Smooth points of the unit sphere in Musielak-Orlicz function spaces equipped with the Luxemburg norm

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Abstract. There is given a criterion for an arbitrary element from the unit sphere of Musielak-Orlicz function space equipped with the Luxemburg norm to be a point of smoothness. Next, as a corollary, a criterion of smoothness of these spaces is given.

Keywords: Musielak-Orlicz function, Musielak-Orlicz space, support functional, smooth point, smooth space

Classification: Primary 46E30; Secondary 45B20

Introduction

In the following, (T, Σ, μ) denotes a non-atomic σ-finite measure space, $\mathbb{R}$ denotes the set of reals, $\mathbb{R}_+$ denotes the set of nonnegative reals, $\mathbb{N}$ denotes the set of natural numbers, χ_A stands for the characteristic function of a set $A \in \Sigma$, X denotes a Banach space and X^* denotes its dual space. Their unit balls and spheres are denoted by $B(X)$, $B(X^*)$ and $S(X)$, $S(X^*)$, respectively.

A map $\Phi : T \times \mathbb{R} \to [0, +\infty]$ is said to be a Musielak-Orlicz function if for μ-a.e. $t \in T$, $\Phi(t, \cdot)$ is vanishing and continuous at zero, left-hand side continuous on the whole $\mathbb{R}_+$, not identically equal to zero, convex and even and if for any $u \in \mathbb{R}$, $\Phi(\cdot, u)$ is a Σ-measurable function.

For a given Musielak-Orlicz function Φ we define

$$a(t, \Phi) = \sup\{u > 0 : \Phi(t, u) < +\infty\}$$

for any $t \in T$.

We denote by Φ'_- and Φ'_+ the left-hand side and the right-hand side derivatives of Φ with respect to the second variable, respectively. For any $u \in \mathbb{R}$ we define

$$\partial\Phi(t,u) = \begin{cases} [\Phi'_+(t,u), \Phi'_-(t,u)] & \text{if } -a(t,\Phi) < u < a(t,\Phi) \\ [\Phi'_-(t,u), +\infty) & \text{if } u = a(t,\Phi) \text{ and } \Phi'_-(t, a(t,\Phi)) < +\infty \\ (-\infty, \Phi'_+(t,u)] & \text{if } u = -a(t,\Phi) \text{ and } \Phi'_+(t, -a(t,\Phi)) > -\infty \\ \{+\infty\} & \text{if } u \geq a(t,\Phi) \text{ and } \Phi'_-(t, a(t,\Phi)) = +\infty \\ \{-\infty\} & \text{if } u \leq -a(t,\Phi) \text{ and } \Phi'_+(t, -a(t,\Phi)) = -\infty \end{cases}$$

I am very much indebted to Professor H. Hudzik for many valuable suggestions

for any $t \in T$ (cf. [GH]).

We have for any $u \in \mathbb{R}$:

$$\partial\Phi(t,u) = \{v \in \mathbb{R} : \Phi(t,u) + \Phi^*(t,v) = uv\}\,,$$

where Φ^* is the Musielak-Orlicz function complementary to Φ in the sense of Young, i.e.

$$\Phi^*(t,u) = \sup_{v>0}\{|u|v - \Phi(t,v)\}$$

for any $t \in T$ and $u \in \mathbb{R}$, under the convention $\Phi^*(t,\pm\infty) = +\infty$.

Denote by $L^0(\mu)$ the space of all (μ-equivalence classes of) Σ-measurable real functions defined on T. Given a Musielak-Orlicz function Φ we can define on $L^0(\mu)$ a convex functional I_Φ by the formula

$$I_\Phi(x) = \int_T \Phi(t,x(t))\,d\mu\,.$$

The Musielak-Orlicz space generated by a Musielak-Orlicz function Φ is defined to be the set of all $x \in L^0(\mu)$ for which $I_\Phi(\lambda x) < +\infty$ for some $\lambda > 0$ depending on x and it is denoted by $L^\Phi(\mu)$. This space endowed with the Luxemburg norm $\|\ \|_\Phi$ defined by

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \le 1\}$$

is a Banach space (cf. [M]). In the case when $\Phi(t_1,\cdot) = \Phi(t_2,\cdot)$ for μ-a.e. $t_1,t_2 \in T$, Φ is a usual Orlicz function and $L^\Phi(\mu)$ is called an Orlicz space.

We can define in $L^\Phi(\mu)$ another norm $\|\ \|_\Phi^0$, called the Orlicz norm, by the formula

$$\|x\|_\Phi^0 = \sup\{|\int_T x(t)y(t)\,d\mu| : I_{\Phi^*}(y) \le 1\}\,,$$

where Φ^* is the Musielak-Orlicz function complementary to Φ in the sense of Young (cf. [M] and in the case of Orlicz spaces also [KR], [L] and [RR]). The Amemiya formula for the Orlicz norm is the following:

$$\|x\|_\Phi^0 = \inf_{k>0}\frac{1}{k}(1 + I_\Phi(kx))$$

(cf. [KR] and [RR]).

We say that a Musielak-Orlicz function Φ satisfies the Δ_2-condition if there exist a constant $K \ge 2$, a set T_0 of measure zero and a Σ-measurable function $h : T \to \mathbb{R}_+$ such that $\int_T h(t)\,d\mu < +\infty$ and the inequality

$$\Phi(t,2u) \le K\Phi(t,u) + h(t)$$

holds for any $u \in \mathbb{R}$ and $t \in T \setminus T_0$ (cf.[K] and [M]).

Recall that a functional $x^* \in X^*$ is said to be a support functional at $x \in X$ if $\|x^*\| = 1$ and $x^*(x) = \|x\|$. The set of all support functionals at x is denoted by Grad(x). A point $x \in X$ is said to be smooth if Card(Grad$(x)) = 1$ (cf. [D] and [P]).

It is known (cf. [HY], [K] and in the case of Orlicz spaces also [A]) that for any finite-valued Musielak-Orlicz function Φ, we have

$$(L^\Phi(\mu))^* = L^{\Phi^*}(\mu) \oplus S\,, \tag{1}$$

where S is the space of all singular functionals over $L^\Phi(\mu)$, i.e. functionals which vanish on the subspace $E^\Phi(\mu)$ of $L^\Phi(\mu)$ defined by

$$E^\Phi(\mu) = \{x \in L^0(\mu) : I_\Phi(\lambda x) < +\infty \text{ for any } \lambda > 0\}\,.$$

Equality (1) means that every $x^* \in (L^\Phi(\mu))^*$ is uniquely represented in the form

$$x^* = T_v + \varphi\,, \tag{2}$$

where $\varphi \in S$ and T_v is the functional generated by an element $v \in L^{\Phi^*}(\mu)$ by the following formula

$$T_v(y) = \int_T v(t)y(t)\,d\mu \quad (\forall y \in L^\Phi(\mu))\,. \tag{3}$$

Every functional T_v of the form (3) is said to be a regular functional. It is well known that if $L^\Phi(\mu)$ is endowed with the Luxemburg norm, then for every $x^* \in (L^\Phi(\mu))^*$ we have

$$\|x^*\| = \|T_v\| + \|\varphi\|\,, \tag{4}$$

where T_v and φ are the regular and singular parts of x^*, respectively (cf. [K] and [A], [N]).

The set of all regular (singular) functionals from Grad(x) will be denoted by RGrad(x) (resp. SGrad(x)).

It is worth to recall at this place that smoothness of Musielak-Orlicz sequence spaces was considered in [HY] and [PY]. Moreover, smoothness of Orlicz function spaces equipped with the Orlicz norm was characterized in [C].

Results

We start with some auxiliary lemmas.

Lemma 1. *Let Φ be a Musielak-Orlicz function such that $\Phi(t,u)/u \to +\infty$ as $u \to +\infty$ for μ-a.e. $t \in T$. Then there exists a constant $l > 0$ such that*

$$\|x\|^0_\Phi = \frac{1}{l}(1 + I_\Phi(lx))\,. \tag{5}$$

Proof: It can be proceeded in an analogous way as the proof of Lemma 1 in [GH]. □

Lemma 2 (A.Kamińska [Ka])**.** *Let Φ be a Musielak-Orlicz function. Then there exists an increasing sequence (T_i) such that $\mu(T_i) < +\infty$, $\mu(T \setminus \bigcup_{i=1}^{\infty} T_i) = 0$ and $\sup_{t\in T_i} \Phi(t,u) < +\infty$ for every $u \in \mathbb{R}_+$, $i \in \mathbb{N}$.*

Lemma 3. *Assume Φ is a finite-valued Musielak-Orlicz function, $x \in S(L^\Phi(\mu))$ and $I_\Phi(\lambda x) < +\infty$ for some $\lambda > 1$. Then every $x^* \in \mathrm{Grad}(x)$ must be regular.*

Proof: In virtue of Lemma 2, we can replay the proof of Lemma 2 in [GH]. □

Lemma 4. *Assume Φ is a finite-valued Musielak-Orlicz function and $I_\Phi(\lambda x/\|x\|_\Phi) < +\infty$ for some $\lambda > 1$. Then*

1° $\mathrm{RGrad}(x) \neq \emptyset$,

2° $x^* \in \mathrm{RGrad}(x)$ *if, and only if it is of the form*

$$x^*(y) = T_w(y) = \int_T w(t)y(t)\,d\mu \quad (\forall y \in L^\Phi(\mu))\,, \tag{8}$$

where

$$w(t) = z(t)/\int_T z(t)(x(t)/\|x\|_\Phi)\,d\mu \tag{9}$$

and

(10) *z is a Σ-measurable function such that $z(t) \in \partial\Phi(t, x(t)/\|x\|_\Phi)$*

for μ-a.e. $t \in T$.

Proof: We can repeat here the proof of Lemma 3 from [GH]. □

Lemma 5. *Let Φ be a finite-valued Musielak-Orlicz function, $x \in S(L^\Phi(\mu))$ and $I_\Phi(\lambda x) = +\infty$ for every $\lambda > 1$. Then there are sets $A, B \in \Sigma$ of positive measure such that $\mu(A\cap B) = 0$, $A \cup B = \operatorname{supp} x$ and*

$$\|x\chi_A\|_\Phi = \|x\chi_B\|_\Phi = 1\,.$$

Proof: Although Lemma 5 is an analogue of Lemma 6 from [GH], we cannot repeat its proof which was suitable only for Orlicz spaces which are rearrangement invariant spaces. We will present here a completely new proof.

Let $(T_n)_{n=1}^{\infty}$ be the sequence of sets from Lemma 2. Then we have

$$I_\Phi(u\chi_{T_n}) < +\infty \quad (\forall u > 0, n \in \mathbb{N})\,.$$

Let $\lambda_1 > \lambda_2 > \dots$ and $\lambda_n \to 1$ as $n \to +\infty$. Since $I_\Phi(\lambda_1 x) = +\infty$, we can find $n_1 \in \mathbb{N}$ such that the set

$$A_1 = \{t \in T_{n_1} : |x(t)| \le n_1\}$$

satisfies the inequality $I_\Phi(\lambda_1 x\chi_{A_1}) \geq 2$.

We have $I_\Phi(\lambda_2 x\chi_{T\setminus A_1}) = +\infty$ because of $I_\Phi(\lambda_2 x\chi_{A_1}) < +\infty$, and we can find $n_2 \in \mathbb{N}$ such that defining

$$A_2 = \{t \in (T \setminus A_1) \cap T_{n_2} : |x(t)| \leq n_2\}$$

we get $A_1 \cap A_2 = \emptyset$ and $I_\Phi(\lambda_2 x\chi_{A_2}) \geq 2$.

We have again $I_\Phi(\lambda_3 x\chi_{T\setminus(A_1\cup A_2)}) = +\infty$. Repeating this procedure by induction we can find a sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint sets such that

$$I_\Phi(\lambda_n x\chi_{A_n}) \geq 2 \quad (n = 1, 2, \dots)\,.$$

We can now decompose every set A_n into the sum

$$A_n = A_n' \cup A_n''$$

of disjoint and measurable sets such that

$$I_\Phi(\lambda_n x\chi_{A_n'}) = I_\Phi(\lambda_n x\chi_{A_n''}) = \frac{1}{2} I_\Phi(\lambda_n x\chi_{A_n}) \geq 1\,.$$

Define now disjoint sets

$$A = \bigcup_{n=1}^\infty A_n'\,, \quad B = \bigcup_{n=1}^\infty A_n''$$

and the functions

$$\begin{aligned} y &= x\chi_A + \frac{1}{2} x\chi_{T\setminus(A\cup B)}, \\ z &= x\chi_B + \frac{1}{2} x\chi_{T\setminus(A\cup B)}\,. \end{aligned}$$

Obviously, we have $x = y + z$ and we need to prove that $\|y\|_\Phi = \|z\|_\Phi = 1$. It is evident that $|y(t)| \leq |x(t)|$ μ-a.e., therefore $I_\Phi(y) \leq I_\Phi(x) \leq 1$. Let us take an arbitrary $\lambda > 1$. We can find $m \in \mathbb{N}$, such that $\lambda \geq \lambda_m$. Hence

$$I_\Phi(\lambda y) \geq I_\Phi(\lambda_m y) \geq I_\Phi(\lambda_m x\chi_{A_m'}) \geq 1\,,$$

which yields together with $I_\Phi(y) \leq 1$ that $\|y\|_\Phi = 1$. In the same way we obtain that $\|z\|_\Phi = 1$. □

Now, we are ready to prove the main results of this paper.

Theorem 6. *Let Φ be a finite-valued Musielak-Orlicz function. A point $x \in S(L^\Phi(\mu))$ is smooth if and only if:*

(i) $I_\Phi(\lambda x) < +\infty$ *for some* $\lambda > 1$,
(ii) Φ *is smooth at* $x(t)$ *for* μ*-a.e.* $t \in T$.

Proof: It follows by Lemmas 3, 4 and 5 in the same way as Theorem 8 in [GH]. □

Theorem 7. *Let Φ be a finite-valued Musielak-Orlicz function. $L^\Phi(\mu)$ is smooth if and only if*

(i) *Φ is smooth,*
(ii) *Φ satisfies the Δ_2-condition.*

PROOF: *Sufficiency.* Note that, in virtue of the Δ_2-condition, $a(t,\Phi)=+\infty$ for μ-a.e. $t\in T$ and $E^\Phi(\mu)=L^\Phi(\mu)$. Therefore, $\mathrm{Grad}(x)=\mathrm{RGrad}(x)$ for every $x\in S(L^\Phi(\mu))$. Thus, condition (i) implies that $\mathrm{Card}(\mathrm{Grad}(x))=1$ for every $x\in S(L^\Phi(\mu))$ (cf. Lemma 4), which means that $L^\Phi(\mu)$ is smooth.

Necessity. Assume that Φ does not satisfy condition (ii). Then there is $x\in S(L^\Phi(\mu))$ such that $I_\Phi(\lambda x)=+\infty$ for any $\lambda>1$ (cf. [H]). Therefore, in view of Lemma 5, there exist $A,B\in\Sigma$ such that $\mu(A\cap B)=0$, $A\cup B=\operatorname{supp}x$ and $\|x\chi_A\|_\Phi=\|x\chi_B\|_\Phi=1$. Therefore, as it was shown on the occasion of the proof of Theorem 6, x is not smooth.

Assume now that Φ satisfies condition (ii) and does not satisfy condition (i), i.e. Φ is not smooth. Thus, there exists a set $K\in\Sigma$, $\mu(K)>0$, such that $\Phi(t,\cdot)$ have in $\mathbb{R}_+$ at least one point of nonsmoothness for any $t\in K$. Define a multifunction Γ by

$$\Gamma(t)=\{u\in\mathbb{R}_+:\Phi'_-(t,u)<\Phi'_+(t,u)\}\quad(\forall t\in K)\,.$$

The Carathéodory conditions for Φ imply the $\Sigma\times\mathcal{B}$-measurability of Φ and this implies the $\Sigma\times\mathcal{B}$-measurability of Φ'_- and Φ'_+, where $\mathcal{B}$ denotes the Σ-algebra of Borel sets. Now, we can apply Theorem 5.2 from [Hi], because $\mathrm{Grad}\,\Gamma=\{(t,u)\in T\times\mathbb{R}_+:u\in\Gamma(t)\}=\{(t,u)\in T\times\mathbb{R}_+:\Phi'_-(t,u)<\Phi'_+(t,u)\}\in\Sigma\times\mathcal{B}$. We get a measurable function (selector) $a:K\to\mathbb{R}_+$, such that $a(t)\in\Gamma(t)$ for μ-a.e. $t\in K$. Take $K_1\in\Sigma$, $K_1\subset K$ such that $I_\Phi(a\chi_{K_1})\le 1$. Choose a function $b:T\setminus K_1\to\mathbb{R}_+$ in such a manner that $I_\Phi(x)=1$, whenever $x=a+b$. Now, we can take two measurable functions $c,d:K_1\to\mathbb{R}_+$, $c(t),d(t)\in\partial\Phi(t,a(t))$, $c(t)\neq d(t)$ for μ-a.e. $t\in K_1$ (we can put for example $c(t)=\Phi'_-(t,a(t))$, $d(t)=\Phi'_+(t,a(t))$, because the $\Sigma\times\mathcal{B}$-measurability of $\Phi'_-(t,u)$ and $\Phi'_+(t,u)$ implies the Σ-measurability of these functions). Take a Σ-measurable function such that $e:T\setminus K_1\to\mathbb{R}_+$, $e(t)\in\partial\Phi(t,b(t))$ for μ-a.e. $t\in T\setminus K_1$. Define two functionals:

$$x^*(y)=\frac{\int_T(c(t)\chi_{K_1}(t)+e(t)\chi_{T\setminus K_1}(t))y(t)\,d\mu}{\int_T(c(t)\chi_{K_1}(t)+e(t)\chi_{T\setminus K_1}(t))x(t)\,d\mu}\quad(\forall y\in L^\Phi(\mu))\,,$$

$$x_1^*(y)=\frac{\int_T(d(t)\chi_{K_1}(t)+e(t)\chi_{T\setminus K_1}(t))y(t)\,d\mu}{\int_T(d(t)\chi_{K_1}(t)+e(t)\chi_{T\setminus K_1}(t))x(t)\,d\mu}\quad(\forall y\in L^\Phi(\mu))\,,$$

belonging, in view of Lemma 4, to $\mathrm{RGrad}(x)$. Since $x^*\neq x_1^*$, x is not smooth. Therefore, $L^\Phi(\mu)$ is not smooth, too. The theorem is proved. □

Corollary 8. *Let Φ be a finite-valued Musielak-Orlicz function. The space $E^\Phi(\mu)$ is smooth if and only if Φ is smooth.*

PROOF: The sufficiency follows by the sufficiency part of the proof of Theorem 7 and the fact that the dual of $E^\Phi(\mu)$ consists of only regular functionals.

To prove necessity we need to find a point $x \in S(E^\Phi(\mu))$ which is not smooth. Define

$$G_{i,n} = \{t \in T_i \cap K : \Phi(t, a(t)) \le n\} \quad i, n = 1, 2, \dots ,$$

where T_i are the sets from Lemma 2.

There exist $i_0, n_0 \in \mathbb{N}$ and a measurable subset G of G_{i_0,n_0} such that

$$0 < I_\Phi(a\chi_G) = \int_G \Phi(t, a(t))\, d\mu < 1 .$$

Denote $I_\Phi(a\chi_G) = \varkappa$ and define

$$H_i = T_i \cap (T \setminus K) \quad i = 1, 2, \dots .$$

There exists $i_1 \in \mathbb{N}$ such that $\mu(H_{i_1}) > 0$. By Lemma 2 it follows that $b\chi_{H_{i_1}} \in E^\Phi(\mu)$ for every $b > 0$. The function $f(b) = I_\Phi(b\chi_{H_{i_1}})$ is convex and finite-valued, so it is continuous on $\mathbb{R}_+$. Moreover, $f(b) \to +\infty$ as $b \to +\infty$, whence it follows that Image$(f) = \mathbb{R}_+$. Thus, we can find $b_1 > 0$ satisfying $I_\Phi(b_1\chi_{H_{i_1}}) = 1 - \varkappa$. Defining x by the formula

$$x(t) = a(t)\chi_G(t) + b_1\chi_{H_{i_1}}(t)$$

we have $I_\Phi(x) = 1$. The fact that x is not smooth can be proved in the same way as in Theorem 7. $\square$

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(*Received* September 4, 1992)