Two cardinal inequalities for functionally Hausdorff spaces

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Abstract. In this paper, two cardinal inequalities for functionally Hausdorff spaces are established. A bound on the cardinality of the $\tau\theta$-closed hull of a subset of a functionally Hausdorff space is given. Moreover, the following theorem is proved: if X is a functionally Hausdorff space, then $|X| \leq 2^{\chi(X)wcd(X)}$.

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A space X is said to be functionally Hausdorff if whenever $x \neq y$ in X there is a continuous real valued function f defined on X such that $f(x) = 0$ and $f(y) = 1$. A well-known Arkhangel'skii's theorem states that if X is a Hausdorff space, then $|X| \leq 2^{\chi(X)L(X)}$ ([1], [6]). Bella and Cammaroto [2] established some cardinal inequalities for Urysohn spaces that improve, for non regular spaces, the Arkhangel'skii's formula. In this paper, a bound on the cardinality of the $\tau\theta$-closed hull of a subset of a functionally Hausdorff space and a bound on the cardinality of a functionally Hausdorff space are given. We refer the reader to [3] and [4] for notations and definitions not explicitly given. All topological spaces considered here are assumed to be infinite. Let E be a set; the cardinality of E is denoted by $|E|$, $\mathcal{P}_k(E)$ is the collection of all subsets of E of cardinality $\leq k$. $\chi(X)$ and $L(X)$ denote respectively the character and the Lindelöf degree of a space X.

Definition 1 [5]**.** *Let A be a subset of a space X. A is called τ-open if A is a union of cozero-sets of X. The τ-closure of A, denoted by $\mathrm{cl}_\tau(A)$, is the set of all points $x \in X$ such that any cozero-set neighbourhood of x intersects A. The τ-interior of A, denoted by $\mathrm{int}_\tau(A)$, is the set of all x such that there is a cozero-set neighbourhood of x contained in A.*

Definition 2. *Let X be a topological space and A a subset of X. The $\tau\theta$-closure of A, denoted by $\mathrm{cl}_{\tau\theta}(A)$, is the set of all points $x \in X$ such that $\mathrm{cl}_\tau(V) \cap A \neq \emptyset$ for every open neighbourhood V of x. A is said to be $\tau\theta$-closed if $A = \mathrm{cl}_{\tau\theta}(A)$.*

As pointed to me by S. Watson, the $\tau\theta$-closure is not in general idempotent.

Definition 3. *Let X be a topological space and A a subset of X. The $\tau\theta$-closed hull of A, denoted by $[A]_{\tau\theta}$, is the smallest $\tau\theta$-closed subset of X containing A.*

Clearly, $[A]_{\tau\theta} = \bigcap\{F : A \subset F \text{ and } \mathrm{cl}_{\tau\theta}(F) = F\}$. For every space X and every $A \subset X$ we have $\overline{A} \subset \mathrm{cl}_{\tau\theta}(A) \subset [A]_{\tau\theta} \subset \mathrm{cl}_{\tau}(A)$. It is obvious that if X is a Tychonoff space, then $\overline{A} = \mathrm{cl}_{\tau\theta}(A) = [A]_{\tau\theta} = \mathrm{cl}_{\tau}(A)$ for any $A \subset X$.

The next result gives some conditions on a functionally Hausdorff space which are equivalent to $\mathrm{cl}_{\tau\theta} = \mathrm{cl}_{\tau}$.

Proposition 4. *For a functionally Hausdorff space X the following conditions are equivalent:*

(i) *For each τ-open set V of X, $\overline{V} = \mathrm{cl}_{\tau}(V)$.*
(ii) *For each open set G of X, $G \subset \mathrm{int}_{\tau}(\mathrm{cl}_{\tau}(G))$.*
(iii) *For each subset A of X, $\mathrm{cl}_{\tau\theta}(A) = \mathrm{cl}_{\tau}(A)$.*
(iv) *For each τ-open subset V of X, $\mathrm{cl}_{\tau\theta}(V) = \mathrm{cl}_{\tau}(V)$.*

Proof: (i) $\Leftrightarrow$ (ii) Lemma 28 in [9]. (ii) $\Rightarrow$ (iii) Let $A \subset X$ and $x \notin \mathrm{cl}_{\tau\theta}(A)$, then there is an open neighbourhood G of x such that $\mathrm{cl}_{\tau}(G) \cap A = \emptyset$. By hypothesis $G \subset \mathrm{int}_{\tau}(\mathrm{cl}_{\tau}(G))$, then there is a cozero set V such that $x \in V \subset \mathrm{cl}_{\tau}(G)$, so $V \cap A = \emptyset$ and $x \notin \mathrm{cl}_{\tau}(A)$. Hence, $\mathrm{cl}_{\tau\theta}(A) = \mathrm{cl}_{\tau}(A)$. (iii) $\Rightarrow$ (iv) is obvious. (iv) $\Rightarrow$ (i) Let V be a τ-open subset of X, by hypothesis $\mathrm{cl}_{\tau\theta}(V) = \mathrm{cl}_{\tau}(V)$. Now let $x \notin \overline{V}$, then there is an open set G such that $x \in G$ and $G \cap V = \mathrm{cl}_{\tau}(V)$. Since V is τ-open, we have $\mathrm{cl}_{\tau}(G) \cap V = \emptyset$, hence $x \notin \mathrm{cl}_{\tau\theta}(V)$. Therefore, $\overline{V} = \mathrm{cl}_{\tau\theta}(V) = \mathrm{cl}_{\tau}(V)$. □

Remark 5. A functionally Hausdorff space X is called weakly absolutely closed [8] provided that every τ-open filter base on X has an adherent point. An SW space is a functionally Hausdorff space X such that every point-separating subalgebra of $C^*(X)$ which contains the constants is uniformly dense in $C^*(X)$ [8]. It is worth noting that by Lemma 25 in [9] and Proposition 4, a functionally Hausdorff space X is weakly absolutely closed iff it is an SW space and $\mathrm{cl}_{\tau\theta}(A) = \mathrm{cl}_{\tau}(A)$ for every $A \subset X$.

The following result gives an upper bound on the $\tau\theta$-closed hull.

Theorem 6. *Let X be a functionally Hausdorff space. If A is a subset of X, then $|[A]_{\tau\theta}| \leq |A|^{\chi(X)}$.*

Proof: Let $m = \chi(X)$ and $k = |A|$. For each $x \in X$ let $\mathcal{B}(x)$ be a base for X at the point x such that $|\mathcal{B}(x)| \leq m$. If $x \in \mathrm{cl}_{\tau\theta}(A)$, choose a point in $\mathrm{cl}_{\tau}(U) \cap A$ for every $U \in \mathcal{B}(x)$ and let B_x be the set so obtained. Clearly, $x \in \mathrm{cl}_{\tau\theta}(B_x)$ and $|B_x| \leq m$. Let $\mathcal{G}_x = \{\mathrm{cl}_{\tau}(U) \cap B_x : U \in \mathcal{B}(x)\}$. For every $U \in \mathcal{B}_x$ we have $x \in \mathrm{cl}_{\tau\theta}(\mathrm{cl}_{\tau}(U) \cap B_x)$, in fact, if $V \in \mathcal{B}(x)$ let $W \in \mathcal{B}(x)$ such that $W \subset V \cap U$, then

$$\emptyset \neq \mathrm{cl}_{\tau}(W) \cap B_x \subset \mathrm{cl}_{\tau}(V \cap U) \cap B_x \subset \mathrm{cl}_{\tau}(V) \cap (\mathrm{cl}_{\tau}(U) \cap B_x).$$

Since X is functionally Hausdorff, then $\bigcap\{\mathrm{cl}_{\tau\theta}(\mathrm{cl}_{\tau}(U) \cap B_x) : U \in \mathcal{B}(x)\} = \{x\}$, in fact let $y \neq x$, then there exist open sets G and H such that $x \in G$, $y \in H$ and $\mathrm{cl}_{\tau}(G) \cap \mathrm{cl}_{\tau}(H) = \emptyset$, now let $U \in \mathcal{B}(x)$ such that $U \subset G$, then $\mathrm{cl}_{\tau}(H) \cap \mathrm{cl}_{\tau}(U) = \emptyset$, so $y \notin \bigcap\{\mathrm{cl}_{\tau\theta}(\mathrm{cl}_{\tau}(U) : U \in \mathcal{B}(x))\}$, and, a fortiori, $y \notin \bigcap\{\mathrm{cl}_{\tau\theta}(\mathrm{cl}_{\tau}(U) \cap B_x) :$

$U \in \mathcal{B}(x)\}$. So the map $\psi : \mathrm{cl}_{\tau\theta}(A) \to \mathcal{P}_m(\mathcal{P}_m(A))$ defined by $\psi(x) = \mathcal{G}_x$ for every $x \in \mathrm{cl}_{\tau\theta}(A)$, is one to one. Since $|\mathcal{P}_m(\mathcal{P}_m(A))| \leq (k^m)^m = k^m$, then $|\mathrm{cl}_{\tau\theta}(A)| \leq k^m = |A|^{\chi(X)}$. Let $A_0 = A$ and, by transfinite induction, define for every $\alpha < m^+$ sets A_α such that $A_\alpha = \mathrm{cl}_{\tau\theta}(\bigcup\{A_\beta : \beta < \alpha\})$. Clearly $\bigcup\{A_\alpha : \alpha < m^+\} \subset [A]_{\tau\theta}$. Now let $x \in \mathrm{cl}_{\tau\theta}(\bigcup\{A_\alpha : \alpha < m^+\})$, for each $V \in \mathcal{B}(x)$ choose a point in $\mathrm{cl}_\tau(V) \cap (\bigcup\{A_\alpha : \alpha < m^+\})$ and let B be the set so obtained, obviously $B \in \mathcal{P}_m(\bigcup\{A_\alpha : \alpha < m^+\})$ and $x \in \mathrm{cl}_{\tau\theta}(B)$. Since m^+ is regular, there is an ordinal $\alpha < m^+$ such that $B \subset A_\alpha$, so

$$x \in \mathrm{cl}_{\tau\theta}(B) \subset \mathrm{cl}_{\tau\theta}(A_\alpha) \subset A_{\alpha+1} \subset \bigcup\{A_\alpha : \alpha < m^+\},$$

therefore $\bigcup\{A_\alpha : \alpha < m^+\}$ is $\tau\theta$-closed. Hence $[A]_{\tau\theta} = \bigcup\{A_\alpha : \alpha < m^+\}$. It remains to show that $|A_\alpha| \leq k^m$ for each $\alpha < m^+$ (this is equivalent to $|\bigcup\{A_\alpha : \alpha < m^+\}| \leq k^m$). Suppose there is an ordinal $\alpha < m^+$ such that $|A_\alpha| > k^m$ and let $\gamma = \min\{\alpha : |A_\alpha| > k^m\}$. Since $|A_\alpha| \leq k^m$ for every $\beta < \gamma$, we have $|\bigcup\{A_\beta : \beta < \gamma\}| \leq k^m$. Now $A_\gamma = \mathrm{cl}_{\tau\theta}(\bigcup\{A_\beta : \beta < \gamma\})$, hence

$$|A_\gamma| = |\mathrm{cl}_{\tau\theta}(\bigcup\{A_\beta : \beta < \gamma\})| \leq |\bigcup\{A_\beta : \beta < \gamma\}|^{\chi(X)} \leq (k^m)^m = k^m,$$

a contradiction. □

Definition 7. *Let X be a topological space. The w-compactness degree of X, denoted by $wcd(X)$, is defined as the smallest infinite cardinal number k with the property that for every open cover $\mathcal{U}$ of X there is a subcollection $\mathcal{V} \in \mathcal{P}_k(\mathcal{U})$ for which $X = \bigcup\{\mathrm{cl}_\tau(V) : V \in \mathcal{V}\}$.*

For every space X we have $wcd(X) \leq L(X)$ and this inequality can be proper.

Example 8. Let X be any infinite T_3-space such that every continuous real valued function defined on X is constant. Clearly $wcd(X) = \aleph_0 < L(X)$.

Example 9. For each $\alpha < \omega_1$ let $I(\alpha) = \{\alpha\}\times$ an open interval in the real line. Set $X = \omega_1 \cup \bigcup\{I(\alpha) : \alpha < \omega_1\}$ and for $x, y \in X$ define $x < y$ if (i) $x, y \in \omega_1$ and $x < y$ in ω_1, or (ii) $x \in \omega_1$, $y \in I(\beta)$ and $x \leq \beta$ in ω_1, or (iii) $x \in I(\gamma)$, $y \in \omega_1$ and $\gamma < y$ in ω_1, or (iv) $x \in I(\alpha)$, $y \in I(\beta)$ and $\alpha < \beta$ in ω_1, or (v) $x, y \in I(\alpha)$ and $x < y$ in $I(\alpha)$. Let σ be the order topology on X. Let $Y = X \cup \{\omega_1\}$, define $x < \omega_1$ for every $x \in X$ and let ϱ be the order topology on Y. If τ is the topology on Y generated by $\varrho \cup \{Y - L : L$ is the set of limit ordinals in $Y - \{\omega_1\}\}$, then (Y, τ) is a functionally Hausdorff H-closed space which fails to be Lindelöf [7], so $wcd(Y) = \aleph_0 < L(Y)$.

Theorem 10. *If X is a functionally Hausdorff space, then $|X| \leq 2^{\chi(X)wcd(X)}$.*

Proof: Let $m = \chi(X)wcd(X)$ and for every $x \in X$ let $\mathcal{B}(x)$ be a base for X at the point x such that $|\mathcal{B}(x)| \leq m$. Construct a family $\{C_\alpha : \alpha < m^+\}$ of subsets of X such that

(1) for any $\alpha < m^+$ C_α is $\tau\theta$-closed;
(2) for any $\alpha < m^+$ $|C_\alpha| \le 2^m$;
(3) if $\alpha < \beta < m^+$, then $C_\alpha \subset C_\beta$;
(4) for any $\alpha < m^+$, if $\mathcal{U} \subset \bigcup\{\mathcal{B}(x) : x \in \bigcup\{C_\beta : \beta < \alpha\}\}$, $|\mathcal{U}| \le m$ and $X - \bigcup\{\mathrm{cl}_\tau(U) : U \in \mathcal{U}\} \ne \emptyset$, then $C_\alpha - \bigcup\{\mathrm{cl}_\tau(U) : U \in \mathcal{U}\} \ne \emptyset$.

The construction is done by transfinite induction. Let $p \in X$ and $C_0 = \{p\}$. Let $0 < \alpha < m^+$ and assume that C_β has been constructed for every $\beta < \alpha$. Let $\mathcal{B}_\alpha = \bigcup\{\mathcal{B}(x) : x \in \bigcup\{C_\beta : \beta < \alpha\}\}$, clearly $|\mathcal{B}_\alpha| \le 2^m$. For any $\mathcal{U} \subset \mathcal{B}_\alpha$ such that $|\mathcal{U}| \le m$ and $X - \bigcup\{\mathrm{cl}_\tau(U) : U \in \mathcal{U}\} \ne \emptyset$, choose a point in $X - \bigcup\{\mathrm{cl}_\tau(U) : U \in \mathcal{U}\}$ and let A be the set so obtained, obviously $|A| \le 2^m$. Let $C_\alpha = [A \cup (\bigcup\{C_\beta : \beta < \alpha\})]_{\tau\theta}$, C_α satisfies (1), (3), (4) and, by Theorem 6, also (2). The set $C = \bigcup\{C_\alpha : \alpha < m^+\}$ is $\tau\theta$-closed, in fact let $x \in \mathrm{cl}_{\tau\theta}(C)$, for every $V \in \mathcal{B}(x)$ choose a point in $\mathrm{cl}_\tau(V) \cap C$ and let K be the set so obtained, clearly $|K| \le m$, therefore there exists an $\alpha < m^+$ such that $K \subset C_\alpha$, then $x \in \mathrm{cl}_{\tau\theta}(K) \subset \mathrm{cl}_{\tau\theta}(C_\alpha) = C_\alpha \subset C$. Obviously $|C| \le 2^m$, so to complete the proof it suffices to show that $C = X$. Let us suppose that $y \in X - C$, since X is functionally Hausdorff, then for any $x \in C$ there is a $U_x \in \mathcal{B}(x)$ such that $y \notin \mathrm{cl}_\tau(U_x)$; for every $x \in X - C$ let $U_x \in \mathcal{B}(x)$ such that $\mathrm{cl}_\tau(U_x) \cap C = \emptyset$ (C is $\tau\theta$-closed). $\{U_x\}_{x \in X}$ is an open cover of X, since $wcd(X) \le m$ there is a $B \subset X$ such that $|B| \le m$ and $X = \bigcup\{\mathrm{cl}_\tau(U_x) : x \in B\}$, clearly $C \subset \bigcup\{\mathrm{cl}_\tau(U_x) : x \in B \cap C\}$. Since $|B \cap C| \le m$, there is an $\alpha < m^+$ such that $B \cap C \subset C_\alpha$. Let $\mathcal{U} = \{U_x : x \in B \cap C\}$, $\mathcal{U} \subset \bigcup\{\mathcal{B}(x) : x \in \bigcup\{C_\beta : \beta < \alpha + 1\}\}$, $|\mathcal{U}| \le m$, $y \in X - \bigcup\{\mathrm{cl}_\tau(U_x) : U_x \in \mathcal{U}\}$ and $C_{\alpha+1} - \bigcup\{\mathrm{cl}_\tau(U_x) : U_x \in \mathcal{U}\} = \emptyset$, a contradiction. Hence $C = X$ and the proof is complete. □

Remark 11. Let X be a functionally Hausdorff space and let wX be the completely regular space which has the same points and continuous real valued functions as those of X. Clearly $L(wX) \le wcd(X)$ for every functionally Hausdorff space X. On the other hand, there exist functionally Hausdorff spaces X such that $\chi(X) < \chi(wX)$ (see e.g. [9, Example 36]). I do not know if $\chi(wX)L(wX) \le \chi(X)wcd(X)$ for every functionally Hausdorff space X; if this is the case, then Theorem 10 is a consequence of the Arkhangel'skii's inequality quoted at the beginning.

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