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On Cohen-Macaulay rings

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Abstract. In this paper, we use a characterization of R-modules N such that $fd_RN = pd_RN$ to characterize Cohen-Macaulay rings in terms of various dimensions. This is done by setting N to be the dth local cohomology functor of R with respect to the maximal ideal where d is the Krull dimension of R.

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Classification: 13C14, 13D45, 13H10, 18G10

1. Introduction

R will denote an associative ring with a unit element, R-module will mean left R-module, and noetherian will mean left noetherian.

Let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of R-modules. Then we recall that if A and B are objects in $\mathcal{A}$ and $\mathcal{B}$ respectively, then the *A-injective dimension* of B (denoted $A - idB$) or *B-projective dimension* of A (denoted $B - pdA$) is the smallest nonnegative integer n such that $\mathrm{Ext}^i_R(A, B) = 0$ for all $i > n$. Otherwise, we set $A - idB = B - pdA = \infty$.

We define $A - id\mathcal{B}$ to be the $\sup\{A - idB : B \in \mathcal{B}\}$. Note that $A - id\mathcal{B} = \mathcal{B} - pdA$. Similarly, $\mathcal{A} - idB = B - pd\mathcal{A}$ can be defined. If $\mathcal{A} - idB = 0$, we will say that B is *$\mathcal{A}$-injective*. We define the *$\mathcal{A}$-injective dimension* of $\mathcal{B}$ (denoted by $\mathcal{A} - id\mathcal{B}$) to be $\sup\{A - idB : A \in \mathcal{A}, B \in \mathcal{B}\}$.

Likewise, if $\mathcal{A}$ is a subcategory of right R-modules and $\mathcal{B}$ is a subcategory of left R-modules, then we can define $\mathcal{A} - fdB$ and $A - fd\mathcal{B}$ using $\mathrm{Tor}^R_i(A, B)$. B is *$\mathcal{A}$-flat* if $\mathcal{A} - fdB = 0$.

Now let M be an R-module and $\mathcal{A}$ be a full subcategory of R-modules. Then a map $\psi : M \to A$ with A in $\mathcal{A}$ is said to be an *$\mathcal{A}$-preenvelope* of M if any diagram

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d] & A \arrow[dl, dotted] \\ A' & \end{tikzcd}$$

with A' in $\mathcal{A}$ can be completed.

If $\mathcal{A}$ contains all injective R-modules, then the preenvelopes are monomorphisms. So that in the case $\mathcal{A}$-preenvelopes exist for all R-modules, we can define an *$\mathcal{A}$-resolution* of M to be an exact sequence

$$0 \to M \to A^\circ \to A^1 \to \dots \text{ where}$$

$M \to A^\circ$, $Coker(M \to A^\circ) \to A^1$, $Coker(A^{n-1} \to A^n) \to A^{n+1}$ for $n \geq 1$ are $\mathcal{A}$-preenvelopes. We will say that M has *$\mathcal{A}$-resolution dimension* (denoted $\mathcal{A} - rndimM) \leq n$ if there is an $\mathcal{A}$-resolution $0 \to M \to A^\circ \to A^1 \to \dots \to A^n \to 0$. The *$\mathcal{A}$-resolution global dimension* of R (denoted by $\mathcal{A} - rngldimR$) is to be the $\sup\{\mathcal{A} - rndimM : M \in Mod\}$ where Mod is the category of R-modules.

If the preenvelopes are not necessarily exact, we get a sequence, not necessarily exact, called an *$\mathcal{A}$-resolvent* of M, and so *$\mathcal{A}$-resolvent dimension* (denoted $\mathcal{A} - rtdim$) and $\mathcal{A} - rtgldimR$ can be defined similarly.

We start in Section 2 by extending the results in Enochs-Jenda [3] to an arbitrary ring R. In particular, we get a characterization of R-modules N such that $fd_R N = pd_R N$ in terms of the various dimensions defined above (Theorem 2.1). By choosing an appropriate N, this theorem specializes to n-Gorenstein rings (Corollary 2.3) and Cohen-Macaulay local rings (Theorem 3.7).

If (R, m, k) is a commutative noetherian local ring of Krull dimension d and M is a finitely generated R-module, then $H^i_m(M)$ denotes the i^{th} local cohomology functor with respect to the maximal ideal m. A finitely generated R-module K is said to be a *canonical module* of R if the completion of K_R with respect to the m-adic topology

$$\hat{K}_R \cong \mathrm{Hom}(H^d_m(R), E(k))$$

where $E(k)$ denotes the injective envelope of k (see Herzog-Kunz [7]).

A finitely generated R-module M is said to be *maximal Cohen-Macaulay* if depth $M = d$. If R is a Cohen-Macaulay ring with a canonical module, then every finitely generated R-module M has a *maximal Cohen-Macaulay precover* (see Auslander-Buchweitz [1] or Yoshino [13]), that is, a surjective map $\psi : C \to M$ with C maximal Cohen-Macaulay such that any diagram

$$\begin{tikzcd} & C' \arrow[d] \arrow[dl, dotted] \\ C \arrow[r] & M \end{tikzcd}$$

with C' maximal Cohen-Macaulay can be completed.

We can therefore form a *Cohen-Macaulay* resolution

$$\dots \to C_1 \to C_0 \to M \to 0 \text{ where}$$
$$C_0 \to M, C_1 \to \mathrm{Ker}(C_0 \to M), C_{n+1} \to \mathrm{Ker}(C_n \to C_{n-1}),\ n \geq 1$$

are maximal Cohen-Macaulay precovers. If there is a Cohen-Macaulay resolution $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to M \to 0$, we say that M has *Cohen-Macaulay dimension* (denoted $CM - \dim$) $\leq n$. We define the *Cohen-Macaulay global dimension* of R (denoted $CM - gldimR$) to be the $\sup\{CM - dimM : M \in \mathcal{FG}Mod\}$ where $\mathcal{FG}Mod$ denotes the full subcategory of finitely generated R-modules.

The aim of Section 3 is to give characterizations of Cohen-Macaulay rings in terms of the local cohomology functor and the various dimensions that we have defined above. One of the consequences of this section is that the length of a Cohen-Macaulay resolution of a finitely generated R-module does not exceed the Krull dimension of R when R has a canonical module.

In this paper, $\mathrm{Ext}^i(A,B)$, $\mathrm{Tor}_i(A,B)$ will denote $\mathrm{Ext}^i_R(A,B)$, $\mathrm{Tor}^R_i(A,B)$ respectively, and for a local ring (R,m,k), the Matlis dual $\mathrm{Hom}(M,E(k))$ will be denoted by M^v where $E(k)$ is the injective envelope of k.

2. Resolutions and resolvents

Let N be a fixed R-module. Then $\mathcal{A}_N$ will denote the full subcategory of all N-injective R-modules and $\mathcal{B}_N$ will denote the full subcategory of all N-flat right R-modules.

In [3], we showed the existence of copure injective preenvelopes over noetherian rings, and copure flat preenvelopes over commutative artinian rings. For an arbitrary ring R, the same proofs show the existence of $\mathcal{A}_N$-preenvelopes, and $\mathcal{B}_N$-preenvelopes in the case N is of finite type for then $\mathrm{Tor}_i(-,N)$ preserves direct products by Lenzing [10]. So straight forward modifications to the proofs of the results in Section 3 and Theorem 4.1 of [3] give the following result which holds for any ring R.

Theorem 2.1. *Let N be an R-module such that $fdN = pdN$. Then the following are equivalent for an integer n.*

(1) $pdN \leq n$.
(2) $\mathcal{A}_N - rngldimR \leq n$.
(3) *Every n^{th} cosyzygy of an R-module is in $\mathcal{A}_N$.*
(4) $N - fdMod_R \leq n$.
(5) $N - fd\mathcal{FG}Mod_R \leq n$.
(6) *Every n^{th} syzygy of a right R-module is in $\mathcal{B}_N$.*

Furthermore, if N is of finite type, then each of the above statements is equivalent to (7) $\mathcal{B}_N - rtgldimR \leq n$.

To see that Theorem 4.1 of [3] for n-Gorenstein rings (that is, R is left and right noetherian and is of finite injective dimension at most n over itself on either side) is a consequence of the above theorem, one observes the following:

Proposition 2.2. *Let R be noetherian and $\{X_\alpha\}$ be a representative set of indecomposable injective R-modules. Set $X = \oplus X_\alpha$. Then the following are* *equivalent for an integer n.*

(1) $idR_R = n$.
(2) $pd_R X = n$.
(3) $fd_R X = n$.

PROOF: $1 \Leftrightarrow 2$. Suppose $idR_R = n$. Then $pd_R X \leq n$ by Jensen [9, Theorem 5.9]. If $pd_R X < n$, then $pd_R E < n$ for all injective R-modules E since $E = \oplus X'_\beta$ where $X'_\beta \in \{X_\alpha\}$, and so $idR_R < n$ by Jensen [9]. So $pd_R X = n$, and conversely.

$1 \Leftrightarrow 3$. $idR_R = n$ implies that $fd_R X \leq n$ by Enochs-Jenda [4, Theorem 4.4] and so $fd_R X = n$ as above, and conversely. □

Now we simply note that $\mathcal{A}_X$-injective dimension is the copure injective dimension (*cid*), X-flat dimension is the copure flat dimension (cfd), and $\mathcal{B}_X$-resolvent dimension is the copure flat resolvent dimension. Furthermore, R is n-Gorenstein if and only if $pd_R X \leq n$ and $pdX_R \leq n$. So if we set $N = X$ in Theorem 2.1 above, we get the following result using Proposition 2.2 above.

Corollary 2.3 ([3, Theorem 4.1]). *The following are equivalent for a left and right noetherian ring R.*

(1) *R is n-Gorenstein.*
(2) *$cidM \leq n$ for all R-modules (left and right) M.*
(3) *Every nth cosyzygy of an R-modules (left and right) is in $\mathcal{A}_X$.*
(4) *$cfdM \leq n$ for all R-modules (left and right) M.*
(5) *$cfdM \leq n$ for all finitely generated R-modules (left and right) M.*
(6) *Every nth syzygy of an R-module (left and right) is in $\mathcal{B}_X$.*

Furthermore, if R is commutative artinian, then each of the above statements is equivalent to

(7) *Copure flat resolvent dimension of each R-module is at most n.*

Remark. We note that if R is a commutative artinian ring, then R_p is quasi-Frobenius for each prime ideal P of R. Therefore R is quasi-Frobenius and so $n = 0$ in this case.

3. Local rings

Throughout this section, R will denote a commutative noetherian local ring with maximal ideal m and residue field k.

We start with the following.

Lemma 3.1. *The following are equivalent for a ring R and integer $d \geq 1$*

(1) *R is Cohen-Macaulay of dimension d.*
(2) *$fdH^d_m(R) = d$.*
(3) *$pdH^d_m(R) = d$.*

PROOF: $1 \Rightarrow 2$. See Strooker [12, Proposition 9.1.4].

$2 \Rightarrow 1$. $H_m^d(R)$ is artinian and so is an $\hat{R}$-module naturally. So $fdH_m^d(R) = d$ implies that $fd_{\hat{R}}H_m^d(R) = d$ and thus $id_{\hat{R}}H_m^d(R)^v = d$. Therefore, $H_m^d(R)^v$ is a noetherian $\hat{R}$-module of finite injective dimension. Thus $\hat{R}$ is Cohen-Macaulay (see Strooker [12, Theorem 13.1.7]) and so R is Cohen-Macaulay. Furthermore, the dimension is d for otherwise $H_m^d(R) = 0$.

$2 \Rightarrow 3$. $fdH_m^d(R) = d$ implies that Krull $dimR = d$ by the above. So $pdH_m^d(R) \leq d$ by Foxby [5, Corollary 3.4]. So $d = fdH_m^d(R) \leq pdH_m^d(R) \leq d$. Thus $pd_m^d(R) = d$.

$3 \Rightarrow 2$ is trivial since $fd \leq pd$. □

For $d = 0$, we have the following which is surely known and we present it here for completeness.

Lemma 3.2. *The following are equivalent for a ring R.*

(1) *R is artinian.*
(2) *$H_m^0(R) = R$.*
(3) *$H_m^0(R) \neq 0$ and $H_m^0(R)$ is flat.*
(4) *$H_m^0(R) \neq 0$ and $H_m^0(R)$ is projective.*

Proof: $1 \Rightarrow 2, 3, 4$.

$$\begin{aligned} H_m^0(R)^v &= \mathrm{Hom}(\varinjlim \mathrm{Hom}(R/m^t, R), E(k)) \\ &= \varprojlim \mathrm{Hom}(\mathrm{Hom}(R/m^t, R), E(k)) \\ &= \varprojlim R/m^t \otimes E(k) \\ &= E(k) \end{aligned}$$

since R is complete. Thus $H_m^0(R)$ is nonzero and flat. But then $H_m^0(R)$ is free and so $H_m^0(R) = R$.

$2 \Rightarrow 1$ $H_m^0(R)^v = E(k)$ is noetherian and so R is artinian.

$3 \Rightarrow 1$ follows as in Lemma 3.1 and $4 \Rightarrow 3$ is trivial. □

Corollary 3.3. *R is Gorenstein if and only if*

$$fd_{R_p}E(k(P)) = pd_{R_p}E(k(P)) = htP$$

for all $P \in SpecR$ where $k(P)$ is the quotient ring of R/p.

Proof: We first recall that R-Gorenstein means that $idR_p < \infty$ for all $P \in SpecR$ (see Bass [2]). If R_p has finite injective dimension, then $H_{mR_p}^d(R_P) = E(k(P))$ where $d =$ Krull $dimR_p = htP$. So the result follows from the Lemmas above. Conversely, if $fd_{R_p}E(k(P)) = htP$, then $id\hat{R}_p = htP < \infty$, and so $idR_p < \infty$. □

Now let $\mathcal{I}$ be the full subcategory of finitely generated R-modules with finite injective dimension. We state the following, noting that if R is Cohen-Macaulay, then $\mathcal{I} \neq 0$.

Lemma 3.4. *Let R be Cohen-Macaulay. Then the following are equivalent for a finitely generated R-module M.*

(1) *M is a maximal Cohen-Macaulay R-module.*
(2) *Every R-module in $\mathcal{I}$ is M-injective.*
(3) *$\mathcal{I}$ has a nonzero M-injective R-module.*

Furthermore, if R has a canonical module K, then each of the above statements is equivalent to

(4) *K is M-injective.*
(5) *$\hat{K}$ is M-injective.*

Proof: $1 \Leftrightarrow 2$. We recall that $idI = depthR$ for each $I \in \mathcal{I}, I \neq 0$. Furthermore, $depthM + M - idI = idI$ (see Roberts [11]). So the result follows.

Similarly (3) implies (1), and (3) follows from (2) trivially.

$1 \Leftrightarrow 4$. We use the local duality $\mathrm{Ext}^i(M,K) \otimes_R \hat{R} \cong \mathrm{Hom}_R(H_m^{d-i}(M), E(k))$ (see Yoshino [13, Proposition 1.12] or Grothendieck [6, Theorem 6.3]). M is maximal Cohen-Macaulay if and only if $H_m^{d-i}(M) = 0$ for all $i > 0$ and so if and only if $\mathrm{Ext}^i(M,K) = 0$ for $i > 0$.

$4 \Leftrightarrow 5$. We simply note that $\mathrm{Ext}^i(M,K) \otimes_R \hat{R} \cong \mathrm{Ext}^i(M, \hat{K}_R)$ by Ishikawa [8, Corollary 1.2], and so the result follows. $\square$

Now let $\mathcal{C}$ be the full subcategory of $\mathcal{FG}$ *Mod* consisting of all maximal Cohen-Macaulay R-modules, and $\overline{\mathcal{I}}$ be the full subcategory of $\mathcal{FG}$ *Mod* consisting of all $\mathcal{C}$-injective R-modules. It follows from Lemma 3.4 above that if R is Cohen-Macaulay, then $\mathcal{I}$ is a full subcategory of $\overline{\mathcal{I}}$.

If $I \in \overline{\mathcal{I}}$ and $0 \to I \to E^\circ \to E' \to \cdots$ is an injective resolution of I, then $0 \to \mathrm{Hom}(C,I) \to \mathrm{Hom}(C,E^\circ) \to \mathrm{Hom}(C,E') \to \cdots$ is exact for all C in $\mathcal{C}$. Furthermore, if $\cdots \to C_1 \to C_0 \to M \to 0$ is a Cohen-Macaulay resolution of a finitely generated R-module M, then $0 \to \mathrm{Hom}(M,E) \to \mathrm{Hom}(C_0,E) \to \cdots$ is exact for each injective E. So $\mathrm{Hom}(-,-)$ is right balanced by $(\mathcal{C}, Inj)$ on $\mathcal{FG}Mod \times \overline{\mathcal{I}}$ (see Enochs-Jenda [4]). So we obtain right derived functors $\overline{\mathrm{Ext}}^i(M,I)$. We note that $\overline{\mathrm{Ext}}^i(M,I) = \mathrm{Ext}^i(M,I)$.

We are now in a position to prove the following.

Theorem 3.5. *The following are equivalent for a ring R with a canonical module.*

(1) *R is Cohen-Macaulay of dimension d.*
(2) *Every finitely generated R-module has a maximal Cohen-Macaulay precover and $CM - gldimR = d$.*
(3) *$\mathcal{I} \neq 0$ and $\sup_{I \in \overline{\mathcal{I}}}\{idI\} = d$.*

Proof: $1 \Rightarrow 2$. The first part was mentioned in Section 1. Now let K be the canonical module. Then $id\hat{K}_R = d$ by Lemmas 3.1 and 3.2 since $\hat{K}_R \cong H_m^d(R)^v$. But then $idK_R = d$ since $\hat{R}$ is faithfully flat. Now consider a Cohen-Macaulay resolution $\cdots \to C_1 \to C_0 \to M \to 0$ of a finitely generated R-module M. Let

$0 \to T_{i+1} \to C_i \to T_i \to 0$ where $i \geq n$ be the short exact sequence. Then we have $\mathrm{Ext}^1(T_i, K) \cong \mathrm{Ext}^2(T_{i-1}, K) \cong \cdots \cong \mathrm{Ext}^{i+1}(M, K)$ since $\mathrm{Ext}^i(C, K) = 0$ for $i > 0$ for all maximal Cohen-Macaulay R-modules C by Lemma 3.4. But $\mathrm{Ext}^{i+1}(M, K) = 0$ for all $i \geq d$ since $idK = d$. So $\mathrm{Ext}^1(T_i, K) = 0$ for all $i \geq d$. But we also have that $\mathrm{Ext}^j(T_d, K) \cong \mathrm{Ext}^{j-1}(T_{d+1}, K) \cong \cdots \cong \mathrm{Ext}^1(T_{d+j-1}, K)$ for $j \geq 1$. So $\mathrm{Ext}^j(T_d, K) = 0$ for all $j \geq 1$. Therefore, T_d is maximal Cohen-Macaulay, again by Lemma 3.4. Thus $CM - gldimR \leq d$.

Suppose $CM - gldimR = n < d$. Then $\overline{\mathrm{Ext}}^i(M, I) = 0$ for all $i > n$ and for all $M \in \mathcal{FG}Mod, I \in \overline{\mathcal{I}}$. But $K \in \overline{\mathcal{I}}$. So $\mathrm{Ext}^i(M, K) = \overline{\mathrm{Ext}}^i(M, K) = 0$ for all $i > n$ and for all $M \in \mathcal{FG}Mod$. Thus $idK \leq n < d$, a contradiction.

$2 \Rightarrow 3$. Let $C \to R \to 0$ be a maximal Cohen-Macaulay precover. Then R is a direct summand of C. So depth $C \leq$ depth R. But depth $C = \dim R$. So R is Cohen-Macaulay. Thus $\mathcal{I} \neq 0$.

$CM - gldimR = d$ implies that $\overline{\mathrm{Ext}}^i(M, I) = 0$ for all $i > d$ for all $\mathcal{FG}ModM$, $I \in \overline{\mathcal{I}}$. So if $I \in \overline{\mathcal{I}}$, then $idI \leq d$. Thus $\sup_{I \in \overline{\mathcal{I}}}\{idI\} \leq d$. If it were less than d, then it is easy to see that $CM - gldimR < d$.

$3 \Rightarrow 1$ $\mathcal{I} \neq 0$ means R is Cohen-Macaulay. So let Krull $dimR = n$. Then $\sup_{I \in \overline{\mathcal{I}}}\{idI\} = n$ since $1 \Rightarrow 3$. So Krull $dimR = d$. □

Remark. It follows from part (3) of the theorem above that if R is a Cohen-Macaulay ring with a canonical module, then $\mathcal{I} = \overline{\mathcal{I}}$.

Corollary 3.6 (Auslander-Buchweitz [1]). *Let R be a Cohen-Macaulay ring with a canonical module. Then in $\mathcal{FG}Mod$, the full subcategories $\mathcal{C}$ and $\mathcal{I}$ are orthogonal. In particular, $\mathcal{C} = ^{\perp}(\mathcal{C})^{\perp}$ and $\mathcal{I} = (^{\perp}\mathcal{I})^{\perp}$.*

Proof: $\mathcal{C} - id\mathcal{I} = 0$ by Lemma 3.4. Furthermore, if C is a finitely generated R-module such that $C - id\mathcal{I} = 0$, then $C \in \mathcal{C}$, by the same lemma. So $\mathcal{C}$ consists precisely of all R-modules C in $\mathcal{FG}Mod$ such that $C - id\mathcal{I} = 0$. So $\mathcal{C} = ^{\perp}\mathcal{I}$. But by the preceding remark, $\mathcal{I}$ consists of precisely of R-modules I in $\mathcal{FG}Mod$ such that $\mathcal{C} - idI = 0$. So $\mathcal{I} = \mathcal{C}^{\perp}$. Thus $\mathcal{C}$ and $\mathcal{I}$ are orthogonal. So $\mathcal{C} = ^{\perp}\mathcal{I} = ^{\perp}(\mathcal{C}^{\perp})$ and $\mathcal{I} = (^{\perp}\mathcal{I})^{\perp}$. □

We now finally have the following version of Theorem 2.1 for Cohen-Macaulay rings.

Theorem 3.7. *The following are equivalent for a ring R and for an integer $d \geq 1$.*

(1) *R is Cohen-Macaulay of dimension d.*
(2) $H_m^d(R) - idMod = d$.
(3) $\mathcal{A}_{H_m^d(R)} - rngldimR = d$.
(4) $H_m^d(R) - fdMod = d$.
(5) $H_m^d(R) - fd\mathcal{FG}Mod = d$.

Furthermore, if R has a canonical module, then each of the above statements is equivalent to

(6) *Every finitely generated R-module has a maximal Cohen-Macaulay precover and $CM - gldimR = d$.*

PROOF: The equivalence of 1 to 5 follows from Theorem 2.1 and Lemma 3.1 above.

$1 \Leftrightarrow 6$ is part of Theorem 3.5. □

For $d = 0$, we have the following which easily follows from Lemma 3.2 and Theorem 3.5.

Proposition 3.8. *The following are equivalent for a ring R.*

(1) *R is artinian.*
(2) *$H^0_m(R) \neq 0$ and every R-module is $H^0_m(R)$-flat.*
(3) *$H^0_m(R) \neq 0$ and every R-module is $H^0_m(R)$-injective.*
(4) *Every finitely generated R-module is maximal Cohen-Macaulay.*

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