The fixed point index for noncompact mappings in non locally convex topological vector spaces

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Abstract. We introduce the relative fixed point index for a class of noncompact operators on special subsets of non locally convex spaces.

Keywords: fixed point index, admissible sets, compact reducible and (φ, γ)-condensing operators, φ-measure of noncompactness
Classification: 47H10

Introduction

Nagumo introduced the Brouwer-Leray-Schauder-degree for compact vector fields in locally convex spaces.

Kaballo [8], Hahn-Riedrich [6] and Kayser [10] generalized this notion for not necessarily locally convex topological vector spaces. In the last twenty years the degree, the fixed point index and the equivalent notion of the rotation were defined for various classes of noncompact vector fields, for example for condensing, k-set-contractions $(0 < k < 1)$, ultimatively compact and related vector fields. However, the considered spaces must be normed or locally convex in all known spaces. In this paper we introduce the relative fixed point index of compact reducible operators on special subsets of general topological vector spaces.

1. Notions and definitions

In this paper the topological spaces are separated and the topological vector spaces E are real and separated. Let $K \subseteq E$ and $M \subseteq K$. We denote the boundary of M with respect to K and the closure of M with respect to K by $\partial_K M$ and $\operatorname{cl}_K M$ respectively. Further, we denote the closed convex hull of K and the zero of E by $\overline{\operatorname{co}}\, K$ and $\underline{o}$ respectively. Let X be a topological space. A mapping $F : X \to E$ is called compact, if F is continuous and $F(X)$ is relatively compact.

We recall that $K \subseteq E$ is called an admissible set provided that for every compact subset $N \subseteq K$ and every neighbourhood V of $\underline{o}$ in E there are a finite dimensional subspace E_V of E and a continuous mapping $h_V : N \to K$ with $x - h_V(x) \in V$ $(x \in N)$. Each convex subset of a locally convex space is admissible. An open question is the following: Does there exist a convex subset of a (non locally convex) topological vector space, which is not admissible? Some examples of admissible sets in non locally convex spaces can be found in [2]. Krauthausen [12] introduced the notion of the locally convex set (see [2] too), which was defined by Jerofsky [7] as follows.

Definition 1. Let E be a topological vector space and $K \subseteq E$. K is said to be a locally convex subset of E iff for any $x \in K$ there exists a base of neighbourhoods U_x of x with respect to K such that $U_x = (x + W) \cap K$ and W is a convex subset of E.

Clearly, each subset of a locally convex set is locally convex too and each subset of a locally convex space is locally convex. Jerofsky proved in [7] the following result (in a more general form):

Remark 1. Let E be a topological vector space and $K \subseteq E$ closed and convex. If K is locally convex, then K is admissible.

This result generalizes a theorem of Krauthausen [12] for metrizable spaces (see [2] too). Special classes of locally convex subsets can be found in [2], for example.

We need the following well-known result (see [7]) in Section 3.

Remark 2. Let E be a topological vector space, $M \subseteq E$, N a closed subset of M and $F : M \to E$ a compact mapping. Then the set $\{z \in E : z = x - Fx,\ x \in N\}$ is closed.

2. Compact reducible and (φ, γ)-condensing operators

Some well-known classes of noncompact mappings are special cases of the following class of operators (see [11] for Banach spaces, for example).

Definition 2. Let E be a topological vector space, $\emptyset \neq M \subseteq E$ and T a topological space. A continuous $F : T \times M \to E$ will be called compact reducible, if there exists a closed convex set $S \subseteq E$ such that the following conditions hold.

(1) $H(T \times (M \cap S)) \subseteq S$.
(2) $x \in \overline{\mathrm{co}}\,(\{H(t, x)\} \cup S)$ for some $t \in T$ implies $x \in S$.
(3) $H(T \times (M \cap S))$ is relatively compact.

Every set S with the properties (1)–(3) is said to be a fundamental set of H.

It is clear that $S_1 \cap S_2$ is a fundamental set, if S_1 and S_2 are such. From (2) it follows that $x = H(t, x)$ for some $t \in T$ implies $x \in S$ and the empty set is a fundamental set of H iff we have $x \neq H(t, x)$ for each $x \in M$, $t \in T$.

We remark that we can identify $T \times M$ with M if we suppose that T contains one element only.

Every compact mapping $H : T \times M \to E$ is compact reducible, the set $S = \overline{\mathrm{co}}\, H(T \times M)$ is a fundamental set.

Ultimatively compact mappings, which were investigated also for non locally convex spaces in [5], are compact reducible too. The limit range of them is a fundamental set. Especially, this holds for condensing mappings in locally convex spaces and for k-set contractions $(0 < k < 1)$ in Banach spaces.

In general topological vector spaces the notions of the φ-measure of noncompactness of the (φ, γ)-condensing mappings are suitable, which were introduced by Hadzic [4], [3].

Definition 3. Let E be a topological vector space, $\emptyset \neq K \subseteq E$, $(A, \leq)$ a partially ordered set, $\varphi : A \to A$ a mapping and $\mathcal{M}$ a system of subsets of $\overline{\mathrm{co}}\, K$ such that $M \in \mathcal{M}$ implies $\overline{M} \in \mathcal{M}$, $\overline{\mathrm{co}}\, M \in \mathcal{M}$, $N \in \mathcal{M}$ $(N \subseteq M)$ and $M \cup \{a\} \in \mathcal{M}$ $(a \in K)$. The mapping $\gamma : \mathcal{M} \to A$ is said to be a φ-measure of noncompactness on K if the following conditions are satisfied.

(N1) $\gamma(\mathrm{co}\ M) \leq \varphi(\gamma(M))$ $(M \in \mathcal{M})$.

(N2) $\gamma(N) \leq \gamma(M) = \gamma(\overline{M}) = \gamma(M \cup \{a\})$ $(a \in K,\ M \in \mathcal{M},\ N \subseteq M)$.

Further let $M \subseteq K$ be nonvoid, T a topological space, $H : T \times M \to K$ a continuous operator and γ a φ-measure of noncompactness on K. H is called a (φ, γ)-condensing operator provided that $H(T \times N) \in \mathcal{M}$ $(N \subseteq M)$ and if $\gamma(N) \leq \varphi(\gamma(H(T \times N)))$ $(N \subseteq M)$ implies that $H(T \times N)$ is relatively compact.

Condensing mappings or k-set contractions in Banach spaces are special classes of (φ, γ)-condensing operators. The study of (φ, γ)-condensing operators is suitable in non locally convex spaces, because we cannot find nontrivial measures of noncompactness in such general spaces. Some examples of φ-measure of noncompactness and of (φ, γ)-condensing mappings in non locally convex topological vector spaces can be found in [9], [4]. In these papers K will be assumed to be of "Zima's type", these are special cases of locally convex sets. In the essential cases these φ-measures of noncompactness have the following property too:

(N3) If $M \in \mathcal{M}$, then $M \cup (-M) \in \mathcal{M}$ and $\gamma(M) = \gamma(M \cup (-M))$.

Theorem 1. *Let E, M, K, T, φ and γ be stated as in Definition* 3.

Let $H : T \times M \to K$ be a (φ, γ)-condensing operator and $a \in K$. Then H is compact reducible and H has a fundamental set S with $a \in S$. If K is symmetric and (N3) *holds for γ, then H has a nonvoid and symmetric fundamental set.*

Proof: (1) Let $a \in K$ and $\mathscr{S} := \{S \subseteq K : a \in S,\ S = \overline{\mathrm{co}}\, S,\ S$ satisfies the conditions (1), (2) in Definition 2$\}$. Since $K \in \mathscr{S}$, $\mathscr{S} \neq \emptyset$. We define $S_0 := \bigcap_{S \in \mathscr{S}}$ and have $S_0 \subseteq S$ $(S \in \mathscr{S})$.

Clearly $a \in S_0$, $S_0 = \overline{\mathrm{co}}\, S_0$ and $H(T \times (M \cap S_0)) \subseteq S_0$.

Moreover, from $x \in \overline{\mathrm{co}}\,(\{H(t,x)\} \cup S_0) \subseteq \overline{\mathrm{co}}\,(\{H(t,x)\} \cup S)$ for some $t \in T$ it follows that $x \in S$ for each $S \in \mathscr{S}$ and therefore $x \in S_0$.

Hence we have $S_0 \in \mathscr{S}$. Let $S_1 := \overline{\mathrm{co}}\,(H(T \times (M \cap S_0)) \cup \{a\})$. Since $a \in S_0$, $H(T \times (M \cap S_0)) \subseteq S_0$, we obtain $S_1 \subseteq S_0$. Furthermore $H(T \times (M \cap S_1)) \subseteq H(T \times (M \cap S_0)) \subseteq S_1$ and from $x \in \overline{\mathrm{co}}\,(\{H(t,x)\} \cup S_1) \subseteq \overline{\mathrm{co}}\,(\{H(t,x)\} \cup S_0)$ for some $t \in T$ it follows that $x \in S_0$ therefore $x \in \overline{\mathrm{co}}\,(H(T \times (M \cap S_0)) \cup S_1) \subseteq S_1$. Hence $S_1 \in \mathscr{S}$ and therefore $S_0 \subseteq S_1$.

Altogether, we obtain $S_0 = \overline{\mathrm{co}}\,[H(T \times (M \cap S_0)) \cup \{a\}]$. Since γ is a φ-measure of noncompactness, we have $\gamma(M \cap S_0) \leq \gamma(S_0) \leq \varphi(\gamma(H(T \times (M \cap S_0))))$. Because H is a (φ, γ)-condensing operator, we obtain that $H(T \times (M \cap S_0))$ is relatively compact, S_0 is a fundamental set and H is compact reducible.

(2) Now we suppose that γ has the property (N3) and K is symmetric. Then we define

$$\mathscr{S} = \{S \subseteq K : \underline{o} \in S,\ S = \overline{\mathrm{co}}\, S = (-S),\ S \text{ satisfies (1), (2) in Definition 2}\}.$$

Since K is symmetric, we have $\underline{o} \in K$, $K \in \mathscr{S}$ and $\mathscr{S} \neq \emptyset$. Now we define S_0 so as in part (1), however we set $S_1 := \overline{\mathrm{co}}\,(H(T \times (M \cap S_0)) \cup \{\underline{o}\})$. Since S_0 is symmetric, we obtain $S_1 = S_0$ again. Now, from (N1), (N2), (N3) and property of H it follows similarly as in part (1) that S_0 is a (symmetric) fundamental set. $\square$

3. The relative fixed point index of compact reducible operators

Now we will define the relative fixed point index with respect to locally convex subsets K of general topological vector spaces for compact reducible operators. Let E be a topological vector space, T a nonempty closed, convex and admissible subset of E, $M \subseteq E$ open and $M_T = M \cap T$. Let $F : \mathrm{cl}_T\, M_T \to T$ be a compact mapping with $Fx \neq x$ $(x \in \partial_T M_T)$ and $f(x) = x - F(x)$ $(x \in \mathrm{cl}_T\, M_T)$. From Remark 2 it follows that there are a symmetric, starshaped neighbourhood V of zero, a finite dimensional subspace E_V of E and a compact mapping F_V : $\mathrm{cl}_T\, M_T \to E_V \cap T$ with $F(\partial_T M_T) \cap (V + V) = \emptyset$ and $F_V x - Fx \in V$ $(x \in \mathrm{cl}_T\, M_T)$. Then Kayser [10] defined an integer as the relative rotation $\gamma(f, \partial_T M_T)$ by

$$\gamma(f, \partial_T M_T) := \gamma_V(f_V, \partial_{T_V} M_{T_V}),$$

where $T_V = T \cap E_V$, $f_V = f \mid \mathrm{cl}_{T_V}\, M_{T_V}$, and the integer $\gamma_V(f_V, \partial_{T_V} M_{T_V})$ is the relative rotation of f_V in the finite dimensional space E_V which is defined by Borisovitch [1] and has the known properties of a degree.

In the following we denote this Kayser-rotation and the Borisovitch-rotation by $d(F, M_T)$ and by $d_V(F_V, M_T)$, respectively. Let $T_0 \subseteq T$ be closed, convex and admissible and $G = F \mid \mathrm{cl}_T\, M_T$. Then we set $d(F, M_{T_0}) := d(G, M_{T_0})$, where $M_{T_0} = M \cap T_0$. The relative rotation of Kayser has the well-known properties of the degree of compact vector fields. We need the following properties (see [10]).

- (R1) If $d(F, M_T) \neq 0$, then there exists a $x \in M_T$ with $Fx = x$.
- (R2) If S is a closed, convex, admissible subset of E with $F(\mathrm{cl}_T\, M_T) \subseteq S \subseteq T$, then $d(F, M_T) = d(F, M_S)$.
- (R3) If $H : [0, 1] \times \mathrm{cl}_T\, M_T \to T$ is compact with $x \neq H(t, x)$ $(t \in [0, 1]$, $x \in \partial_T M_T)$, $H_0(x) = H(0, x)$, $H_1(x) = H(1, x)$, then $d(H_0, M_T) = d(H_1, M_T)$.
- (R4) If $M_T = \emptyset$, then $d(F, M_T) = 0$. If $M_T \neq \emptyset$ and $\partial_T M_T = \emptyset$, then $d(F, M_T) = 1$.

Now we can define our new rotation.

Definition 4. Let E be a topological vector space, $K \subseteq E$ nonempty, closed, convex and locally convex, $M \subseteq E$ nonempty and open, $M_K = M \cap K$. Let $F : \mathrm{cl}_K\, M_K \to K$ be a compact reducible operator with $Fx \neq x$ $(x \in \partial_K M_K)$. Then we define the relative fixed point index $i(F, M_K)$ of F on M_K by

$$i(F, M_K) := d(F, M_T),$$

where $T = K \cap S$ and S is a fundamental set of F.

The right-hand side is defined, because $F \mid \mathrm{cl}_T M_T$ is a compact mapping with values in T and T is admissible, because T is a closed, convex subset of the locally convex set K. Now we must prove that this definition is independent of the special choice of the fundamental set S. This fact is based on the following

Lemma. *Let E be a topological vector space, $K \subseteq E$ convex, closed, locally convex, $A \subseteq K$ nonvoid, closed and $F : A \to K$ compact reducible. Let S_1, S_2 be fundamental sets and $S_0 = S_1 \cap S_2$ with $A \cap S_0 \neq \emptyset$. Further, let V be a neighbourhood of zero. Then there are a finite subspace E_V of E and a compact mapping*

$$F_V : (A \cap S_1) \cup (A \cap S_2) \longrightarrow S_0 \cap K \cap E_V \quad \textit{with}$$
$$F_V x - Fx \in V \quad (x \in A \cap S_0).$$

Proof: We define $N = F(A \cap S_0)$ and $M = F(A \cap S_1) \cup F(A \cap S_2)$. Then $\emptyset \neq N \subseteq M$. The set $\overline{\mathrm{co}}\, N$ is admissible, because K is locally convex and $\overline{\mathrm{co}}\, N \subseteq K$. Therefore there are a finite subspace E_V of E and a compact mapping $h : N \to \overline{\mathrm{co}}\, N \cap E_V$ with $h(y) - y \in V$ $(y \in N)$, because N is a compact subset of $\overline{\mathrm{co}}\, N$. M is a normal topological space, N a closed subset of M and E_V a finite dimensional normed space. Therefore we can apply a known extension theorem on h and there exists a continuous mapping $\widetilde{h} : M \to E_V$ with $\widetilde{h}(x) = h(x)$ for each $x \in N$. Since the set $\overline{\mathrm{co}}\, N \cap E_V$ is a retract of E_V, there exists a continuous mapping $r : E_V \to \overline{\mathrm{co}}\, N \cap E_V$ with $r(z) = z$ for each $z \in \overline{\mathrm{co}}\, N \cap E_V$. We define

$$F_V(x) := (r \circ h \circ F)(x) \quad (x \in (A \cap S_1) \cup (A \cap S_2)).$$

Then $F_V : (A \cap S_1) \cup (A \cap S_2) \to S_0 \cap K \cap E_V$ is continuous and $F_V((A \cap S_1) \cup (A \cap S_2)) \subseteq \overline{\mathrm{co}}\, N \cap E_V$ implies that F_V is compact.

This implies $F_V(x) - F(x) \in V$ for each $x \in A \cap S_0$. □

Now we can show that our definition is independent of the choice of the fundamental set.

Theorem 2. *Let E, K, M and F be stated as in Definition 4. Let S_1, S_2 be fundamental sets of F. Then*

$$d(F, M \cap K \cap S_1) = d(F, M \cap K \cap S_2).$$

Proof: Let $S_0 = S_1 \cap S_2$, $T_i = K \cap S_i$ $(i = 0, 1, 2)$, $A := \mathrm{cl}_K M_K$, $A_i := \mathrm{cl}_{T_i} M_{T_i}$, $\partial_i A_i := \partial_{T_i} M_{T_i}$ $(i = 1, 2)$. If $A \cap S_0 = \emptyset$, then F has no fixed point on A, because S_0 is a fundamental set of F.

Then from (R1) it follows that $d(F, M \cap T_1) = 0 = d(F, M \cap T_2)$.

Now we suppose that $A \cap S_0 \neq \emptyset$. Since $F \mid A_1 \cup A_2$ is compact and $Fx \neq x$ $(x \in \partial_1 A_1 \cup \partial_2 A_2)$, we find, applying Remark 2, a neighbourhood W of zero with

$$x - Fx \notin W \quad (x \in \partial_1 A_1 \cup \partial_2 A_2). \tag{1}$$

Let V be a symmetric, starshaped neighbourhood of $\underline{o}$ with $V + V \subseteq W$. It follows from the lemma that there are a finite dimensional subspace E_V of E and a compact mapping

(2) $F_V : (A \cap S_1) \cup (A \cap S_2) \longrightarrow S_0$ with $F_V(x) - F(x) \in V \quad (x \in A \cap S_0)$.

(1) and (2) imply $x \neq F_V(x)$ $(x \in \partial_1 A_1 \cup \partial_2 A_2)$. Now, using the Kayser's definition, we obtain

(3) $$d(F, M \cap T_0) = d_V(F_V, M \cap T_0 \cap E_V)$$

and

(4) $$d(F_V, M \cap T_i) = d_V(F_V, M \cap T_i \cap E_V) \quad (i = 1, 2).$$

Since $F_V(A \cap S_i) \subseteq (T_0 \cap E_V) \subseteq (T_i \cap E_V)$ $(i = 1, 2)$, it follows from (R2) that

(5) $$d_V(F_V, M \cap T_0 \cap E_V) = d_V(F_V, M \cap T_i \cap E_V) \quad (i = 1, 2).$$

From (3), (4) and (5) we obtain

(6) $$d(F, M \cap T_0) = d(F_V, M \cap T_i) \quad (i = 1, 2).$$

Now we show that

(7) $$d(F_V, M \cap T_i) = d(F, M \cap T_i) \quad (i = 1, 2).$$

We consider the compact mappings $H : [0, 1] \times A_i \to T_i$ $(i = 1, 2)$, defined by

$$H_i(t, x) := tF(x) + (1 - t)F_V(x) \quad (x \in A_i,\ i = 1, 2,\ t \in [0, 1]).$$

Then $H_i(0, x) = F_V(x)$ and $H_i(1, x) = F(x)$ $(x \in A_i,\ i = 1, 2)$.

We claim that there are a $t_i \in [0, 1]$ and a $x_i \in \partial_i A_i$ with $x_i = H_i(t_i, x_i)$ $(i = 1, 2)$. Since $F_V x_i \in S_0$, we obtain $x_i \in \overline{\mathrm{co}}\,(\{Fx_i\} \cup S_0)$ $(i = 1, 2)$. Since S_0 is a fundamental set, this implies $x_i \in S_0$ $(i = 1, 2)$.

Then we obtain $x_i \in t_i F x_i + (1 - t_i)(Fx_i + V) \subseteq Fx_i + V$ $(i = 1, 2)$. This is a contradiction to (1).

Now it follows from (R3) that (7) holds. Then by (6) and (7) we obtain

$$d(F, M \cap T_1) = d(F, M \cap T_0) = d(F, M \cap T_2).$$

□

Now we give some properties of the relative fixed point index of compact reducible operators.

Theorem 3. *Let E be a topological vector space, $K \subseteq E$ nonempty, convex, closed and locally convex, $M \subseteq E$ open, nonempty and $M_K := M \cap K$. Let $F : \mathrm{cl}_K M_K \to K$ be a compact reducible mapping with $Fx \neq x$ $(x \in \partial_K M_K)$. Then the relative fixed point index $i(F, M_K)$ has the following properties.*

(I1) *If* $i(F, M_K) \neq 0$, *then there exists a* $x \in M_K$ *with* $Fx = x$.
(I2) *If* S *is a closed, convex subset of* E *with* $F(\mathrm{cl}_K\, M_K) \subseteq S \subseteq K$, *then* $i(F, M_K) = i(F, M \cap S)$.
(I3) *If* $H : [0,1] \times \mathrm{cl}_K\, M_K \to K$ *is a compact reducible operator with* $H(t,x) \neq x$ $(t \in [0,1],\ x \in \partial_K M_K)$ *and* $H_0(x) = H(0,x)$, $H_1(x) = H(1,x)$ $(x \in \mathrm{cl}_K\, M_K)$, *then* $i(H_0, M_K) = i(H_1, M_K)$.
(I4) *If* $M_K = \emptyset$, *then* $i(F, M_K) = 0$. *If* $M_K \neq \emptyset$ *and* $\partial_K M_K = \emptyset$, *then* $i(F, M_K) = 1$.
(I5) *Let* $x_0 \in K$ *and* $F(x) = x_0$ $(x \in \mathrm{cl}_K\, M_K)$. *Then*

$$i(F, M_K) = \begin{cases} 1 & \text{if } x_0 \in M_K \\ 0 & \text{if } x_0 \notin \mathrm{cl}_K\, M_K. \end{cases}$$

(I6) *Let* $M_i \subseteq E$ $(i = 1, \ldots, n)$ *be open subsets with* $M_i \cap M_j = \emptyset$ $(i \neq j)$ *and* $M_{iK} := M_i \cap K$ $(i = 1, \ldots, n)$.
If $\bigcup_{i=1}^n M_{iK} \subseteq M_K$ *and* $F(x) \neq x$ $(x \in \mathrm{cl}_K\, M_K \setminus \bigcup_{i=1}^n M_{iK})$, *then* $i(F, M_K) = \sum_{i=1}^n i(F, M_{iK})$.

Proof: These properties follow from the properties of the relative rotation of compact vector fields. (see [10, Satz 3]) directly. We prove for instance (I3).

Let S be a fundamental set of $H : [0,1] \times \mathrm{cl}_K\, M_K \to K$. Then S is a fundamental set of H_0 and H_1 too. Since $H(t,x) \neq x$ for each $t \in [0,1]$ and each $x \in \partial_K M_K$, we obtain $H(t,x) \neq x$ $(t \in [0,1],\ x \in \partial_T M_T)$, where $T = K \cap S$.

Since $H_S := H \mid [0,1] \times (K \cap S)$ is a compact mapping with $H_S([0,1] \times (K \cap S)) \subseteq (K \cap S)$, we can apply (R3) on H_S. From this and Definition 4 it follows that

$$i(H_0, M_K) = d(H_0, M \cap K \cap S) = d(H_1, M \cap K \cap S) = i(H_1, M_K).$$

$\square$

Suppose that $M = E$. Then from (I1) and (I4) it follows directly that the compact reducible operator $F : K \to K$ has a fixed point.

If F is (φ, γ)-condensing, then the Borsuk's theorem on odd index holds even.

Theorem 4. *Let* E *be a topological vector space,* $K \subseteq E$ *nonvoid, closed, symmetric, convex and locally convex,* $M \subseteq E$ *open, nonvoid and symmetric. Let* γ *be a* φ*-measure of noncompactness, for which the property* (N3) *holds. Further let* $F : \mathrm{cl}_K(M \cap K) \to K$ *be a* (φ, γ)*-condensing mapping. Suppose*

$$x - F(x) \neq \beta(-x - F(-x)) \quad (x \in \partial_K M_K,\ \beta \in [0,1]).$$

Then $i(F, M_K)$ *is odd and* F *has a fixed point in* $M \cap K$.

Proof: From Theorem 1 it follows that F is compact reducible and there exists a nonvoid symmetric fundamental set S for F. Since $T = K \cap S$ is nonvoid, symmetric, convex and admissible, $F(\mathrm{cl}_T\, M_T) \subseteq T$, $F(\mathrm{cl}_T\, M_T)$ is relatively compact and $x - F(x) \neq \beta(-x - F(x))$ $(x \in \partial_T M_T,\ \beta \in [0,1])$, we can apply the analogue theorem for the relative rotation for compact vector fields of Kayser ([10, Satz 5]). Therefore $d(F, M_T)$ is odd and, by Definition 4, $i(F, M_K)$ is odd. From Theorem 3 (I1) it follows that F has a fixed point. □

Corollary 1. *Let E, K, M, φ be stated as in Theorem 4. Let $F : \mathrm{cl}_K M_K \to K$ be a (φ,γ)-condensing operator with $x \neq tFx + (1-t)(-F(-x))$ $(x \in \partial_K M_K,$ $t \in [0,1])$. Then F has a fixed point.*

Proof: We can easily see that the condition for F on $\partial_K M_K$ implies the condition on $\partial_K M_K$ in Theorem 4. Clearly, Theorem 4 implies that $i(F, M_K)$ is odd if the vector fields $f = I - F$ is odd on $\partial_K M_K$. Theorem 4 is the first generalization on the Borsuk's theorem for a class of noncompact mappings in non locally convex topological vector spaces. □

The following fixed point theorem of the Leray-Schauder type is an application of our fixed point index.

Theorem 5. *Let E be a topological vector space, $K \subseteq E$ convex, locally convex, closed, $M \subseteq E$ open with $M \cap K \neq \emptyset$ and $F : \mathrm{cl}_K M_K \to K$ a compact reducible mapping such that there are a fundamental set S and a $x_0 \in M \cap K \cap S$ with*

$$x \neq tFx + (1-t)x_0 \quad (x \in \partial_K M_K,\ t \in [0,1]).$$

Then F has a fixed point.

Proof: Let $H(t,x) = tF(x) + (1-t)\cdot x$ $(t \in [0,1],\ x \in \partial_K M_K)$ and $G(x) = x_0$ $(x \in \mathrm{cl}_K M_K)$. Then we obtain $H(0,x) = G(x)$, $H(1,x) = F(x)$ $(x \in \mathrm{cl}_K M_K)$ and $x \neq H(t,x)$ $(t \in [0,1],\ x \in \partial_K M_K)$.

We show that S is a fundamental set for $H : [0,1] \times \mathrm{cl}_K M_K \to K$ and therefore H is compact reducible. Since $F(\mathrm{cl}_K M_K \cap S) \subseteq S$ and $x_0 \in S$, we have $H([0,1] \times (\mathrm{cl}_K M_K \cap S)) \subseteq \overline{\mathrm{co}}\,(F(\mathrm{cl}_K M_K \cap S) \cup \{x_0\}) \subseteq S$.

Further $H([0,1] \times (\mathrm{cl}_K M_K \cap S))$ is relatively compact, because this set is a subset of the set $[0,1] \cdot F(\mathrm{cl}_K M_K \cap S) + [0,1] \cdot \{x_0\}$ which is relatively compact. Now we prove that the condition (2) in Definition 2 holds for S and H.

Let $x \in \overline{\mathrm{co}}\,(\{H(t,x)\} \cup S)$ for some $t \in [0,1]$. From $\mathrm{co}\,(\{H(t,x)\} \cup S) \subseteq \overline{\mathrm{co}}\,(\overline{\mathrm{co}}\,(\{F(x)\} \cup \{x_0\}) \cup S) \subseteq \overline{\mathrm{co}}\,(\{F(x)\} \cup \{x_0\} \cup S) = \overline{\mathrm{co}}\,(\{F(x)\} \cup S)$ and the properties of S and F it follows that $x \in S$. Now we can apply Theorem 3 (I3), (I5) and (I1). Therefore $i(F, M_K) = i(G, M_K) = 1$ and F has a fixed point. □

Corollary 2. *Let E be a topological vector space. $K \subseteq E$ closed, convex, locally convex, $M \subseteq K \neq \emptyset$ and $F : \mathrm{cl}_K M_K \to K$ a (φ,γ)-condensing operator such that there exists a $x_0 \in M \cap K$ with $x \neq tF(x) + (1-t)x_0$ $(x \in \partial_K M_K,\ t \in [0,1])$. Then F has a fixed point.*

Proof: Theorem 1 and Theorem 5 imply this result directly. □

Corollary 2 is a special case of a theorem of Kaniok ([9, Theorem 1]). Kaniok proved this result without index theory.

References

[1] Borisovitch Iu.G., *On the relative rotation of compact vector fields in linear spaces* (in Russian), Trudy seminara pro funkc. analyzu **12** (1969), 3–27 (Voronesh).

[2] Hadzic O., *Fixed Point Theory in Topological Vector Spaces*, Novi Sad, 1984.

[3] ______, *Fixed point theorems for multivalued mappings in not necessarily locally convex topological vector spaces*, Zb. rad. Prir.-mat. fak. Novi Sad, ser. mat. **14** (1984).

[4] ______, *Some properties of measures of noncompactness in paranormed spaces*, Proc. Amer. Math. Soc. **102** (1988), 843–849.

[5] Hahn S., *Fixpunktsätze für limeskompakte mengenwertige Abbildungen in nicht notwendig lokalkonvexen topologischen Vektorräumen*, Comment. Math. Univ. Carolinae **27** (1986), 189–204.

[6] Hahn S., Riedrich T., *Der Abbildungsgrad kompakter Vektorfelder in nicht notwendig lokalkonvexen topologischen Vektorräumen*, Wiss. Z. Techn. Univ. Dresden **22** (1973), 37–42.

[7] Jerofsky T., *Zur Fixpunkttheorie mengenwertiger Abbildungen*, Dissertation, TU Dresden, 1983.

[8] Kaballow W., *Zum Abbildungsgrad in Hausdorffschen topologischen Vektorräumen*, Manuscripta Math. **8** (1973), 209–216.

[9] Kaniok L., *On measures of noncompactness in general topological vector spaces*, Comment. Math. Univ. Carolinae **31** (1990), 479–487.

[10] Kayser G., *Zur relativen Drehung in Hausdorffschen topologischen Vektorräumen*, Wiss. Z. Techn. Univ. Dresden **23** (1974), 335–340.

[11] Krasnoselskii M.A., Zabreiko P.P., *Geometrical Methods in the Nonlinear Analysis* (in Russian), Moscow, 1975.

[12] Krauthausen C., *Der Fixpunktsatz von Schauder in nicht notwendig lokalkonvexen Räumen sowie Anwendungen auf Hammersteinsche Gleichungen*, Dissertation, TH Aachen, 1976.

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(*Received* March 9, 1992)