On function spaces of Corson-compact spaces

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Abstract. We apply elementary substructures to characterize the space $C_p(X)$ for Corson-compact spaces. As a result, we prove that a compact space X is Corson-compact, if $C_p(X)$ can be represented as a continuous image of a closed subspace of $(L_\tau)^\omega \times Z$, where Z is compact and L_τ denotes the canonical Lindelöf space of cardinality τ with one non-isolated point. This answers a question of Archangelskij [2].

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1. Elementary substructures

We begin with a brief exposition of some definitions and facts concerning elementary substructures (see also A. Dow [5], K. Kunen [7]). A non-empty subset $\mathcal{M}$ of a set $\mathcal{H}$ is said to be an elementary substructure of $\mathcal{H}$, if for any formula $\phi(x_1,\dots,x_n)$ of the language of set theory with the only free variables $x_1,\dots,x_n$ and for any $a_1,\dots,a_n \in \mathcal{M}$ $\phi[a_1,\dots,a_n]$ is true in $\mathcal{M}$ if and only if it is true in $\mathcal{H}$.

A frequently used argument is Tarski's Criterion:

> A subset $\mathcal{M}$ of $\mathcal{H}$ forms an elementary substructure of $\mathcal{H}$ if and only if for every formula $\phi(x_0,x_1,\dots,x_n)$ of the language of set theory and every $a_1,\dots,a_n \in \mathcal{M}$ such that there exists an $a \in \mathcal{H}$ such that $\phi[a,a_1,\dots,a_n]$ is true in $\mathcal{H}$, there is a $b \in \mathcal{M}$ such that $\phi[b,a_1,\dots,a_n]$ is true in $\mathcal{H}$ (and therefore in $\mathcal{M}$).

The base of all our applications of elementary substructures is the following

Theorem 1.1 (Löwenheim-Skolem-Tarski)**.** *For every infinite set $\mathcal{H}$ and each subset $X \subseteq \mathcal{H}$ there exists an elementary substructure $\mathcal{M}$ of $\mathcal{H}$ such that $X \subseteq \mathcal{M}$ and $|\mathcal{M}| = \max\{|X|,\omega\}$.*

If θ is a cardinal, then $\mathcal{H}(\theta)$ denotes the collection of all sets hereditarily of cardinality $< \theta$. We will usually be interested in elementary substructures of $\mathcal{H}(\theta)$, where θ is a sufficiently large regular cardinal. The main reason is that for any sentence ϕ there exists a large enough regular cardinal θ such that ϕ is true (in V) if and only if it is true in $\mathcal{H}(\theta)$. In practice, one "chooses" θ without discussion how large it needs to be.

The following fact is well known and useful.

Proposition 1.2. *If θ is a regular uncountable cardinal, $\mathcal{M}$ an elementary substructure of $\mathcal{H}(\theta)$ and $A \in \mathcal{M}$ a countable set, then $A \subseteq \mathcal{M}$.*

If $\mathcal{H}$ is an uncountable $[\mathcal{H}]^\omega$ denotes the set of all countable subsets of $\mathcal{H}$. $C \subseteq [\mathcal{H}]^\omega$ is said to be unbounded if for every $X \in [\mathcal{H}]^\omega$ there is a $Y \in C$ with $X \subseteq Y$. We say C is closed if, whenever $X_n \in C$ and $X_n \subseteq X_{n+1}$ for each $n \in \omega$, then $\bigcup\{X_n : n \in \omega\} \in C$. A consequence of Theorem 1.1 is that the family of all countable elementary substructures of $\mathcal{H}$ is closed and unbounded. Remark, that the intersection of two closed unbounded subsets of $[\mathcal{H}]^\omega$ is closed unbounded, too.

For the sake of simplicity we shall often write "Let $\mathcal{M}$ be a suitable elementary substructure ... ". This means that all "information" we need to investigate an object, say a topological space, can be found in $\mathcal{M}$. For example, if X is a dyadic compact space, we suppose that there is a continuous mapping $f : D^\tau \to X$, which is an element of $\mathcal{M}$. Obviously there is always a closed unbounded family of "suitable" countable elementary substructures. On the other hand, " ... if for any suitable countable elementary substructure $\mathcal{M}$ the following condition is satisfied ... " means that the condition is satisfied for all countable substructures from a closed unbounded subset of $[\mathcal{H}(\theta)]^\omega$, where θ is a large enough (with respect to the object we investigate) regular cardinal.

2. The main construction

Now we are going to describe a construction for arbitrary uniform spaces. Let $\langle X,\mathcal{U}\rangle$ be a uniform space (see Engelking [6]). If $\mathcal{M}$ is an elementary substructure with $X,\mathcal{U} \in \mathcal{M}$, the equivalence relation on X is defined by

$$x \approx y(\mathcal{M}) \;\text{ iff }\; |x-y| < V \;\text{ for each }\; V \in \mathcal{U} \cap \mathcal{M}.$$

If $x \approx y(\mathcal{M})$, we say that x and y are $\mathcal{M}$-equivalent. Let $X(\mathcal{M},\mathcal{U})$ denote the set of all equivalence classes and $\phi^{X,\mathcal{U}}_{\mathcal{M}}$ the canonical mapping of X onto $X(\mathcal{M})$. (For short, we often write $X(\mathcal{M})$ and $\phi^X_{\mathcal{M}}$ and drop the $\mathcal{U}$.) A uniformity $\mathcal{U}(\mathcal{M})$ is given on $X(\mathcal{M})$ by all entourages $V_{\mathcal{M}}$ of the diagonal in $X(\mathcal{M})$. $\mathcal{U}(\mathcal{M})$ is defined by

$$|\phi^X_{\mathcal{M}}(x) - \phi^X_{\mathcal{M}}(y)| < V_{\mathcal{M}} \;\text{ iff }\; |x'-y'| < V \;\text{ for all }\; x',y' \in X$$
$$\text{such that }\; x' \approx x(\mathcal{M}) \;\text{ and }\; y' \approx y(\mathcal{M}),$$

where V is an arbitrary element of $\mathcal{U} \cap \mathcal{M}$.

$\phi^X_{\mathcal{M}}$ is uniformly continuous with respect to the uniformities $\mathcal{U}$ and $\mathcal{U}(\mathcal{M})$ on X and $X(\mathcal{M})$ respectively (see Bandlow [3]). Now we shall give some easy assertions used in the sequel.

Fact 2.1. Let $f : \langle X,\mathcal{U}\rangle \to \langle Y,\mathcal{V}\rangle$ be a uniformly continuous mapping, suppose $f \in \mathcal{M}$. Then there exists a uniformly continuous mapping $f_{\mathcal{M}} : X(\mathcal{M}) \to Y(\mathcal{M})$

which makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi^X_{\mathcal{M}}\big\downarrow & & \big\downarrow\phi^X_{\mathcal{M}} \\ X(\mathcal{M}) & \xrightarrow[f_{\mathcal{M}}]{} & Y(\mathcal{M}) \end{array} \tag{1}$$

Fact 2.2. Let $\langle X,\mathcal{U}\rangle$ be the product of the uniform spaces $\{\langle X_t,\mathcal{U}_t\rangle : t \in T\}$; suppose $\{\langle X_t,\mathcal{U}_t\rangle : t \in T\} \in \mathcal{M}$. Then $\langle X(\mathcal{M}),\mathcal{U}(\mathcal{M})\rangle$ is uniformly homeomorphic to the product of the uniform spaces $\{\langle X_t(\mathcal{M}),\mathcal{U}_t(\mathcal{M})\rangle : t \in T\cap\mathcal{M}\}$ in a natural way.

Fact 2.3. If $\langle Y,\mathcal{U}_Y\rangle$ is a uniform subspace of $\langle X,\mathcal{U}\rangle$, then $Y(\mathcal{M})$ is uniformly homeomorphic to $\phi^X_{\mathcal{M}}(Y)$ in a natural way.

Fact 2.4. If X is a compact Hausdorff space, there is a unique uniformity $\mathcal{U}$ on X which induces the original topology on X. $\mathcal{U}$ is generated by all sets of the form

$$\mathcal{U}^{\varepsilon}_{f_1,\ldots,f_n} = \{\langle x,y\rangle : |f_i(x) - f_i(y)| < \varepsilon, \quad i = 1,\ldots,n\},$$

where $f_1,\ldots,f_n$ are arbitrary real-valued continuous functions on X, $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $\mathcal{M}$ be a suitable elementary substructure. It is easy to see that for any pair of distinct points $x,y \in X$ we have $\phi^X_{\mathcal{M}}(x) \neq \phi^X_{\mathcal{M}}(y)$ if and only if there is a function $f \in C(X)\cap\mathcal{M}$ with $f(x) \neq f(y)$. Consequently, $\phi^X_{\mathcal{M}}$ corresponds to the mapping which relates each point $x \in X$ to $(fx)_{C(X)\cap\mathcal{M}}$ from the product space $\mathbb{R}^{C(X)\cap\mathcal{M}}$. An easy consequence is that for each function $f \in C(X)\cap\mathcal{M}$ there is a continuous function $f_{\mathcal{M}} : X(\mathcal{M}) \to \mathbb{R}$ such that $f = f_{\mathcal{M}} \circ \phi^X_{\mathcal{M}}$. If $U \in \mathcal{M}$ is a functionally open subset of X, i.e. if there is a function $f \in C(X)\cap\mathcal{M}$ with $U = f^{-1}(0,1)$, then let $U_{\mathcal{M}}$ be defined by $U_{\mathcal{M}} = f^{-1}_{\mathcal{M}}(0,1)$. Remark that $U = (\phi^X_{\mathcal{M}})^{-1}U_{\mathcal{M}}$. The family consisting of all open subsets of $X(\mathcal{M})$ of the form $U_{\mathcal{M}}$, where U is a functionally open subset of X and belongs to $\mathcal{M}$, is a base for the topology of $X(\mathcal{M})$.

We now prove that if X is a Lindelöf space, then $X(\mathcal{M})$ as a topological space does not depend on the uniformity on X.

Proposition 2.5. *Let X be a Lindelöf space and let $\mathcal{U}$ and $\mathcal{V}$ be uniformities on X, which induces the topology on X. If $\mathcal{M}$ is a suitable elementary substructure, then:*

(a) $\phi^{X,\mathcal{U}}_{\mathcal{M}}(x) = \phi^{X,\mathcal{U}}_{\mathcal{M}}(y)$ *if and only if* $\phi^{X,\mathcal{V}}_{\mathcal{M}}(x) = \phi^{X,\mathcal{V}}_{\mathcal{M}}(y)$ *for arbitrary points* $x,y \in X \quad (X(\mathcal{M}) = X(\mathcal{M},\mathcal{U}) = X(\mathcal{M},\mathcal{V}))$.

(b) $\mathcal{U}(\mathcal{M})$ *and* $\mathcal{V}(\mathcal{M})$ *generate the same topology on* $X(\mathcal{M})$.

PROOF: W.l.o.g. we may assume that $\mathcal{V} \subseteq \mathcal{U}$. Let x_0, y_0 be a pair of distinct points of X. Suppose that $\phi^{X,\mathcal{U}}_{\mathcal{M}}(x_0) \neq \phi^{X,\mathcal{U}}_{\mathcal{M}}(y_0)$, i.e. $|x_0 - y_0| \geq U$ for some

$U \in \mathcal{U} \cap \mathcal{M}$. We can choose a $W \in \mathcal{U} \cap \mathcal{M}$ such that $4W \subseteq U$.
$\eta = \{\mathrm{Int}\, B(x, W) : x \in X\}$ is an open cover of the space X and belongs to $\mathcal{M}$. Here $B(x, W)$ denotes the set $\{y \in X : |x - y| < V\}$. Therefore there exists a countable subcover $\eta' = \{\mathrm{Int}\, B(x_n, W) : n = 1, 2, 3, \dots\}$ which also belongs to $\mathcal{M}$. By Proposition 1.2, we may assume that all points x_n, $n = 1, 2, 3, \dots$, are elements of $\mathcal{M}$. Hence we can fix $z_1, z_2 \in X \cap \mathcal{M}$ such that $x_0 \in B(z_1, W)$ and $y_0 \in B(z_2, W)$. $F = \mathrm{cl}\,(B(z_1, W))$ and $G = \mathrm{cl}\,(B(z_2, W))$ are elements of $\mathcal{M}$ too, and we have $x_0 \in F$, $y_0 \in G$ and $F \cap G = \emptyset$. For every point $x \in F$ there exists an entourage $V_x \in \mathcal{V}$ such that $B(x, 2V_x) \cap G = \emptyset$. Similarly as above we consider now the open cover $\xi = \{\mathrm{Int}\, B(x, V_x) : x \in F\}$ of F. Since F and G are elements of $\mathcal{M}$, ξ also belongs to $\mathcal{M}$. Let $\xi' = \{\mathrm{Int}\, B(x_n, V_{x_n}) n = 1, 2, 3, \dots\}$ be a countable subcover of ξ. We may assume that ξ' belongs to $\mathcal{M}$ and, consequently, that all x_n and V_{x_n} are elements of $\mathcal{M}$. Hence, there exists a point $z \in F \cap \mathcal{M}$ and an entourage $V \in \mathcal{V} \cap \mathcal{M}$ such that $x_0 \in B(z, V)$ and $B(z, 2V) \cap G = \emptyset$, i.e. $|x_0 - y_0| > V$ and $\phi^{X,\mathcal{V}}_{\mathcal{M}}(x_0) \neq \phi^{X,\mathcal{V}}_{\mathcal{M}}(y_0)$. This proves the assertion (a) and we may identify $X(\mathcal{M}) = X(\mathcal{M}, \mathcal{U}) = X(\mathcal{M}, \mathcal{V})$. Let $T_{\mathcal{U}}$ and $T_{\mathcal{V}}$ denote the topologies on $X(\mathcal{M})$ generated by $\mathcal{U}(\mathcal{M})$ and $\mathcal{V}(\mathcal{M})$, respectively. We have to prove that the (identical) mapping

$$\langle X(\mathcal{M}), T_\gamma \rangle \longrightarrow \langle X(\mathcal{M}), T_{\mathcal{U}} \rangle$$

is continuous with respect to these topologies. Suppose $x_0 \in X$ and $\mathcal{O}$ is a neighborhood of $\phi^X_{\mathcal{M}}(x_0)$ in $\langle X(\mathcal{M}), T_{\mathcal{U}} \rangle$. Then there is a $U \in \mathcal{U} \cap \mathcal{M}$ such that $\phi^X_{\mathcal{M}}(B(x_0, u)) \subseteq \mathcal{O}$. It is sufficient to check that there exists a $V \in \mathcal{V} \cap \mathcal{M}$ with $B(x_0, V) \subseteq B(x_0, U)$. To do this we take a $W \in \mathcal{U} \cap \mathcal{M}$ with $2W \subseteq U$ and consider the open cover $\{\mathrm{Int}\, B(x, W) : x \in X\}$ of $\langle X, T_{\mathcal{U}} \rangle$. For every $x \in X$ there exists a $V_x \in \mathcal{V}$ such that $B(x, 2V_x) \subseteq \mathrm{Int}\, B(x, W)$. One can assume that the open cover of $\langle X, T_\gamma \rangle$ $\{\mathrm{Int}\, B(x, V_x) : x \in X\}$ is an element of $\mathcal{M}$. Using the same arguments as above we can find a point $z \in X \cap \mathcal{M}$ and an entourage $V \in \mathcal{V} \cap \mathcal{M}$ such that $x_0 \in B(z, V)$ and $B(z, 2V) \subseteq B(z, W)$. Hence, $B(x_0, V) \subseteq B(z, 2V) \subseteq B(z, W) \subseteq B(x_0, 2W) \subseteq B(x_0, U)$. The proof is now complete. $\square$

3. Corson-compact spaces

A compact Hausdorff space X is called Corson-compact, if it is homeomorphic to a subset of

$$\sum(\mathbb{R}^T) = \left\{ x \in \mathbb{R}^T : \mathrm{supp}\,(x) \ \text{ is countable} \right\},$$

where $\mathrm{supp}\,(x) = \{t \in T : x(t) \neq 0\}$ for $x \in \mathbb{R}^T$, for some set T. Corson-compact spaces have been extensively studied by various authors (for a detailed information see Archangelskij [2], and Negrepontis [8]).

It is easy to verify that for any suitable elementary substructure $\mathcal{M}$, $\phi^X_{\mathcal{M}}$ maps $\mathrm{cl}\,(X \cap \mathcal{M})$ homeomorphic on $X(\mathcal{M})$. The main result of the previous paper is the following characterization of Corson-compact spaces.

Theorem 3.1 (Bandlow [4]). *A compact Hausdorff space X is Corson-compact iff for any suitable countable elementary substructure $\mathcal{M}$, $\phi^X_{\mathcal{M}}$ maps $\mathrm{cl}\,(X \cap \mathcal{M})$ homeomorphically on $X(\mathcal{M})$.*

4. The construction and $C_p(X)$

Our aim now is to consider the space $C_p(X)$ of all real-valued continuous functions on a completely regular space X in the topology of pointwise convergence. The natural uniformity is given on $C_p(X)$ by all sets of the form

$$\mathcal{U}^{\varepsilon}_{x_1,\dots,x_n} = \{\langle f, g\rangle : |f(x_i) - g(x_i)| < \varepsilon, \quad i = 1, \dots, n\},$$

with $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$ and $\varepsilon > 0$.

It is easy to see that two continuous functions f and g on X are $\mathcal{M}$-equivalent if and only if $f\mid_{X\cap\mathcal{M}} = g\mid_{X\cap\mathcal{M}}$.

In connection with $C_p(X)$, we consider on X the uniformity generated by all real-valued continuous functions. There is a natural embedding i from X into the product space $\mathbb{R}^{C(X)}$ and therefore an embedding $i_{\mathcal{M}}$ from $X(\mathcal{M})$ into $\mathbb{R}^{C(X)\cap\mathcal{M}}$.

Let $\overline{X(\mathcal{M})}$ denote the image of X by the projection mapping onto $\mathbb{R}^{\mathrm{cl}\,(C(X)\cap\mathcal{M})}$. We get the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & \mathbb{R}^{C(X)} \\ {\scriptstyle \overline{\phi}^X_{\mathcal{M}}}\downarrow & & \downarrow \\ \overline{X(\mathcal{M})} & \xrightarrow{\ \overline{i}_{\mathcal{M}}\ } & \mathbb{R}^{\mathrm{cl}\,(C(X)\cap\mathcal{M})} \\ {\scriptstyle \psi^X_{\mathcal{M}}}\downarrow & & \downarrow \\ X(\mathcal{M}) & \xrightarrow{\ i_{\mathcal{M}}\ } & \mathbb{R}^{C(X)\cap\mathcal{M}} \end{array} \tag{2}$$

Lemma 4.1. *$\psi^X_{\mathcal{M}}$ is a bijection from $\overline{X(\mathcal{M})}$ onto $X(\mathcal{M})$.*

PROOF: Let x and y be a pair of different points of X with $\overline{\phi}^X_{\mathcal{M}}(x) \neq \overline{\phi}^X_{\mathcal{M}}(y)$. Then there exists a function $g \in \mathrm{cl}\,(C_p(X) \cap \mathcal{M})$ such that $g(x) \neq g(y)$. Let $\varepsilon = |g(x) - g(y)|/2$. Then there is a function $f \in C_p(X) \cap \mathcal{M}$ with $|f(x) - g(x)| < \varepsilon$ and $|f(y) - g(y)| < \varepsilon$. Hence, $g(x) \neq g(y)$ and $\varphi^X_{\mathcal{M}} \neq \phi^X_{\mathcal{M}}(y)$. $\square$

Remark 4.2. If X is compact, then $\psi^X_{\mathcal{M}}$ is a homeomorphism and we identify $\overline{X(\mathcal{M})}$ and $X(\mathcal{M})$. It is possible to show that $\psi^X_{\mathcal{M}}$ is a homeomorphism, if X is a Lindelöf p-space or pseudocompact.

If $f : X \to Y$ is a continuous mapping, then let $f^* : C(Y) \to C(X)$ denote the mapping induced by f.

Lemma 4.3. $(\overline{\phi}^X_{\mathcal{M}})^* C_p(\overline{X(\mathcal{M})}) = \mathrm{cl}\,(C_p(X) \cap \mathcal{M})$.

PROOF: Obviously, we have $C_p(X) \cap \mathcal{M} \subseteq (\phi^X_{\mathcal{M}})^* C_p(X(\mathcal{M}))$ and $\mathrm{cl}\,(C_p(X) \cap \mathcal{M}) \subseteq (\overline{\phi}^X_{\mathcal{M}})^* C_p(\overline{X(\mathcal{M})})$ (see diagram (2)). Since $\psi^X_{\mathcal{M}}$ is a bijection mapping,

$(\psi^X_{\mathcal{M}})^* C_p(X(\mathcal{M}))$ is a dense subspace of $C_p(\overline{X(\mathcal{M})})$ and therefore $(\phi^X_{\mathcal{M}})^* C_p(X(\mathcal{M}))$ is a dense subspace of $(\overline{\phi}^X_{\mathcal{M}})^* C_p(\overline{X(\mathcal{M})})$. Now, it is sufficient to show that $C_p(X) \cap \mathcal{M}$ is a dense subset of $(\phi^X_{\mathcal{M}})^* C_p(X(\mathcal{M}))$. Let g' be an arbitrary continuous function on $X(\mathcal{M})$ and $g = g' \circ \phi^X_{\mathcal{M}}$. Also, let $x_1, \dots, x_n$ be points of X and ε a positive real number. Choose points $y_1, \dots, y_k$ of X such that $\phi^X_{\mathcal{M}}(y_i) \neq \phi^X_{\mathcal{M}}(y_j)$ for distinct $i, j \leq k$ and $\phi^X_{\mathcal{M}}\{x_1, \dots, x_n\} = \phi^X_{\mathcal{M}}\{y_1, \dots, y_k\}$. It is enough to show that there is a function $f \in C_p(X) \cap \mathcal{M}$ with $|f(y_i) - g(y_i)| < \varepsilon$ for all $i = 1, \dots, k$. Since $\phi^X_{\mathcal{M}}(y_i) \neq \phi^X_{\mathcal{M}}(y_j)$ for distinct $i, j \leq k$, we find functionally open neighborhoods $V_1, \dots, V_k$ of $y_1, \dots, y_k$, respectively, in $\mathcal{M}$ with $\mathrm{cl}\,(V_i) \cap \mathrm{cl}\,(V_j) = \emptyset$ for distinct i, j. There exist functions $f_1, \dots, f_k$ in $C_p(X) \cap \mathcal{M}$ such that $V_i = f_i^{-1}(0,1)$ for every $i = 1, \dots, k$. Fix rational numbers $q_1, \dots, q_k$ such that $|q_i \cdot f_i(y_i) - g(y_i)| < \varepsilon$ for all $i = 1, \dots, k$. Remark that all rational numbers are elements of $\mathcal{M}$. $f = q_1 \cdot f_1 + \dots + q_k \cdot f_k$ is the desired function. $\square$

Corollary 4.4. *Let X be a compact Hausdorff space. Then*

$$(\phi^X_{\mathcal{M}})^* C_p(X(\mathcal{M})) = \mathrm{cl}\,(C_p(X) \cap \mathcal{M}).$$

A characterization of $C_p(X)$ for Corson-compact spaces

Let X be a completely regular space. We consider X with the uniform structure induced by all real-valued continuous functions on X. The following definition plays the decisive role in what follows.

Definition 5.1. One says that the completely regular space X has the property Ω, if for sufficiently large regular cardinals θ there exists a closed unbounded family C of countable elementary substructures of $\mathcal{H}(\theta)$ such that $\phi^X_{\mathcal{M}}\,\mathrm{cl}\,(X \cap \mathcal{M}) = X(\mathcal{M})$ for every $\mathcal{M} \in C$.

More briefly $X \in \Omega$, if $\phi^X_{\mathcal{M}}\,\mathrm{cl}\,(X \cap \mathcal{M}) = X(\mathcal{M})$ for every suitable countable elementary substructure $\mathcal{M}$.

It is easy to verify that every compact Hausdorff space satisfies this condition. More generally, we have the following

Proposition 5.2. *Every Lindelöf p-space has the property Ω.*

Proof: Since $\phi^X_{\mathcal{M}}(X \cap \mathcal{M})$ is always a dense subset of $X(\mathcal{M})$, it is enough to prove that $\phi^X_{\mathcal{M}}$ is a perfect mapping. X is a Lindelöf p-space if and only if there is a perfect mapping $g : X \to Y$, where Y has a countable base. By Proposition 2.5 we may assume that g is uniformly continuous. Since $\phi^Y_{\mathcal{M}}$ is a homeomorphism, g can be represented as a composition of $\phi^X_{\mathcal{M}}$ and $g_{\mathcal{M}}$. This implies that $\phi^X_{\mathcal{M}}$ is perfect. $\square$

Proposition 5.3. *Let X be a Lindelöf space, satisfying Ω. If f is a continuous mapping from X onto a completely regular space Y, then Y satisfies Ω, too.*

Proof: Like in the proof of previous proposition, we assume that f is uniformly continuous and consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi^X_{\mathcal{M}} \big\downarrow & & \big\downarrow \phi^Y_{\mathcal{M}} \\ X(\mathcal{M}) & \xrightarrow[f_{\mathcal{M}}]{} & Y(\mathcal{M}) \end{array} \tag{3}$$

It is easy to see that $f(X \cap \mathcal{M}) = Y \cap \mathcal{M}$ and, consequently, $f(\mathrm{cl}\,(X \cap \mathcal{M}) \subseteq \mathrm{cl}\,(Y \cap \mathcal{M}))$. Hence, $Y(\mathcal{M}) = f_{\mathcal{M}}(\phi^X_{\mathcal{M}}\,\mathrm{cl}\,(X \cap \mathcal{M})) = \phi^Y_{\mathcal{M}}(\mathrm{cl}\,(Y \cap \mathcal{M}))$. □

Theorem 5.4. *If $X \in \Omega$, then each compact subspace of $C_p(X)$ is a Corson-compact space.*

Proof: Let $\mathcal{M}$ be a suitable countable elementary substructure. Set $Z = C_p(X)$. By Theorem 3.1, it suffices to show that $\phi^Z_{\mathcal{M}}(f) \neq \phi^Z_{\mathcal{M}}(g)$ for any pair $f, g \in \mathrm{cl}\,(C_p(X) \cap \mathcal{M}) = (\overline{\phi}^X_{\mathcal{M}})^* C_p(\overline{X(\mathcal{M})})$, $f \neq g$. There exist continuous functions f', g' on $\overline{X(\mathcal{M})}$ such that $f = f' \circ \overline{\phi}^X_{\mathcal{M}}$ and $g = g' \circ \overline{\phi}^X_{\mathcal{M}}$. From $X \in \Omega$ it follows that $\overline{\phi}^X_{\mathcal{M}}(X \cap \mathcal{M})$ is a dense subset of $\overline{X(\mathcal{M})}$. Hence, there exists a point $x \in X \cap \mathcal{M}$ with $f'(\overline{\phi}^X_{\mathcal{M}}(x)) \neq g'(\overline{\phi}^X_{\mathcal{M}}(x))$, i.e. $f(x) \neq g(x)$ and therefore $\phi^Z_{\mathcal{M}}(f) \neq \phi^Z_{\mathcal{M}}(g)$. □

Remark 5.5. A compact subspace of a space $C_p(X)$, where X is an arbitrary compact Hausdorff space, is called an Eberlein space. Every Eberlein space is a Corson-compact space (Amir and Lindenstrauß [1]). Gul'ko proved Theorem 5.4 for Lindelöf Σ-spaces (see Negrepontis [8]), i.e. for continuous images of Lindelöf p-spaces.

Theorem 5.6. *Let X be a compact Hausdorff space. Then X is a Corson-compact space if and only if $C_p(X) \in \Omega$.*

Proof: The sufficiency follows from Theorem 5.4 and the easy fact that there is a natural embedding of X into $C_p(C_p(X))$.

To prove the necessity, let X be a Corson-compact space and let $\mathcal{M}$ be a suitable countable elementary substructure. By Theorem 3.1, $\phi^X_{\mathcal{M}}$ maps $\mathrm{cl}\,(X \cap \mathcal{M})$ homeomorphic on $X(\mathcal{M})$. Hence, for every function $f \in C_p(X)$ there is a function $h \in C_p(X(\mathcal{M}))$ such that

$$f\mid_{\mathrm{cl}\,(X\cap\mathcal{M})} = (h \circ \phi^X_{\mathcal{M}})\mid_{\mathrm{cl}\,(X\cap\mathcal{M})}\ .$$

Consequently, $\phi^Z_{\mathcal{M}}(f) = \phi^Z_{\mathcal{M}}(h \circ \phi^X_{\mathcal{M}})$, where $Z = C_p(X)$. Since, by Corollary 4.4, $h \circ \phi^X_{\mathcal{M}} \in \mathrm{cl}\,(C_p(X) \cap \mathcal{M})$, this proves that $C_p(X) \in \Omega$. □

6. Archangelskij's question

Let D_τ be the discrete space of cardinality τ. Let L_τ denote the space $D_\tau \cup \{\xi\}$, where $\xi \notin D_\tau$ is the only non-isolated point and every neighborhood of ξ has the form $\{\xi\} \cup D_\tau \setminus A$, where A is an arbitrary countable subset of D_τ. It is easy to see that L_τ is a Lindelöf space.

Theorem 6.1 (R. Pol [9], see Archangelskij [2])**.** *A compact Hausdorff space is Corson-compact if and only if $C_p(X)$ is a continuous image of a closed subspace of $(L_\tau)^\omega$ for some cardinal τ.*

It was asked by Archangelskij [2, Problem IV.3.16], whether X is Corson-compact, if $C_p(X)$ is a continuous image of a closed subspace of $(L_\tau)^\omega \times Z$ for some compact space Z.

Theorem 6.2. *Let Z and X be compact Hausdorff spaces; suppose that $C_p(X)$ can be represented as a continuous image of a closed subspace of $(L_\tau)^\omega \times Z$. Then X is Corson-compact.*

The idea to prove this theorem is very easy. By virtue of Theorem 5.6 and Proposition 5.3, it suffices to show that every closed subspace $Y \subseteq (L_\tau)^\omega \times Z$ satisfies the condition Ω. Remark that $(L_\tau)^\omega$ is always a Lindelöf space (see Step 3).

Step 1. Set $L = L_\tau$. It is easy to see that for every continuous function $f : L \to \mathbb{R}$ there is a neighborhood $O(\xi) = \{\xi\} \cup D_\tau \setminus A$, where A is a countable subset of D_τ, such that $f(t) = f(\xi)$ for every $t \in O(\xi)$. If $\mathcal{M}$ is a suitable elementary substructure and $f \in \mathcal{M}$, we may assume that $A \in \mathcal{M}$ and, by Proposition 1.2, $A \subset \mathcal{M}$. Hence, by the definition of $\phi^L_\mathcal{M}$, $\phi^L_\mathcal{M}(\xi) = \phi^L_\mathcal{M}(t)$ for every $t \in D_\tau \setminus \mathcal{M}$ and $\phi^L_\mathcal{M}(t_1) \neq \phi^L_\mathcal{M}(t_2)$ for all $t_1, t_2 \in L \cap \mathcal{M}$, $t_1 \neq t_2$. Consequently, we may identify $L(\mathcal{M})$ with $\{\xi\} \cup (D_\tau \cap \mathcal{M})$. Here all points from $D_\tau \cap \mathcal{M}$ are isolated and the neighborhoods of ξ are of the form $\{\xi\} \cup ((D_\tau \cap \mathcal{M}) \setminus A)$, where A is a countable subset of D_τ and $A \in \mathcal{M}$.

Step 2. Suppose $\eta \in L^\omega \cap \mathcal{M}$. Then, by Proposition 1.2, $\eta(n) \in \mathcal{M}$ for any $n \in \omega$. Hence, $\mathrm{cl}\,(L^\omega \cap \mathcal{M})$ is the following subspace of L^ω:

$$(\{\xi\} \cup (D_\tau \cap \mathcal{M}))^\omega.$$

The family of all subsets of L^ω of the form

$$A^{n_1,\dots,n_k}_{\eta_1,\dots,\eta_k} = \{\eta \in L^\omega : \eta(n_i) = \eta_i, \quad i = 1, \dots, k\},$$

where $n_1, \dots, n_k \in \omega$ and $\eta_1, \dots, \eta_k \in (\{\xi\} \cup (D_\tau \cap \mathcal{M}))$, is a network for $\mathrm{cl}\,(L^\omega \cap \mathcal{M})$ in L^ω. This means that for any $\eta \in \mathrm{cl}\,(L^\omega \cap \mathcal{M})$ and any open set $W \subseteq L^\omega$ with $\eta \in W$ there exists a set $A^{n_1,\dots,n_k}_{\eta_1,\dots,\eta_k}$, such that $\eta \in A^{n_1,\dots,n_k}_{\eta_1,\dots,\eta_k} \subseteq W$. Remark that $A^{n_1,\dots,n_k}_{\eta_1,\dots,\eta_k} \in \mathcal{M}$.

By Proposition 2.1, $L^\omega(\mathcal{M}) \simeq L(\mathcal{M})^\omega$ and $\phi^{L^\omega}_\mathcal{M}(\eta) = (\phi^L_\mathcal{M}(\eta(n)))_{n\in\omega}$ for every $\eta \in L^\omega$. For short, we write ϕ instead of $\phi^{L^\omega}_\mathcal{M}$. Remark that $\phi(\mathrm{cl}\,(L^\omega \cap \mathcal{M})) = L^\omega(\mathcal{M})$.

Step 3. Now we are going to prove that L^ω is a Lindelöf space for every cardinal τ (see also Archangelskij [2]). Let γ be an open cover of L^ω. W.l.o.g. we can assume that $\gamma \in \mathcal{M}$. We can also assume that every element of γ is a finite intersection of sets of the form

$$V(n,t) = \{\eta \in L^\omega : \eta(n) = t\},$$

where $n \in \omega$ and $t \in D_\tau$, or of the form

$$W(m,A) = \{\eta \in L^\omega : \eta(m) \in \{\xi\} \cup D_\tau \setminus A\},$$

where $m \in \omega$ and A is a countable subset of D_τ. If $V(n,t) \in \mathcal{M}$, then $t \in \mathcal{M}$ and, analogously, if $W(m,A) \in \mathcal{M}$ and, by Proposition 1.2, $A \subset \mathcal{M}$. Consequently, $(\phi)^{-1}\phi V(n,t) = V(n,t)$ and $(\phi)^{-1}\phi W(n,t) = W(n,t)$. Hence, $(\phi)^{-1}\phi U = U$ for every $U \in \gamma \cap \mathcal{M}$.

Now we claim that $\gamma \cap \mathcal{M}$ is a countable subcover of γ. By the result of Step 2, there exists a system σ of subsets of L^ω satisfying the following conditions:

(a) $\sigma \subset \mathcal{M}$,
(b) $\mathrm{cl}\,(L^\omega \cap \mathcal{M}) \subseteq \cup\sigma$,
(c) for every $A \in \sigma$ there exists a $U \in \gamma \cap \mathcal{M}$ with $A \subseteq U$.

Hence, $\cup(\gamma \cap \mathcal{M}) = \cup\{(\phi)^{-1}\phi U : U \in \gamma \cap \mathcal{M}\} = (\phi)^{-1}\phi(\cup(\gamma \cap \mathcal{M})) = L^\omega$.

Step 4. Let Z be a compact Hausdorff space and let Y be a closed subspace of $L^\omega \times Z$. Since $L^\omega \times Z$ is a Lindelöf space, Y is Lindelöf, too. By Proposition 2.3, we may think Y with the uniform structure induced by the uniform structure on $L^\omega \times Z$. Let $\mathcal{M}$ be a suitable countable elementary substructure and let h denote the product of the mappings ϕ and $\phi^Z_{\mathcal{M}}$:

$$h : L^\omega \times Z \longrightarrow L(\mathcal{M})^\omega \times Z(\mathcal{M}).$$

It suffices to prove that $h(\mathrm{cl}\,(Y \cap \mathcal{M})) = h(Y)$ (see Fact 2.3).

Step 5. Let $x = \langle \eta, z\rangle$ be a point of $Y \subseteq L^\omega \times Z$. We define a point $\overline{\eta} \in L^\omega$ by setting $\overline{\eta}(n) = \eta(n)$, if $\eta(n) \in \mathcal{M}$, and $\overline{\eta}(n) = \xi$, if $\eta(n) \notin \mathcal{M}$. Set $A_z = (\phi^Z_{\mathcal{M}})^{-1}\phi^Z_{\mathcal{M}}(z)$. Remark that $h(\overline{\eta}, z') = h(\eta, z)$ for every $z' \in A_z$. Now, it is enough to prove that $\{\overline{\eta}\} \times A_z \cap \mathrm{cl}\,(Y \cap \mathcal{M}) \neq \emptyset$.

(a) At first, we prove that $\{\overline{\eta}\} \times A_z \cap Y \neq \emptyset$.

Assume, on the contrary, that $\{\overline{\eta}\} \times A_z \cap Y = \emptyset$. Since $\{\overline{\eta}\} \times A_z$ is compact, there exist open sets $W \subseteq L^\omega$ and $V \subseteq Z$ such that $\overline{\eta} \in W$, $A_z \subseteq V$ and $W \times V \cap Y = \emptyset$. We may assume that $V \in \mathcal{M}$ (see Fact 2.4). Further, assume that W is a member of the canonical base for L^ω. Then there is a natural number n such that W depends only on the coordinates $i \leq n$. Let $\{i_1, \dots, i_k\} = \{i \leq n : \overline{\eta}(i) = \xi\}$, where $i_1 < i_2 < \cdots < i_k \leq n$. If $i \in n \setminus \{i_1, \dots, i_k\}$, then $\overline{\eta}(i) = \eta(i) = a_i \in \mathcal{M}$. For every $j \in \{1, \dots, k\}$, we can fix a countable set $A_j \subseteq D_\tau$ such that (w.l.o.g.) W is the set of all $\xi \in L^\omega$ such that $\xi(i) = a_i$ for all $i \leq n$, $i \notin \{i_1, \dots, i_k\}$, and

$\xi(i_j) \notin A_j$ for all $j \in \{1, \ldots, k\}$. Let H denote the set of all countable subsets of D_τ. For the sake of simplicity, we define for every collection $B_1, \ldots, B_k \in H$

$$W(B_1, \ldots, B_k) = \{\xi \in L^\omega : \xi(i) = a_i \text{ for } i \leq n,\ i \notin \{i_1, \ldots, i_k\} \text{ and } \xi(i_j) \notin B_j \text{ for } j = 1, \ldots, k\}.$$

The following assertion is true:

$$(\exists\, B_1, \ldots, B_k \in H)\ (W(B_1, \ldots, B_k) \times V \cap Y = \emptyset).$$

Since $n, i_1, \ldots, i_k, V$ and Y are elements of $\mathcal{M}$, we find $\overline{B}_1, \ldots, \overline{B}_k \in H \cap \mathcal{M}$ satisfying this condition. If $\overline{B}_j \in \mathcal{M}$, then $\overline{B}_j \subset \mathcal{M}$. Since, by the definition of $\overline{\eta}$, $\eta(i_j) \notin D_\tau \cap \mathcal{M}$, $\eta(i_j) \notin \overline{B}_j$ for every $j \in \{1, \ldots, k\}$. Now, it is easy to see that $\eta \in W(\overline{B}_1, \ldots, \overline{B}_k)$. Hence, $x \in W(\overline{B}_1, \ldots, \overline{B}_k) \times V$, contradicting $W(\overline{B}_1, \ldots, \overline{B}_k) \times V \cap Y = \emptyset$.

(b) Now, we are going to prove that $\{\overline{\eta}\} \times A_z \cap \mathrm{cl}\,(Y \cap \mathcal{M}) \neq \emptyset$. In assuming that the intersection is empty, we find open sets $W \subseteq L^\omega$ and $V \subseteq Z$, $V \in \mathcal{M}$, such that $\overline{\eta} \in W, A_z \subset V$ and $W \times V \cap \mathrm{cl}\,(Y \cap \mathcal{M}) = \emptyset$. There exists a natural number n such that

$$C = \{\vartheta \in L^\omega : \vartheta(i) = \overline{\eta}(i) \text{ for } i = 1, \ldots, n\} \subseteq W.$$

Remark that $C \in \mathcal{M}$ and $\overline{\eta} \in C$. From $C \subseteq W$ it follows that $C \times V \cap (Y \cap \mathcal{M}) = \emptyset$. Since C, V and Y are elements of $\mathcal{M}$, this implies that $C \times V \cap Y = \emptyset$. Hence, $\{\overline{\eta}\} \times A_z \cap Y = \emptyset$, contradicting the result of (a). This completes the proof of Theorem 6.2.

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