Note on special arithmetic and geometric means

Horst Alzer

Abstract. We prove: If $A(n)$ and $G(n)$ denote the arithmetic and geometric means of the first n positive integers, then the sequence $n \mapsto nA(n)/G(n) - (n-1)A(n-1)/G(n-1)$ $(n \geq 2)$ is strictly increasing and converges to $e/2$, as n tends to ∞.

Keywords: arithmetic and geometric means, discrete inequality

Classification: 26D15

In this paper we denote by $A(n)$ and $G(n)$ the arithmetic and geometric means of the first n positive integers, that is,

$$A(n) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2} \quad \text{and} \quad G(n) = \prod_{i=1}^{n} i^{1/n} = (n!)^{1/n}\,.$$

In 1964 H. Minc and L. Sathre [2] published several remarkable inequalities involving $G(n)$. "Probably the most interesting of them, and certainly the hardest to prove" [2, p. 41], is

$$1 < n\frac{G(n+1)}{G(n)} - (n-1)\frac{G(n)}{G(n-1)} \quad (n \geq 2)\,. \tag{1}$$

It is the aim of this paper to present a closely related result. We prove the following counterpart of inequality (1):

$$\frac{3}{\sqrt{2}} - 1 \leq n\frac{A(n)}{G(n)} - (n-1)\frac{A(n-1)}{G(n-1)} < \frac{e}{2} \quad (n \geq 2)\,. \tag{2}$$

Both bounds are best possible. The double-inequality (2) is an immediate consequence of the following

Theorem. *The sequence*

$$n \mapsto n\frac{A(n)}{G(n)} - (n-1)\frac{A(n-1)}{G(n-1)} \quad (n \geq 2)$$

is strictly increasing and converges to $e/2$, as n tends to ∞.

Proof: In the first part of the proof we show that the function

$$f(x) = x(x+1)(\Gamma(x+1))^{-1/x} \quad (0 < x \in \mathbb{R})$$

is strictly convex on $[4,\infty)$. In what follows we assume $x \geq 4$. Differentiation yields

$$
\begin{aligned}
x^2(x+1)\frac{f''(x)}{f(x)} =& 2x - 2x\Psi(x+1) + 2\log\Gamma(x+1) + (x+1)(\Psi(x+1))^2 \\
& - \frac{2(x+1)}{x}\Psi(x+1)\log\Gamma(x+1) + \frac{x+1}{x^2}(\log\Gamma(x+1))^2 \\
& - x(x+1)\Psi'(x+1)\,,
\end{aligned}
\tag{3}
$$

where $\Psi = \Gamma'/\Gamma$ designates the logarithmic derivative of the gamma function. Using the inequalities

$$
\begin{aligned}
0 &< (x-1/2)\log(x) - x + \log\sqrt{2\pi} \\
&< \log\Gamma(x) < 1/(12x) + (x-1/2)\log(x) - x + \log\sqrt{2\pi}\,, \\
0 &< \log(x) - 1/(2x) - 1/(12x^2) < \Psi(x) < \log(x) - 1/(2x)\,,
\end{aligned}
$$

and

$$\Psi'(x) < 1/x + 1/(2x^2) + 1/(6x^3)\,,$$

(see [1, p. 820 ff.]), we get from (3):

(4)

$$
\begin{aligned}
x^2(x+1)\frac{f''(x)}{f(x)} >& 2x - 2x\left[\log(x+1) - \frac{1}{2(x+1)}\right] \\
& + 2\left[(x+1/2)\log(x+1) - (x+1) + \log\sqrt{2\pi}\right] \\
& + (x+1)\left[\log(x+1) - \frac{1}{2(x+1)} - \frac{1}{12(x+1)^2}\right]^2 \\
& - \frac{2(x+1)}{x}\left[\log(x+1) - \frac{1}{2(x+1)}\right]\times \\
& \times\left[\frac{1}{12(x+1)} + (x+1/2)\log(x+1) - (x+1) + \log\sqrt{2\pi}\right] \\
& + \frac{x+1}{x^2}\left[(x+1/2)\log(x+1) - (x+1) + \log\sqrt{2\pi}\right]^2 \\
& - x(x+1)\left[\frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{6(x+1)^3}\right] \\
=& \frac{1}{2} + \frac{1}{x}\left[\frac{25}{12} - \frac{3}{2}\log(2\pi) + (\log\sqrt{2\pi})^2\right] - \frac{1}{2(x+1)} \\
& + \frac{1}{x^2}\left[1 - \log(2\pi) + (\log\sqrt{2\pi})^2\right] + \frac{1}{4(x+1)^2} + \frac{1}{144(x+1)^3} \\
& + \frac{x+1}{4x^2}(\log(x+1))^2 + \log(x+1)\left\{\frac{1}{x}\left[\frac{1}{2}\log(2\pi) - \frac{5}{3}\right]\right. \\
& \left. - \frac{1}{6(x+1)} + \frac{1}{x^2}\left[\frac{1}{2}\log(2\pi) - 1\right]\right\}.
\end{aligned}
$$

Since
$$1-\log(2\pi)+(\log\sqrt{2\pi})^2>0$$
and
$$\frac{x+1}{4x^2}(\log(x+1))^2>\frac{1}{2x}\log(x+1)$$
we conclude from (4):
$$x^2(x+1)\frac{f''(x)}{f(x)}>\frac{1}{2}+\frac{a}{x}-\frac{1}{2(x+1)}-\log(x+1)\left[\frac{b}{x}+\frac{1}{6(x+1)}+\frac{c}{x^2}\right], \tag{5}$$
where
$$\begin{aligned}a=&\frac{25}{12}-\frac{3}{2}\log(2\pi)+(\log\sqrt{2\pi})^2=0.170\ldots,\\ b=&\frac{7}{6}-\frac{1}{2}\log(2\pi)=0.247\ldots,\quad c=1-\frac{1}{2}\log(2\pi)=0.081\ldots.\end{aligned}$$
Using $\log(x+1)<x$, we obtain from (5):
$$\begin{aligned}x^2(x+1)\frac{f''(x)}{f(x)}&>\frac{1}{3}-b+\left(a-c-\frac{1}{3}\right)\frac{1}{x}\\ &\geq\frac{1}{3}-b+\left(a-c-\frac{1}{3}\right)\frac{1}{4}=0.024\ldots,\end{aligned}$$
valid for all $x\geq 4$.

Thus, f is strictly convex on $[4,\infty)$. From Jensen's inequality we get
$$2f(n)<f(n-1)+f(n+1)$$
for all integers $n\geq 5$. This implies that the sequence
$$n\mapsto[f(n)-f(n-1)]/2=nA(n)/G(n)-(n-1)A(n-1)/G(n-1)$$
is strictly increasing for $n\geq 5$. The approximate values of $nA(n)/G(n)-(n-1)A(n-1)/G(n-1)$ for $n=2,3,4,5$, are 1.121, 1.180, 1.216, 1.239, respectively. Hence, $n\mapsto nA(n)/G(n)-(n-1)A(n-1)/G(n-1)$ is strictly increasing for all $n\geq 2$.

Finally we prove that
$$\lim_{n\to\infty}[nA(n)/G(n)-(n-1)A(n-1)/G(n-1)]=e/2. \tag{6}$$
If we set
$$\alpha(n)=n/G(n)\quad\text{and}\quad\beta(n)=G(n)/G(n-1),$$
then we have for $n\geq 2$:
$$\begin{aligned}n\frac{A(n)}{G(n)}-&(n-1)\frac{A(n-1)}{G(n-1)}\\ &=\frac{1}{2}\left[\alpha(n)+\frac{n}{n-1}\alpha(n-1)-\frac{n}{n-1}\alpha(n)\frac{\beta(n)-1}{\log\beta(n)}\log\alpha(n)\right].\end{aligned}$$
Since
$$\lim_{n\to\infty}\alpha(n)=e\quad\text{and}\quad\lim_{n\to\infty}\beta(n)=1$$
we obtain (6). This completes the proof of the Theorem. □

References

[1] Fichtenholz G.M., *Differential – und Integralrechnung, II*, Dt. Verlag Wissensch., Berlin, 1979.

[2] Minc H., Sathre L., *Some inequalities involving* $(r!)^{1/r}$, Edinburgh Math. Soc. **14** (1964/65), 41–46.

Morsbacher Str. 10, 51545 Waldbröl, Germany

(*Received* September 24, 1993)