On local and global injectivity of noncompact vector fields in non necessarily locally convex vector spaces

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Abstract. We give in this paper conditions for a mapping to be globally injective in a topological vector space.

Keywords: fixed point index, locally injective mappings, (φ,γ)-condensing mappings

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Introduction

Using the relative fixed point index of compact reducible mappings in [1], we give in this paper conditions for a mapping to be globally injective whenever the mapping is locally injective.

Our results do not follow from the well-known theorem of Banach-Mazur [3], because our assumptions on the range of the mapping are more simple.

Furthermore, we prove a uniqueness theorem for the fixed point in the Schauder fixed point theorem for (φ,γ)-condensing mappings in topological vector spaces. This result generalizes a theorem of Talmann [16] and a theorem of Alex/Hahn [2] for a special case. In [2] we proved the following

Theorem A. *Let E be an admissible topological vector space, $a \in E$, W an open and connected neighbourhood of a and $F : \overline{W} \to E$ a compact mapping. Suppose*

(a) $Fx \neq \beta x + (1-\beta)\cdot a \quad (x \in W,\ \beta \geq 1)$,
(b) *$f = I - F$ is locally injective on W.*

Then F has a unique fixed point.

Our uniqueness theorem implies the following

Proposition. *Let E be a complete, locally convex and metrizable vector space, $K \subseteq E$ nonempty, closed and convex. $M \subseteq E$ nonempty, open and $M_K := M \cap K$ connected, $a \in M_K$. Let $F : \mathrm{cl}_K M_K \to K$ be a condensing mapping with respect to a measure of noncompactness γ (e.g. γ can be the measure of noncompactness of Kuratowski). Suppose*

(a)′ $Fx \neq \beta x + (1-\beta)\cdot a \quad (x \in \partial_K M_K,\ \beta \geq 1)$,
(b)′ *$f = I - F$ is locally injective on M_K,*
(c)′ $F(\mathrm{cl}_K M_K) + f(\mathrm{cl}_K M_K) \subseteq K$.

Then F has a unique fixed point.

In the following example, we give a mapping for which the assumptions of the proposition hold, but not the assumptions of Theorem A.

Example. Let $E = R^2$, $M = \{(x, y) : x^2 + y^2 < 1\}$, $F : \overline{M} \to E$ with $F(x, y) = (xy, \frac{1}{2}xy)$ $((x, y) \in \overline{M})$.

Obviously F has the unique fixed point $(0, 0)$, however we cannot apply Theorem A:

With $f = I - F$ we obtain $f(x, y) = (x - xy, y - \frac{1}{2}xy)$ $((x, y) \in \overline{M})$.

Using the derivative of f, it is easy to show that f is locally injective on $M \setminus \{(x, y) \in M : y = -\frac{x}{2} + 1,\ 0 < x < \frac{4}{5}\}$.
However, we have $f(\frac{1}{2}, \frac{3}{4}+\varepsilon) = f(\frac{1}{2} - 2\varepsilon, \frac{3}{4})$ for each $\varepsilon \in R$ and hence f is not locally injective in $(\frac{1}{2}, \frac{3}{4}) \in M$. The assumption (b) of Theorem A does not hold for F.

Now we set $K = \{(x, y) : 0 \le 2y \le x\}$ and $M_K := M \cap K$. With $F \mid \mathrm{cl}_K M_K$ and $f \mid \mathrm{cl}_K M_K$ we denote the restriction on $\mathrm{cl}_K M_K$ of F and f, respectively. Clearly, the assumptions of the proposition for M, K and M_K hold.

Since K is a cone, we have $K + K \subseteq K$.

Furthermore $M_K \cap \{(x, y) : y = -\frac{x}{2}+1,\ 0 < x < \frac{4}{5}\} = \emptyset$ and we have f is locally injective on M_K. Obviously we have $F(\mathrm{cl}_K M_K) \subseteq M_K$ and $f(\mathrm{cl}_K M_K) \subseteq K$. Hence the assumptions of the proposition hold for $F \mid \mathrm{cl}_K M_K$ and the uniqueness of the fixed point follows from the proposition.

1. Notations and definitions

We use all notations and definitions of the paper of Alex, Hahn, Kaniok [1] in this journal in the same kind.

Furthermore we need the following notations. Let X be a real, separated topological space; X is called connected if and only if $X = X_1 \cup X_2$, $X_1 \neq \emptyset$, $X_2 \neq \emptyset$ and X_1, X_2 open in X implies $X_1 \cap X_2 \neq \emptyset$.

X is called pathwise connected, if for each $x_1, x_2 \in X$ there exists a continuous mapping $s = [0, 1] \to X$ with $s(0) = x_1$, $s(1) = x_2$.

X is called locally (pathwise) connected, if for each $x \in X$ there exists a (pathwise) connected neighbourhood U of x with $U \subseteq X$. It is well known that if X is connected and locally pathwise connected, then X is pathwise connected (see [14, p. 162]). This implies

Lemma 1. *Let E be a topological vector space, $K \subseteq E$ nonempty, convex. If $M \subseteq K$ is connected and open in K, then M is pathwise connected.*

Proof: With the relative topology M is a topological space. We must show, that M is locally pathwise connected. Let $a \in M$. Then there exists a neighbourhood V of a, which is starshaped relative a, with $V \cap K \subseteq M$, because M is open in K. Since K is convex, $U := V \cap K$ is a starshaped neighbourhood of a in K. Hence U is pathwise connected and M locally pathwise connected. $\square$

It is also well known that the continuous image of a (pathwise) connected set is also (pathwise) connected.

Let X, Y be topological spaces, $M \subseteq X$ nonempty, open. A continuous mapping $f : M \to Y$ is called

(1) locally injective, if for each $x \in M$ there exists a neighbourhood $U \subseteq M$ of x such that f is injective on U,

(2) locally topological, if for each $x \in M$ there exist neighbourhoods $U \subseteq M$ of x and $V \subseteq Y$ of $f(x)$ such that f is a homeomorphism of U onto V,

(3) open, if $N \subseteq M$ open in M implies $f(N)$ is open in $f(M)$,

(4) proper, if $K \subseteq Y$ compact implies $f^{-1}(K)$ is compact.

Remark. If f is a locally injective and open mapping, then f is locally topological.

The local index of (φ, γ)-condensing vector fields.

The notions φ-measure of noncompactness γ on K and (φ, γ)-condensing mapping are defined such as in [1]. The partially ordered set A and the system $\mathcal{M}$ of subsets of $\overline{\text{co}}\, K$ we use in the same kind. Furthermore we need the following properties of γ and φ.

(N4) If $0 \in A$, $0 \leq a$ $(a \in A)$, then $\gamma(M) = 0 \Leftrightarrow \overline{M}$ is compact $(M \in \mathcal{M})$.

(N5) If $M, N \in \mathcal{M}$ implies $M + N \in \mathcal{M}$, then $\gamma(M + N) \leq \gamma(M)$ whenever N is compact.

(N6) If $a_1, a_2 \in A$, $a_1 \leq a_2$, then $a_1 \leq \varphi(a_1) \leq \varphi(a_2)$.

Now we give an example of a nontrivial φ-measure of noncompactness γ with the properties (N1)–(N6).

Let E be a complete metric space, $M \subseteq E$ a bounded subset of E. The Kuratowski measure of noncompactness $\mathcal{L}(M)$ of the set M is defined by

$$\begin{aligned}\mathcal{L}(M) := \inf\{\varepsilon > 0 : &\text{ there exists a finite cover } \{B_j\}_{j \in J} \text{ of } M \text{ such that} \\ &\operatorname{diam}(B_j) < \varepsilon \ (j \in J)\}.\end{aligned}$$

It is well known that $\mathcal{L}$ has the properties (N1), (N3), (N4) and (N5). If E is a complete metrizable and locally convex vector space, then $\mathcal{L}$ has also the property (N2) with $\varphi(t) = t$ $(t \in A = [0, \infty))$. If E is non locally convex, $\mathcal{L}$ does not have this property with $\varphi(t) = t$.

Hadzic proved that $\mathcal{L}$ is a φ-measure of noncompactness on special subsets of a paranormed space [6].

Proposition. *Let $(E, \|\cdot\|^*)$ be a complete paranormed space, $\varphi : [0, \infty) \to [0, \infty)$ a continuous monotone nondecreasing mapping with $f(t) \geq t$ $(t \in [0, \infty))$, $K \subseteq E$ a nonempty, bounded and convex subset of E which is of $Z\varphi$-type, e.g. for each neighbourhood of zero $V_r = \{x \in E : \|x\|^* < r\}$ is $\text{co}\,(V_r \cap (K - K)) \subseteq V_{\varphi(r)}$.*

Then $\mathcal{L}$ is a $\widetilde{\varphi} = \varphi \circ \varphi$-measure of noncompactness with the properties (N1)–(N6).

Remark. 1. Obviously the properties (N1), (N3), (N4) and (N5) hold for $\mathcal{L}$ and the assumptions for φ imply (N6) also for $\widetilde{\varphi}$. Property (N2) is proved by Hadzic in [6, Lemma 2].

2. If K is a convex set of $Z\varphi$-type and $\inf_{t>0}\varphi(t)=0$, then K is a locally convex set. (This follows from the remarks following Definition 2 in [6] and Proposition 3 in [5, p. 30].)

3. We can find a subset of $Z\varphi$-type in the paranormed space $S[0,1]$ of finite real measurable functions on $[0,1]$ by Hadzic [6].

In this paper the φ-measure of noncompactness γ has always the properties (N1)–(N6). Let E be a topological vector space, $K\subseteq E$ nonempty, convex, closed and locally convex, $M\subseteq E$ nonempty, open and $M_K:=M\cap K$.

Let $F:\operatorname{cl}_K M_K\to K$ be a (φ,γ)-condensing mapping with respect to a φ-measure of noncompactness γ on K:

The mapping $f:=I-F$ is called a (φ,γ)-condensing vector field.

If $x\neq Fx$ $(x\in\partial_K M_K)$, then the relative fixed point index of F, $i(F,M_K)$, is defined [1].

A point $x_0\in M_K$ is called an isolated point of zero of the (φ,γ)-condensing vector field $f:=I-F$, if there exists a neighbourhood U of x_0 with $U\subseteq M$ such that $f(x)=\underline{o}$ $(x\in\operatorname{cl}_K U_K,\ U_K:=U\cap K)$ implies $x=x_0$. (x_0 is an isolated fixed point of F.) In this case, the relative fixed point index $i(F,U_K)$ is independent of the choice of U.

We define the local index of the isolated point of zero x_0 of f, $i(x_0,f,\underline{o})$, with

$$i(x_0,f,\underline{o}):=i(F,U_K).$$

Now let $F(\operatorname{cl}_K M_K)+f(\operatorname{cl}_K M_K)\subseteq K$, $y\in f(M_K)$. A point $x_0\in M_K$ is called an isolated y-point of f, if there exists a neighbourhood U of x_0 such that $f(x)=y$ $(x\in\operatorname{cl}_K U_K)$ implies $x=x_0$. Then x_0 is an isolated point of zero of f_y with $f_y(x)=f(x)-y$ $(x\in\operatorname{cl}_K M_K)$. Since $y+Fx\in K$ $(x\in\operatorname{cl}_K M_K)$, the local index of the isolated y-point of f is well defined with

$$i(x_0,f,y):=i(x_0,f_y,\underline{o}).$$

If the set $Y=\{x\in\operatorname{cl}_K M_K : f(x)=y\}=\{x_1,\dots,x_n\}\subseteq M_K$ is finite, then by [1, Theorem 3 (I6)] we obtain

$$\text{(I7)}\qquad i(F_y,M_K)=\sum_{j=1}^{n} i(x_j,f,y)$$

with $F_y(x)=F(x)+y$ $(x\in\operatorname{cl}_K M_K)$.

2. Local and global injectivity of (φ,γ)-condensing vector fields

In this chapter we give conditions for the global injectivity of a (φ,γ)-condensing vector field, whenever the vector field is locally injective. Then, with a simple additional assumption, the vector field is a homeomorphism.

A well-known theorem of Banach-Mazur [3], [13] implies the following

Theorem 1. *Let E be a topological vector space, $f : E \to E$ a locally topological and proper mapping of E onto E. Then f is a homeomorphism of E onto E.*

The assumption $f(E) = E$ in this theorem is essential. Plastock proved a theorem which guarantees that f is a homeomorphism of D onto $f(D)$ where D is a connected open subset of a Banach space. However, Plastock needed a complicated assumption on the range $f(D)$ (see [12]). Plastock investigated the question of the global injectivity of f when we have not exact informations about $f(D)$. Our results are an answer to this question for a special class of mappings.

Theorem 2. *Let E be a topological vector space, $K \subseteq E$ nonempty, closed, $M \subseteq K$ nonempty, closed.*

Let $F : M \to K$ be a (φ, γ)-condensing mapping with respect to a φ-measure of noncompactness γ on K, $f := I - F$. Then f is a proper mapping.

Proof: Let $A \subseteq E$ be compact. $f^{-1}(A) := N$ is closed, because f is continuous. $(I - F)(N) = A$ implies $N \subseteq F(N) + A$. Hence, by the properties of φ and γ, $\gamma(N) \leq \gamma(F(N) + A) \leq \gamma(F(N)) \leq \varphi(\gamma(F(N)))$.

Since F is (φ, γ)-condensing, $\overline{F(N)}$ is compact and hence $N = \overline{N}$ is compact. □

Now we prove the following

Lemma 2. *Let E be a topological vector space, $K \subseteq E$ nonempty, closed and convex, $M \subseteq E$ nonempty, open and $M_K := M \cap K$. Let $f : M_K \to E$ be a locally injective mapping. Then for each $x \in M_K$ there exists an open neighbourhood $U \subseteq E$ of x, $U_K := U \cap K$, such that we have*

(1) *$\overline{U} \subseteq M$ and $f \mid \mathrm{cl}_K U_K$ is injective,*
(2) *$f(U_K)$ is pathwise connected,*
(3) *$f(U_K) \cap f(\partial_K U_K) = \emptyset$.*

Proof: Let $x \in M_K$. Then there exists an open neighbourhood $B \subseteq M$ of x such that $f \mid B \cap K$ is injective.

Let U be an open starshaped neigbourhood of x with $\overline{U} \subseteq B$.

Then $U_K := U \cap K$ is starshaped with respect to x and hence $f(U_K)$ is pathwise connected. Furthermore $f \mid \mathrm{cl}_K U_K$ is injective, because $\mathrm{cl}_K U_K \subseteq B \cap K$. Since U_K is open in K, we obtain $U_K \cap \partial_K U_K = \emptyset$. $(*)$

Suppose that $f(U_K) \cap f(\partial_K U_K) \neq \emptyset$. Then there exists $z \in f(U_K) \cap f(\partial_K U_K)$ and $x_1 \in U_K$, $x_2 \in \partial_K U_K$ with $z = f(x_1) = f(x_2)$. This implies $x_1 = x_2$, because $f \mid \mathrm{cl}_K U_K$ is injective. This is a contradiction to $(*)$.

Hence U has the properties (1)–(3). □

We denote by $S(x)$ the system of all neighbourhoods of x for which (1)–(3) from Lemma 2 hold.

Lemma 3. *Let E be a topological vector space, $K \subseteq E$ nonempty, convex, closed and locally convex, $M \subseteq E$ nonempty, open and $M_K := M \cap K$ be connected. Let*

$F : \mathrm{cl}_K M_K \to K$ be a (φ, γ)-condensing mapping with respect to a φ-measure of noncompactness γ on K, $f := I - F$. Suppose that

(1) f is locally injective on M_K,
(2) $F(\mathrm{cl}_K M_K) + f(\mathrm{cl}_K M_K) \subseteq K$.

Then for each $x_1, x_2 \in M_K$ is $i(x_1, f, f(x_1)) = i(x_2, f, f(x_2))$.

Proof: By the assumptions, $i(x, f, f(x))$ is well defined for each $x \in M_K$.

(1) Let $x_0 \in M_K$, $U \in S(x_0)$, $y \in U_K := U \cap K$.

We show that $i(x_0, f, f(x_0)) = i(x_0, f, f(y)) := i(F(\cdot) + f(y), U_K)$.

We define a mapping $H : [0,1] \times \mathrm{cl}_K U_K \to K$ with $H(t,x) = Fx + s(t)$ ($t \in [0,1]$, $x \in \mathrm{cl}_K U_K$), where $s : [0,1] \to f(U_K)$ is pathwise connected.

There is $H([0,1] \times \mathrm{cl}_K U_K) \subseteq K$, by the assumption (2), $H(0,\cdot) = F(\cdot) + f(x_0)$ and $H(1,\cdot) = F(\cdot) + f(y)$.

Now we show that H is a (φ, γ)-condensing mapping. We have for $N \subseteq \mathrm{cl}_K U_K$ $H([0,1] \times N) \subseteq F(N) + s([0,1])$ and, hence, $\gamma(H([0,1] \times N)) \leq \gamma(F(N))$, because $s([0,1])$ is compact. If $\gamma(N) \leq \varphi(\gamma(H([0,1] \times N)))$, then we obtain by the properties of φ and γ

$$\gamma(N) \leq \varphi(\gamma(F(N))).$$

Since F is (φ, γ)-condensing, $\overline{F(N)}$ is compact and this implies $\overline{H([0,1] \times N)}$ is compact. Furthermore $f(\partial_K U_K) \cap f(U_K) = \emptyset$ implies $z \neq H(t,z) \Leftrightarrow f(z) \neq s(t)$ for each $z \in \partial_K U_K$, $t \in [0,1]$, because $s(t) \in f(U_K)$ ($t \in [0,1]$). Hence the assumptions of (I3) (see [1, Theorem 3]) hold for H and we have

$$\begin{aligned}(1) \qquad i(x_0, f, f(x_0)) &= i(F(\cdot) + f(x_0), U_K) = \\ &= i(F(\cdot) + f(y), U_K) = i(x_0, f, f(y)).\end{aligned}$$

(2) Now, let $x_0 \in M_K$, $U \in S(x_0)$, $y \in U_K$, $W \in S(y)$ and $W_K := W \cap K$. We define $B_1 := \mathrm{cl}_K(U_K \setminus (U_K \cap W_K))$ and $B_2 := \mathrm{cl}_K(W_K \setminus (U_K \cap W_K))$. Then we have $U_K \setminus B_1 = W_K \setminus B_2$. The injectivity of f on U_K and W_K implies $x \neq \widetilde{F}(x)$ ($x \in B_1 \cup B_2$) with $\widetilde{F}(x) = Fx + f(y)$ ($x \in \mathrm{cl}_K M_K$). From (I6) ([1, Theorem 3]) we obtain

$$\begin{aligned}(2) \qquad i(x_0, f, f(y)) &= i(\widetilde{F}, U_K) = i(\widetilde{F}, (U_K \setminus B_1)) = \\ &= i(\widetilde{F}, (W_K \setminus B_2)) = i(\widetilde{F}, W_K) = i(y, f, f(y)).\end{aligned}$$

(1) and (2) imply

$$(3) \qquad i(x_0, f, f(x_0)) = i(y, f, f(y))$$

for $x_0 \in M_K$, $U \in S(x_0)$, $y \in U_K$.

(3) Suppose there are $x, y \in M_K$ with

$$(4) \qquad i(x, f, f(x)) \neq i(y, f, f(y)).$$

We define $A_1 := \{z \in M_K : i(z, f, f(z)) = i(x, f, f(x))\}$ and $A_2 := M_K \setminus A_1$. Since $x \in A_1$, $y \in A_2$, we have $A_1 \neq \emptyset$, $A_2 \neq \emptyset$. If $z_i \in A_i$, $U_i \in S(z_i)$ and $U_{iK} := U_i \cap K$ $(i = 1, 2)$, then (3) implies $U_{1K} \cap U_{2K} = \emptyset$.

Now we choose for each $x \in M_K$ a $U \in S(x)$, $U_K := U \cap K$, and define $M_1 := \bigcup_{x \in A_1} U_K$, $M_2 := \bigcup_{x \in A_2} U_K$.

We obtain $M_1 \neq \emptyset$, $M_2 \neq \emptyset$, $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M_K$. M_1, M_2 are open in K and also in M_K. This contradicts our assumption that M_K is connected. This implies that $i(x, f, f(x)) = i(y, f, f(y))$ for each $x, y \in M_K$. □

Now we prove the following

Theorem 3. *Let E be a topological vector space, $K \subseteq E$ nonempty, closed, convex and locally convex, $M \subseteq E$ nonempty, open and $M_K = M \cap K$ be connected.*

Let $F : \mathrm{cl}_K M_K \to K$ be a (φ, γ)-condensing mapping with respect to a φ-measure of noncompactness γ on K, $f := I - F$. Suppose that

(1) *f is locally injective on M_K,*
(2) *$F(\mathrm{cl}_K M_K) + f(\mathrm{cl}_K M_K) \subseteq K$.*

Then the equation $f(x) = y$ $(x \in M_K)$ has for all $y \in f(M_K)$ with $y \notin f(\partial_K M_K)$ and $i(F(\cdot) + y, M_K) = \pm 1$ exactly one solution.

Proof: Let $y \in f(M_K) \setminus f(\partial_K M_K)$ and $i(F(\cdot) + y, M_K) = \pm 1$. By Theorem 2, f is a proper mapping and this implies that $N := f^{-1}(y)$ is compact.

Applying this fact and the condition that f is locally injective on M_K and $N \cap \partial_K M_K = \emptyset$, we can easily show that N is finite.

Let $N := \{x_1, \dots, x_n\}$ $(n \in N^*)$. Using (I7), we obtain

$$i(F(\cdot) + y, M_K) = \sum_{j=1}^{n} (x_j, f, y).$$

Lemma 3 implies $i(x_j, f, y) = c$ $(j = 1, \dots, n;\ c \in Z)$.

Then $\pm 1 = i(F(\cdot) + y, M_K) = n \cdot c$ and we obtain $n = 1$. Hence the equation $f(x) = y$ has exactly one solution $x \in M_K$. □

Using Theorem 3, we give conditions for a mapping to be a homeomorphism whenever the mapping is locally injective.

Theorem 4. *Let E, K, M, F, f be such as in Theorem 3. Suppose that*

(1) *$f(M_K) \cap f(\partial_K M_K) = \emptyset$,*
(2) *$i(F(\cdot) + y, M_K) = \pm 1$ $(y \in f(M_K))$.*

Then the restriction of f on M_K, $\widetilde{f} := f \mid M_K$, is an injective mapping. If f is additionally an open mapping, then f is a homeomorphism of M_K onto $f(M_K)$.

Proof: (1), (2) and Theorem 3 imply that the equation $f(x) = y$ has exactly one solution $x \in M_K$ for each $y \in f(M_K)$. Hence f is injective. If f is an open mapping, then the inverse mapping f^{-1} of f is continuous. Hence, f is a homeomorphism. □

Remark. The assumption (1) $f(M_K) \cap f(\partial_K M_K) = \emptyset$ is essential. It is easy to show that if f is an open mapping, then $\partial_K f(M_K) \subseteq f(\partial_K M_K)$. A simple example for an open locally injective mapping with $f(\partial_K M_K) \nsubseteq \partial_K f(M_K)$, and hence $f(\partial_K M_K) \cap f(M_K) \neq \emptyset$, can be found in [2].

Corollary 1. *Let E, K, M, F, f be such as in Theorem 3. Suppose that*

(1) $f(M_K) \cap f(\partial_K M_K) = \emptyset$,
(2) *there exists a $y \in K$ with $y \notin f(\partial_K M_K)$ and $i(F(\cdot) + y, M_K) = \pm 1$.*

Then $\widetilde{f} := f \mid M_K$ is injective. If f is additionally an open mapping, then f is a homeomorphism on M_K onto $f(M_K)$.

Proof: We must only show that the assumption (2) of Theorem 3 holds.

Let $y \in K$ with $y \notin f(\partial_K M_K)$ and $i(F(\cdot) + y, M_K) = \pm 1$. Then $y \in f(M_K)$. Since K is convex and M_K is connected and open in K, M_K is pathwise connected (see Lemma 1). Hence $f(M_K)$ is pathwise connected. Let $z \in f(M_K)$. Then there exists a continuous mapping $s : [0,1] \to f(M_K)$ with $s(0) = y$, $s_1 = z$. We define

$$H(t,x) := Fx + s(t) \quad (t \in [0,1],\ x \in \operatorname{cl}_K M_K).$$

H is a (φ,γ)-condensing mapping with $H([0,1] \times \operatorname{cl}_K M_K) \subseteq K$, $H(0,\cdot) = F(\cdot) + y$, $H(1,\cdot) = F(\cdot) + z$.

Furthermore $s([0,1]) \subseteq f(M_K)$ and (1) implies $x \neq H(t,x)$ $(t \in [0,1]$, $x \in \partial_K M_K)$. Using (I3) ([1, Theorem 3]) and (2), we obtain $\pm 1 = i(F(\cdot) + y, M_K) = i(F(\cdot) + z, M_K)$ for each $z \in f(M_K)$. This is the assumption (2) of Theorem 3. □

Corollary 2. *Let E be a topological vector space, $K \subseteq E$ nonempty, closed, convex and locally convex. Let $F : K \to K$ be a (φ,γ)-condensing mapping with respect to a φ-measure of noncompactness γ. Suppose that*

(1) *$f := I - F$ is a locally injective and open mapping on K,*
(2) *$F(K) + f(K) \subseteq K$.*

Then f is a homeomorphism of K onto $f(K)$.

Proof: Setting $M := E$, we obtain $\partial_K M_K = \partial_K K = \emptyset$ and, by (I4) ([1, Theorem 3]), $i(F, M_K) = 1$.

It is easy to see that the assumptions of Corollary 1 hold for E, K, M, F, f with $y = \underline{o}$. □

Remark. (1) The proof of Corollary 2 implies $\underline{o} \in K$.

(2) If $f(K) = K$ in Corollary 2, then Corollary 2 follows from the theorem of Banach-Mazur (see [13, Theorem 4.39, p. 147]), because f is a proper mapping (Theorem 2) and locally topological. Since K is convex, it is easy to show that the assumptions for the domain and the range of f in the theorem of Banach-Mazur hold for K.

(3) If $f(K) \neq K$ and $f(\mathrm{cl}_K M_K) \neq K$ in Theorem 4, respectively, then our results do not follow from the theorem of Banach-Mazur. The identity on the set $\{x \in E : 1 \leq \|x\| \leq 2\}$, where E is a normed space, is a simple example.

(4) Let E be a locally convex vector space. Let $K = E$, then K is convex, closed and locally convex. The assumption $f(K) + F(K) \subseteq K$ holds always in this case.

(5) Let E be a complete locally convex and metrizable vector space, $F : \overline{M} \to E$ a k-set contraction with $0 \leq k < 1$ ($M \subseteq E$ nonempty, open). If $f := I - F$ is locally injective, then f is an open mapping (see [7]).

3. Fixed point theorems

Now we prove a uniqueness theorem for a fixed point of a (φ, γ)-condensing mapping F, whenever a Leray-Schauder-boundary condition holds for the mapping.

Theorem 5. *Let E be a topological vector space, $K \subseteq E$ nonempty, closed, convex and locally convex, $M \subseteq E$ nonempty, open and $M_K := M \cap K$ be connected, $a \in M_K$.*

Let $F : \mathrm{cl}_K M_K \to K$ be a (φ, γ)-condensing mapping with respect to a φ-measure of noncompactness γ on K. Suppose

(a) $Fx \neq x + (1 - \beta)a \quad (x \in \partial_K M_K,\ \beta \geq 1)$,
(b) *$f := I - F$ is locally injective on M_K,*
(c) $F(\mathrm{cl}_K M_K) + f(\mathrm{cl}_K M_K) \subseteq K$.

Then F has a unique fixed point.

Proof: We set $H(t, x) := t \cdot Fx + (1-t) \cdot a$ ($t \in [0, 1]$, $x \in \mathrm{cl}_K M_K$). H is a (φ, γ)-condensing mapping with $H([0, 1] \times \mathrm{cl}_K M_K) \subseteq K$, $H(0, \cdot) = a$, $H(1, \cdot) = F$. Furthermore, from (a), we obtain $x \neq H(t, x)$ ($t \in [0, 1]$, $x \in \partial_K M_K$). Applying (I3) and (15) from [1, Theorem 3], we have $i(F, M_K) = 1$, because $a \in M_K$. Therefore $\underline{o} \in f(M_K)$ and we can apply Theorem 3 for $y = \underline{o}$. Hence the equation $f(x) = \underline{o}$ has exactly one solution $x \in M_K$ and F has a unique fixed point. □

Now the proposition from the introduction follows from Theorem 5.

Proposition. *Let E be a complete, locally convex and metrizable vector space, $K \subseteq E$ nonempty, closed and convex, $M \subseteq E$ nonempty, open and $M_K := M \cap K$ be connected, $a \in M_K$.*

Let $F : \mathrm{cl}_K M_K \to K$ be a condensing mapping with respect to a measure of noncompactness γ. (This means $[N \subseteq M_K \wedge \gamma(F(N)) \geq \gamma(N)] \Rightarrow \overline{F(N)}$ is compact.) Suppose

(a) $Fx \neq \beta x + (1 - \beta) \cdot a$ ($x \in \partial_K M_K$, $\beta \geq 1$).
(b) *$f := I - F$ is locally injective on M_K.*
(c) $F(\mathrm{cl}_K M_K) + f(\mathrm{cl}_K M_K) \subseteq K$.

Then F has a unique fixed point.

PROOF: Since E is locally convex, K is also locally convex. Furthermore F is a (φ, γ)-condensing mapping with $\varphi(t) = t$ $(t \in A)$. Hence all assumptions from Theorem 5 hold. $\square$

Remark. Setting in the proposition $K = E$, then we obtain a generalization of a theorem of Talmann [16] for continuously Fréchet-differentiable k-set contractions in Banach spaces. The assumption "For each $x \in M$ 1 is not an eigen-value of $F'(x)$" by Talmann implies our assumption (b) of Theorem 5.

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