$ as well as $s(a) = 1$. We further have $s(a' \vee b) = s(a') + s(b) - s(a' \wedge b) = 1$.Thus, $s(b) = 1 - s(a' \wedge b)$ and therefore $s(b) = 1 - s(a' \vee b) + s(a') + s(b)$. It follows that $s(a' \vee b) = 1$. Since $s(a') = 0$, we see that $s(b) = 1$. We have obtained $s(a) = s(b) = 1$. It follows that $s(a \wedge b) = 1$ which contradicts our assumption. The proof of Proposition 5 is complete. □

Obviously, the interplay of all the propositions above establishes the required implication — if L is unital with respect to subadditive states, then it has to be Boolean. This completes the proof of Theorem 1.

Let us conclude this note by making two remarks related to the latter theorem. Suppose that s is a valuation on L and suppose that $s(a) = 1$ for an $a \in L$. Does this imply that a has to be central? (An element $a \in L$ is called central if it is compatible to any element of L. Obviously, an affirmative answer to the latter question would give another proof of Theorem 1.) The answer is provided by the following proposition which may be interesting in its own right (see also [9]).

Proposition 6. *Let L do not possess any valuation. Then $L \times \{0, 1\}$, where $\{0, 1\}$ is understood as a two-element lattice, possesses exactly one valuation, s, and moreover, s is two-valued and $s(a, 1) = 1$ for any $a \in L$.*

Proof: Suppose that s is a valuation on $L \times \{0, 1\}$. Then we have $s(1, 0) + s(0, 1) = s(1, 1) = 1$. If the value of $s(1, 0)$ was positive, then we could easily construct a valuation on L. This possibility is, however, excluded by our assumption. Thus, $s(1, 0) = 0$ and therefore $s(a, 0) = 0$ for any $a \in L$. It implies that

$s(0,1) = 1$ and therefore $s(a,1) = 1$ for any $a \in L$. What remains to be verified is that the mapping $s : L \times \{0,1\} \to \langle 0,1 \rangle$ defined by $s(a,0) = 0, s(a,1) = 1\, (a \in L)$ is indeed a valuation, but this is easy.

The latter proposition gives a negative answer to the above stated question (as known, valuationless orthomodular lattices exist in plenties — see [4]).

Our last comment concerns a "non-lattice" version of Theorem 1. Suppose that L is an orthomodular poset (see [10]). Let us call a state s on L subadditive if the following condition is satisfied: If $a, b \in L$, then there is an element $c \in L$ such that $c \geq a, c \geq b$ and $s(c) \leq s(a) + s(b)$. (The latter definition obviously coincides with the former one if L is a lattice.) In view of Theorem 1, a natural question arises whether any orthomodular poset which is unital with respect to subadditive states has to be Boolean. The answer is no — the example by V. Müller (see [8]) shows (among other remarkable things) that a counterexample can be constructed even in the class of set-representable orthomodular posets.

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(*Received* November 16, 1992, *revised* November 23, 1993)