Sequential convergence in $C_p(X)$

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Abstract. I discuss the number of iterations of the elementary sequential closure operation required to achieve the full sequential closure of a set in spaces of the form $C_p(X)$.

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1. Introduction

For a topological space Z and a subset A of Z, let $\tilde{A}$ be the sequential closure of A, that is, the smallest subset of Z including A and containing all limits in Z of sequences in $\tilde{A}$. This may be regarded as the union of a transfinite sequence of sets $s_\xi(A) = s_\xi(A, Z)$, where $s_0(A) = A$ and for each ordinal $\xi > 0$ we take $s_\xi(A)$ to be the set of limits in Z of sequences in $\bigcup_{\eta<\xi} s_\eta(A)$. Clearly $s_{\omega_1}(A) = \bigcup_{\xi<\omega_1} s_\xi(A)$, so that $\tilde{A} = s_{\omega_1}(A)$. If we write $\sigma(A) = \min\{\xi : \tilde{A} = s_\xi(A)\} = \min\{\xi : s_{\xi+1}(A) = s_\xi(A)\}$, we shall have $0 \le \sigma(A) \le \omega_1$ for every A.

In this note I seek to address questions of the form: does Z have a subset A with $\sigma(A) = \omega_1$? or, what is $\Sigma(Z) = \sup_{A\subseteq Z} \sigma(A)$? Definite answers to such questions are frequently illuminating; for instance, 'Fréchet-Urysohn' spaces ([5, p. 53]) are precisely those for which $\overline{A} = s_1(A)$ for every A, and Lebesgue's theorem that there are functions of all Baire classes ([12, § 30.XIV]) can be expressed in the form '$\sigma(C([0,1]), \mathbb{R}^{[0,1]}) = \omega_1$', where here I give $\mathbb{R}^{[0,1]}$ its product topology, and write $C([0,1])$ for the space of continuous real-valued functions on $[0,1]$. Another example is the 'closure ordinal' $\alpha(Y)$ of [9], defined for linear subspaces Y of the dual X^* of a Banach space X, and related to the Pietetski-Shapiro rank on closed sets of uniqueness; this is just $\sigma(Y)$ for the w*-topology of X^*.

Most of the paper is directed towards spaces of the form $Z = C(X)$, where X is a topological space and $C(X)$ is the space of continuous functions from X to $\mathbb{R}$, endowed with the pointwise topology $\mathfrak{T}_p$ induced by the product topology of $\mathbb{R}^X$. In this case we find that

(i) $\Sigma(C(X))$ is either 0 or 1 or ω_1 (Theorem 9);
(ii) if X has a countable network then $\sigma(A) < \omega_1$ for every $A \subseteq C(X)$ (Proposition 2 and Example 3 (b));
(iii) if there is a continuous surjection from X onto a non-meager subset of $\mathbb{R}$, then $\Sigma(B_1(C(X))) = \omega_1$, where $B_1(C(X))$ is the unit ball of $C(X)$ (Theorem 11);

(iv) if X is compact and there is no continuous surjection from X onto $[0,1]$, then $\Sigma(C(X)) \leq 1$ (Corollary 13 (g)).

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2. I begin with a result showing that $\sigma(A) < \omega_1$ in many of the cases of interest here. Recall that if Z is a topological space, then a **network** for its topology is a family $\mathcal{W} \subseteq \mathcal{P}Z$ such that whenever $G \subseteq Z$ is open and $z \in G$ there is a $W \in \mathcal{W}$ such that $z \in W \subseteq G$. (Note that members of $\mathcal{W}$ need not themselves be open sets. See [5, p. 127].)

Proposition. *Let Z be a topological space with a countable network. Then*

(a) *for every $B \subseteq Z$ there is a countable $D \subseteq B$ such that $B \subseteq s_1(D)$;*
(b) *$\sigma(A) < \omega_1$ for every $A \subseteq Z$.*

Proof: **(a)** Let $\mathcal{W}$ be a countable network for the topology of Z; we may suppose that $\mathcal{W}$ is closed under finite intersections. Take $D \subseteq B$ to be a countable set meeting every member of $\mathcal{W}$ which meets B. If $z \in B$, let $\langle W_n \rangle_{n\in\mathbb{N}}$ run over the members of $\mathcal{W}$ containing z. Then for each $n \in \mathbb{N}$, $W'_n = \bigcap_{i\leq n} W_i$ is a member of $\mathcal{W}$ meeting B, so contains a member z_n of D. Now if G is any open set containing z, there is an $n \in \mathbb{N}$ such that $W_n \subseteq G$, so that $z_i \in G$ for every $i \geq n$; thus $\langle z_n \rangle_{n\in\mathbb{N}}$ converges to z and $z \in s_1(D)$.

(b) Now if $A \subseteq Z$ there is a countable $D \subseteq \tilde{A}$ such that $\tilde{A} \subseteq s_1(D)$. There must be a $\xi < \omega_1$ such that $D \subseteq \bigcup_{\eta<\xi} s_\eta(A)$, so that $\tilde{A} \subseteq s_\xi(A)$ and $\sigma(A) \leq \xi$. □

3. Examples

(a) Separable metrizable spaces have countable networks; subspaces, continuous images and countable products of spaces with countable networks have countable networks. ([5, 3.1.J.])

(b) Let X be a topological space with a countable network and give $C(X)$ the topology $\mathfrak{T}_p$ of pointwise convergence inherited from $\mathbb{R}^X$. Then $C(X)$ has a countable network. ([5, 3.4.H(a)].)

(c) Consequently, if X is a separable Banach space, then X^* has a countable network for its w*-topology. (Compare [9, § V.2, Proposition 5].)

4. The cardinal $\mathfrak{b}$

A further general remark about topological spaces of small character will be useful later. Recall that the cardinal $\mathfrak{b}$ is defined as the least cardinal of any set $F \subseteq \mathbb{N}^\mathbb{N}$ which is 'essentially unbounded', that is, for every $g \in \mathbb{N}^\mathbb{N}$ there is an $f \in F$ such that $\{n : f(n) \geq g(n)\}$ is infinite (see [3, § 3]); and that if Z is any topological space and $z \in Z$, then $\chi(z, Z)$ is the least cardinal of any base of neighbourhoods of z in Z. Now we have the following:

Proposition. *Let Z be a topological space such that $\chi(z, Z) < \mathfrak{b}$ for every $z \in Z$. Then $\Sigma(Z) \leq 1$.*

PROOF: Take $A \subseteq Z$ and $z \in s_2(A)$. Then there are $\langle z_{mn} \rangle_{m,n \in \mathbb{N}}$, $\langle z_m \rangle_{m \in \mathbb{N}}$ such that $z_{mn} \in A$ for all m, n, $\langle z_{mn} \rangle_{n \in \mathbb{N}} \to z_m$ for each m, and $\langle z_m \rangle_{m \in \mathbb{N}} \to z$. Let $\mathcal{U}$ be a base of open neighbourhoods of z with $\#(\mathcal{U}) < \mathfrak{b}$. For each $U \in \mathcal{U}$ there are $m_U \in \mathbb{N}$, $f_U \in \mathbb{N}^{\mathbb{N}}$ such that $z_m \in U$ for $m \geq m_U$, $z_{mn} \in U$ for $m \geq m_U$, $n \geq f_U(m)$. Because $\#(\mathcal{U}) < \mathfrak{b}$, there is a $g \in \mathbb{N}^{\mathbb{N}}$ such that $\{n : f_U(n) > g(n)\}$ is finite for every $U \in \mathcal{U}$. Now $\langle z_{m,g(m)} \rangle_{m \in \mathbb{N}} \to z$ so $z \in s_1(A)$.

Thus $s_2(A) \subseteq s_1(A)$ and $\sigma(A) \leq 1$; as A is arbitrary, $\Sigma(Z) \leq 1$. $\square$

5. A note on trees

Recall that a partially ordered set P is **well-founded** if every non-empty subset of P has a minimal element, and that for such P there is a rank function $r : P \to \mathrm{On}$, the class of ordinals, given by

$$r(p) = \min\{\xi : \xi \in \mathrm{On},\ r(q) < \xi\ \ \forall\ \ q < p\}$$

for every $p \in P$. A **tree** is a partially ordered set T such that $\{u : u \leq t\}$ is well-ordered for every $t \in T$; of course a tree must be well-founded, and have a rank function r. I will say that a tree T is **well-capped** if every non-empty subset of T has a maximal element, that is, if $(T, \geq)$ is well-founded; in this case there is a dual rank function r^*. Because all totally ordered subsets of T must now be finite, r must be finite-valued; but r^* need not be, and indeed we have the following well-known fact. (See [13, p. 236].)

Notation. Write Seq for the tree $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$, ordered by inclusion. If $t = (n_0, \dots, n_r) \in \mathrm{Seq}$, write $t^\frown i$ for $(n_0, \dots, n_r, i)$ and $i^\frown t$ for $(i, n_0, \dots, n_r)$.

6. Lemma. *For every ordinal $\alpha < \omega_1$ there is a non-empty well-capped subtree T_α of Seq such that $r^*(\emptyset, T_\alpha) = \alpha$ and every member t of T_α either has no successors in T_α (so that $r^*(t, T_\alpha) = 0$) or has all its successors $t^\frown i$ in T_α, and in this latter case has $r^*(t, T_\alpha) = \lim_{i \to \infty}(r^*(t^\frown i, T_\alpha) + 1)$.*

PROOF: Induce on α. Start with $T_0 = \{\emptyset\}$. For the inductive step to $\alpha > 0$, let $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ be a sequence of ordinals such that $\alpha = \sup_{n \in \mathbb{N}}(\alpha_n + 1) = \lim_{n \to \infty}(\alpha_n + 1)$, and set $T_\alpha = \{\emptyset\} \cup \{n^\frown t : n \in \mathbb{N},\ t \in T_{\alpha_n}\}$. $\square$

7. Embedding trees

Let Z be a Hausdorff space. I will say that a map $t \mapsto z_t : \mathrm{Seq} \to Z$ is a **sequentially regular embedding** if

(i) $\lim_{i \to \infty} z_{t^\frown i} = z_t$ for every $t \in \mathrm{Seq}$;

(ii) whenever $\langle t_i \rangle_{i \in \mathbb{N}}$ is a sequence in Seq such that there are t, $\langle m(i) \rangle_{i \in \mathbb{N}}$ with $t^\frown m(i) < t_i$ and $m(i) < m(i+1)$ for every $i \in \mathbb{N}$, then $\langle z_{t_i} \rangle_{i \in \mathbb{N}}$ has no limit in Z;

(iii) $z_s \neq z_t$ for all distinct $s, t \in \mathrm{Seq}$.

8. Lemma. *Let Z be a Hausdorff space and $t \mapsto z_t : Seq \to Z$ a sequentially regular embedding.*

(a) *If $\alpha < \omega_1$ and $T_\alpha \subseteq Seq$ is a well-capped subtree as constructed in Lemma* 6*, and $A = \{z_t : t \in T_\alpha$ is maximal$\}$, then*

$$s_\beta(A, Z) = \{z_t : t \in T_\alpha,\ r^*(t) \leq \beta\}$$

for every ordinal β; so that $\sigma(A, Z) = r^*(\emptyset) = \alpha$.

(b) *Consequently $\Sigma(Z) = \omega_1$.*

PROOF: **(a)** The point is that if $\langle t_i \rangle_{i\in\mathbb{N}}$ is any sequence in $T = T_\alpha$, then there is a $t \in T$ which is maximal subject to $\{i : i \in \mathbb{N},\ t \leq t_i\}$ being infinite. Now $\langle t_i \rangle_{i\in\mathbb{N}}$ has a subsequence $\langle t'_i \rangle_{i\in\mathbb{N}}$ which is either constant (equal to t), or is a subsequence of $\langle t^\frown i \rangle_{i\in\mathbb{N}}$, or is such that $t'_i > t^\frown m(i)$ for each i, with $\langle m(i) \rangle_{i\in\mathbb{N}}$ strictly increasing. So conditions (i) and (ii) of § 7 tell us that if $\langle z_{t_i} \rangle_{i\in\mathbb{N}}$ is convergent, its limit must be z_t, with infinitely many of the t_i either equal to t or successors of t.

An easy induction on β now shows that $s_\beta(A) = \{z_t : r^*(t) \leq \beta\}$ for every β.

(b) now follows at once. □

9. Theorem. *Let X be any topological space, and give $C(X)$ the topology of pointwise convergence. Then $\Sigma(C(X))$ must be either* 0 *or* 1 *or ω_1.*

PROOF: Suppose that there is an $A \subseteq C(X)$ such that $\sigma(A, C(X)) > 1$. Then there must be a double sequence $\langle f_{ij} \rangle_{i,j\in\mathbb{N}}$ in $C(X)$ such that $f_i = \lim_{j\to\infty} f_{ij}$ is defined in $C(X)$ for each $i \in \mathbb{N}$, $f = \lim_{i\to\infty} f_i$ is similarly defined in $C(X)$, but f is not the limit of any sequence in $\{f_{ij} : i,\ j \in \mathbb{N}\}$. Setting $h_{ij}(x) = i|f_{ij}(x) - f_i(x)|$ for $i,\ j \in \mathbb{N}$ and $x \in X$, we see that each h_{ij} is continuous, that $\lim_{j\to\infty} h_{ij} = 0$ for each i, but that no sequence of the form $\langle h_{m(i),n(i)} \rangle_{i\in\mathbb{N}}$, where $\langle m(i) \rangle_{i\in\mathbb{N}}$ is strictly increasing, can be bounded in $\mathbb{R}^X$, since otherwise

$$|f_{m(i),n(i)} - f| \leq m(i)^{-1} h_{m(i),n(i)} + |f_{m(i)} - f| \to 0.$$

Now, for $t \in \mathrm{Seq}$, take

$$J_t = \{(i,j) : \exists\ u,\ u^\frown i^\frown j \leq t\},$$
$$g_t(x) = \max(\{0\} \cup \{h_{ij}(x) : (i,j) \in J_t\}).$$

Then $g_t \in C(X)$, and the map $t \mapsto g_t : \mathrm{Seq} \to C(X)$ satisfies the conditions (i) and (ii) of § 7. It is not of course injective. However, if we look at the family of rational linear combinations of the g_t, this can contain only countably many constant functions, so there is a real $\delta > 0$ such that the constant function $\delta\chi X$ is not a rational linear combination of the g_t. Choose a family $\langle \delta_t \rangle_{t\in\mathrm{Seq}}$ of distinct rational multiples of δ such that (i) $0 \leq \delta_t \leq 1$ for every t (ii) $\lim_{i\to\infty} \delta_{t^\frown i} = \delta_t$ for every t. Set $e_t = g_t + \delta_t \chi X$ for each $t \in \mathrm{Seq}$. Now $t \mapsto e_t : \mathrm{Seq} \to C(X)$ is a sequentially regular embedding in the sense of § 7. So by Lemma 8 we have $\Sigma(Z) = \omega_1$. □

10. s_1-spaces

The trichotomy above is satisfyingly sharp, and it is natural to look for methods of determining $\Sigma(C(X))$ in terms of other topological properties of X. Of course $\Sigma(C(X)) = 0$ iff $X = \emptyset$. For brevity, I will say that an **s_1-space** is a topological space X such that $\Sigma(C(X)) \leq 1$. Before going further with this, I give a theorem which provides some relevant information and introduces a useful technique.

11. Theorem. *Let X be a topological space such that there is a continuous surjection from X onto a non-meager subset of $\mathbb{R}$. Give $C(X)$ and $\mathbb{R}^X$ the topology of pointwise convergence. Then*

$$\sup\{\sigma(A, C(X)) : A \subseteq C(X) \text{ is uniformly bounded, } s_{\omega_1}(A, \mathbb{R}^X) \subseteq C(X)\} = \omega_1.$$

PROOF: **(a)** I write '$s_{\omega_1}(A, \mathbb{R}^X)$' in order to avoid the difficulty of distinguishing $\tilde{A}$, taken in $\mathbb{R}^X$, from $\tilde{A}$, taken in $C(X)$.

Let me say that a topological space X is **adequate** if there is a function $t \mapsto f_t$ from Seq to a uniformly bounded subset of $C(X)$ which is a sequentially regular embedding of Seq into $\mathbb{R}^X$. The first thing to observe is that in this case X satisfies the conclusion of the theorem; for if $\alpha < \omega_1$ and T_α is the corresponding tree from Lemma 6, then $A = \{f_t : t \in T_\alpha \text{ is maximal}\}$ is a uniformly bounded subset of $C(X)$ such that $s_{\omega_1}(A, \mathbb{R}^X) = \{f_t : t \in T_\alpha\} \subseteq C(X)$ and $\sigma(A, C(X)) = \alpha$. The second point is that if Y is adequate and $h : X \to Y$ is a continuous surjection, then X is adequate. For we have a map $\psi : \mathbb{R}^Y \to \mathbb{R}^X$ given by writing $\psi(g) = g \circ h$ for every $g \in \mathbb{R}^Y$. This map ψ has the properties

(α) it is $\mathfrak{T}_p$-continuous and injective;
(β) for any sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{R}^Y$, $\langle g_n \rangle_{n\in\mathbb{N}}$ is convergent iff $\langle \psi(g_n) \rangle_{n\in\mathbb{N}}$ is convergent;
(γ) $\psi(g)$ is continuous whenever g is continuous;
(δ) $\sup_{x\in X} |\psi(g)(x)| = \sup_{y\in Y} |g(y)|$ for all $g \in \mathbb{R}^Y$.

Now it is easy to see that if $t \mapsto f_t : \mathrm{Seq} \to C(Y)$ witnesses that Y is adequate, then $t \mapsto \psi(f_t) : \mathrm{Seq} \to C(X)$ witnesses that X is adequate.

(b) I begin with a special case. Let Y be the compact metrizable space $\mathbb{N} \cup \{\infty\}$, the one-point compactification of the discrete space $\mathbb{N}$. Set $X_0 = Y^{\mathrm{Seq}}$, with the compact metrizable product topology, and let $D \subseteq X_0$ be any set which meets every non-empty open subset of X_0 in a non-meager set. For each $t \in \mathrm{Seq}$ define $f_t \in C(D)$ by setting

$$\begin{aligned} f_t(x) &= 1 \text{ if there is a } u < t \text{ such that } x(u) \neq \infty \text{ and } u^\frown x(u) \leq t, \\ &= 0 \text{ otherwise.} \end{aligned}$$

(c) The map $t \mapsto f_t : \mathrm{Seq} \to \mathbb{R}^D$ is a sequentially regular embedding in the sense of § 7. To see this, take the conditions in order.

(i) For $t \in \mathrm{Seq}$ and $n \in \mathbb{N}$, $f_{t^\frown n}(x) = 1$ iff either $f_t(x) = 1$ or $x(t) = n$. Consequently $f_t = \lim_{n\to\infty} f_{t^\frown n}$ in $\mathbb{R}^{X_0}$ for every $t \in \mathrm{Seq}$.

(ii) If $t \in \mathrm{Seq}$, $\langle m(i)\rangle_{i\in\mathbb{N}}$ is strictly increasing, $\langle n(i)\rangle_{i\in\mathbb{N}}$ is any sequence in $\mathbb{N}$ and $t^\frown m(i)^\frown n(i) \le t_i$ for every i, set

$$U = \{x : f_t(x) = 0\},$$
$$G_r = \{x : \exists\ i \ge r,\ f_{t_i}(x) = 0,\ f_{t_{i+1}}(x) = 1\};$$

then because all the $m(i)$ are distinct, $U \setminus G_r$ is nowhere dense for every r, and $U \setminus \bigcap_{r\in\mathbb{N}} G_r$ is meager. Accordingly there is a point $x \in D \cap \bigcap_{r\in\mathbb{N}} G_r$; but now $\lim_{i\to\infty} f_{t_i}(x)$ cannot exist, so that $\langle f_{t_i}\rangle_{i\in\mathbb{N}}$ has no limit in $\mathbb{R}^D$.

(iii) Of course all the f_t are distinct, because D is dense in X_0.

(d) Thus D is adequate whenever $D \subseteq X_0$ meets every non-empty open subset of X_0 in a non-meager set. In particular, X_0 itself is adequate. But X_0, being compact, metrizable, zero-dimensional, non-empty and without isolated points, is homeomorphic to the Cantor set $X_1 \subseteq [0,1]$ ([5, 6.2.A(c)]), so X_1 is adequate.

Now observe that there is a linear map $\phi : \mathbb{R}^{X_1} \to \mathbb{R}^{[0,1]}$ such that ϕ has the properties (α)-(δ) of part (a) of this proof. This is a special case of Dugundji's theorem ([4]), but it can be easily proved directly; just take $\phi(f)$ to be the extension of f whose graph is a straight line on the closure of each of the components of $[0,1] \setminus X_1$. So the argument of (a) applies here also, and $[0,1]$ is adequate. Moreover, if X is any topological space such that $[0,1]$ is a continuous image of X, then X will be adequate.

(e) Now let D be any non-meager subset of $\mathbb{R}$. If D includes some non-empty closed interval $[a,b]$, then $[a,b]$ is a continuous image of D (under the map $x \mapsto \max(a, \min(x,b))$), and $[a,b]$, being homeomorphic to $[0,1]$, is adequate; so D is also adequate. So let us suppose that $\mathbb{R} \setminus D$ is dense in $\mathbb{R}$. Next, there must be a non-trivial interval $[a,b]$, with endpoints in D, such that $D \cap U$ is non-meager for every non-empty open $U \subseteq [a,b]$; set $D' = D \cap [a,b]$, so that, as above, D' is a continuous image of D. Now let Q be a countable dense subset of $[a,b] \setminus D$. Then $[a,b] \setminus Q$ is a non-empty G_δ subset of $\mathbb{R}$ without isolated points, so is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ ([5, 6.2.A(a)]; [12, § 36.II]) and therefore to $\mathbb{N}^{\mathrm{Seq}}$, which is a dense G_δ subset of X_0. This homeomorphism carries D' to a subset D'' of X_0 which meets every non-empty open subset of X_0 in a non-meager set, and is therefore adequate. So D' and D are also adequate.

(f) Finally, if X is such that some non-meager subset of $\mathbb{R}$ is a continuous image of X, then X is adequate, putting (a) and (e) together. This proves the theorem. □

12. In particular, if X is an s_1-space, any continuous image of X in $\mathbb{R}$ is meager. But this is by no means the whole story. I continue the argument with some general remarks on s_1-spaces.

Proposition. *Let X be a topological space, and give $C(X)$ the topology of pointwise convergence; write $B_1(C(X))$ for its unit ball, that is, the space of continuous functions from X to $[-1,1]$. Then the following are equivalent:*

(i) *X is an s_1-space;*
(ii) *$\Sigma(B_1(C(X))) \leq 1$, that is, $\sigma(A, C(X)) \leq 1$ for every uniformly bounded set $A \subseteq C(X)$;*
(iii) *whenever $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ is a uniformly bounded double sequence in $C(X)$ such that $\lim_{n\to\infty} f_{mn} = 0$ for each m, there are sequences $\langle m(i)\rangle_{i\in\mathbb{N}}$, $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle m(i)\rangle_{i\in\mathbb{N}}$ is strictly increasing and $\lim_{i\to\infty} f_{m(i),n(i)} = 0$;*
(iv) *whenever $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ is a double sequence in $C(X)$ such that $\lim_{n\to\infty} f_{mn} = 0$ for every m, then there is an infinite $I \subseteq \mathbb{N}$ such that $\lim_{m\to\infty} f_{m,k(m)} = 0$ whenever $\langle k(m)\rangle_{m\in\mathbb{N}}$ is a strictly increasing sequence in I;*
(v) *$h[X]$ is an s_1-space for every continuous $h : X \to \mathbb{R}$.*

Proof: **(a)(i)⇒(iv)** Suppose that X is an s_1-space, and let $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ be a double sequence in $C(X)$ such that $\lim_{n\to\infty} f_{mn} = 0$ for every m. Set

$$g_{mn}(x) = 2^{-m} + 2^{-n} + \max_{i\leq m} |f_{in}(x)|$$

for $m, n \in \mathbb{N}$ and $x \in X$. Then $\lim_{m\to\infty}\lim_{n\to\infty} g_{mn} = 0$ in $C(X)$, so there is a sequence in $A = \{g_{mn} : m, n \in \mathbb{N}\}$ converging to 0, because $0 \in s_2(A) = s_1(A)$. This sequence is of the form $\langle g_{r(i),s(i)}\rangle_{i\in\mathbb{N}}$ where $\langle r(i)\rangle_{i\in\mathbb{N}}$, $\langle s(i)\rangle_{i\in\mathbb{N}}$ are sequences in $\mathbb{N}$; because $g_{mn}(x) \geq 2^{-m} + 2^{-n}$ for all m, n and x, we must have $\lim_{i\to\infty} r(i) = \lim_{i\to\infty} s(i) = \infty$, and we may take it that both sequences are strictly increasing. Set $I = \{s(i) : i \in \mathbb{N}\}$. If $\langle k(m)\rangle_{m\in\mathbb{N}}$ is any strictly increasing sequence in I, then for each $m \in \mathbb{N}$ there is an $i_m \in \mathbb{N}$ such that $s(i_m) = k(m)$, and $m \leq i_m \leq r(i_m)$ for each m, so

$$|f_{m,k(m)}| \leq g_{r(i_m),s(i_m)} \to 0$$

as $m \to \infty$.

(b)(iv)⇒(iii) is trivial.

(c)(iii)⇒(i) Assume (iii); let A be any subset of $C(X)$ and take $g \in s_2(A, C(X))$. Then there is a double sequence $\langle g_{mn}\rangle_{m,n\in\mathbb{N}}$ in A such that $g_m = \lim_{n\to\infty} g_{mn}$ is defined in $C(X)$ for each m and $g = \lim_{m\to\infty} g_m$. Set

$$f_{mn} = \min(1, |g_{mn} - g_m|) \quad \text{for} \quad m, n \in \mathbb{N}.$$

By (iii), there are sequences $\langle m(i)\rangle_{i\in\mathbb{N}}$, $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle m(i)\rangle_{i\in\mathbb{N}}$ is strictly increasing and $\lim_{i\to\infty} f_{m(i),n(i)} = 0$. Then

$$0 = \lim_{i\to\infty} |g_{m(i),n(i)} - g_{m(i)}| = \lim_{i\to\infty} g_{m(i),n(i)} - g,$$

and $g \in s_1(A)$. As A, g are arbitrary, $\Sigma(C(X)) \le 1$, as required.

(d)(i)⇒(ii) is trivial. For **(ii)⇒(iii)**, use the arguments of (a).

(e)(i)⇒(v) If $h : X \to \mathbb{R}$ is continuous and $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ is a double sequence in $C(h[X])$ such that $\lim_{n\to\infty} f_{mn} = 0$ for every m, then $\lim_{n\to\infty} f_{mn} \circ h = 0$ in $C(X)$ for every m, so there are sequences $\langle m(i)\rangle_{i\in\mathbb{N}}$, $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle m(i)\rangle_{i\in\mathbb{N}}$ is strictly increasing and $\lim_{i\to\infty} f_{m(i),n(i)} \circ h = 0$ in $C(X)$; now $\lim_{i\to\infty} f_{m(i),n(i)} = 0$ in $C(h[X])$.

(f)(v)⇒(iii) Assume (v), and let $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ be a double sequence in $C(X)$ such that $\lim_{n\to\infty} f_{mn} = 0$ for each m. Define $h : X \to \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ by setting $h(x)(m,n) = f_{mn}(x)$; then h is continuous. Theorem 11 tells us that $[0,1]$ is not a continuous image of $h[X]$. Thus $h[X]$ is zero-dimensional; being separable and metrizable, it is homeomorphic to a subset of $\mathbb{R}$ ([5, 6.2.16 and 3.1.28]), and is therefore an s_1-space. Setting $g_{mn}(y) = y(m,n)$ for $m,n \in \mathbb{N}$ and $y \in h[X]$, we have $\lim_{n\to\infty} g_{mn} = 0$ for each m, so (because (i)⇒(iv)) there is a sequence $\langle k(m)\rangle_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} g_{m,k(m)} = 0$ in $C(h[X])$, and now $\lim_{m\to\infty} f_{m,k(m)} \to 0$ in $C(X)$. Because (iii)⇒(i), X is an s_1-space, as claimed. □

13. Corollary. (a) *A continuous image of an s_1-space is an s_1-space.*

(b) *Let X be a topological space expressible as $\bigcup_{r\in\mathbb{N}} X_r$ where each X_r is an s_1-space. Then X is an s_1-space.*

(c) *Let X be a normal s_1-space. Then all zero sets and all cozero sets in X are s_1-spaces.*

(d) *Let X be a metrizable s_1-space. Then all open sets, closed sets and F_σ sets in X are s_1-spaces.*

(e) *Let X be a topological space and μ a finite measure defined on the σ-algebra generated by the zero sets in X. If every μ-negligible subset of X is an s_1-space, then X itself is an s_1-space.*

(f) *In particular, if $X \subseteq \mathbb{R}$ meets every Lebesgue negligible subset of $\mathbb{R}$ in a countable set (e.g., if X is a Sierpiński set), then X is an s_1-space.*

(g) *If X is a compact space, then X is an s_1-space iff $[0,1]$ is not a continuous image of X.*

Proof: **(a)** By 12 (v), or otherwise.

(b) Let $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ be a double sequence in $C(X)$ such that $\lim_{n\to\infty} f_{mn} = 0$ for each m. By (i)⇒(iv) of Proposition 12 we may choose inductively a decreasing sequence $\langle I_r\rangle_{r\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that $\lim_{m\to\infty} f_{m,k(m)}(x) = 0$ whenever $x \in X_r$ and $\langle k(m)\rangle_{m\in\mathbb{N}}$ is a strictly increasing sequence in I_r. If we now take $\langle k(m)\rangle_{m\in\mathbb{N}}$ to be a strictly increasing sequence such that $\{m : k(m) \notin I_r\}$ is finite for every r, then $\lim_{m\to\infty} f_{m,k(m)} = 0$ in $C(X)$. By (iii)⇒(i) of Proposition 12, X is an s_1-space.

(c) Let $F \subseteq X$ be a zero set, and $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ a uniformly bounded double sequence in $C(F)$ such that $\lim_{n\to\infty} f_{mn} = 0$ for every $m \in \mathbb{N}$. For each m, n

let f'_{mn} be a continuous extension of f_{mn} to the whole of X, still bounded by the uniform bounds of the f_{mn}. Let $g : X \to \mathbb{R}$ be a continuous function such that $F = g^{-1}[\{0\}]$. For $x \in X$, $n \in \mathbb{N}$ set $g_n(x) = \max(0, 1 - 2^n|g(x)|)$. Set $f''_{mn} = f'_{mn} \times g_n$ for $m, n \in \mathbb{N}$; then $\lim_{n\to\infty} f''_{mn}(x) = 0$ for $x \in X$, $m \in \mathbb{N}$. Because X is an s_1-space, there is a sequence $\langle k(m)\rangle_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} f''_{m,k(m)} = 0$ in $C(X)$, and now $\lim_{m\to\infty} f_{m,k(m)} = 0$ in $C(F)$. Because $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ is arbitrary, F is an s_1-space.

Now a cozero set in X is a countable union of zero sets, so is an s_1-space by (b).

(d) Put (b) and (c) together.

(e) Let $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ be a double sequence in $C(X)$ such that $\lim_{n\to\infty} f_{mn} = 0$ for every m. For $m \in \mathbb{N}$ take $l(m) \in \mathbb{N}$ such that

$$\mu\Big(\bigcup_{i\ge l(m)} \{x : |f_{mi}(x)| \ge 2^{-m}\}\Big) \le 2^{-m}.$$

Set

$$E = \bigcap_{p\in\mathbb{N}} \bigcup_{m\ge p, i\ge l(m)} \{x : |f_{mi}(x)| \ge 2^{-m}\};$$

then $\mu E = 0$, so E is an s_1-space and by (i)⇒(iv) of Proposition 12 there is an infinite $I \subseteq N$ such that $\lim_{m\to\infty} f_{m,k(m)}(x) = 0$ whenever $x \in E$ and $\langle k(m)\rangle_{m\in\mathbb{N}}$ is a strictly increasing sequence in I. Choose such a sequence such that $k(m) \ge l(m)$ for every m; then $\lim_{m\to\infty} f_{m,k(m)}(x) = 0$ for every $x \in X$. By (iii)⇒(i) of Proposition 12, X is an s_1-space.

(f) follows immediately (using (b), if you wish, to deal with the fact that Lebesgue measure is σ-finite rather than totally finite).

(g) If $[0, 1]$ is a continuous image of X, then X cannot be an s_1-space, by Theorem 11. On the other hand, if $[0, 1]$ is not a continuous image of X, then every metrizable continuous image of X is countable, therefore an s_1-space, and X is an s_1-space.

14. The structure of s_1-spaces

Proposition 12 suggests that in order to describe s_1-spaces in general we should investigate their images under real-valued continuous functions. Theorem 11 tells us that if X has a non-meager continuous image in $\mathbb{R}$ then it cannot be an s_1-space; in particular, if $[0, 1]$ is a continuous image of X then X is not an s_1-space. We can go a little further. Suppose that X is a subspace of $\mathbb{N}^{\mathbb{N}}$ which is essentially unbounded in the sense of § 4; then X is not an s_1-space, because if we write $f_{mn}(x) = 1$ if $x(m) \ge n$, 0 otherwise, then $\lim_{n\to\infty} f_{mn} = 0$ in $C(X)$ but $\lim_{m\to\infty} f_{m,k(m)} \not\to 0$ for any sequence $\langle k(m)\rangle_{m\in\mathbb{N}}$. Thus we can say that if X is an s_1-space, then neither $[0, 1]$ nor any essentially unbounded subset of $\mathbb{N}^{\mathbb{N}}$ can be a continuous image of X. We also have a description of the least cardinal of any space which is not an s_1-space. This must be $\mathfrak{b}$; for if $\#(X) < \mathfrak{b}$, then $\chi(f, C(X)) \le \max(\omega, \#(X)) < \mathfrak{b}$ for every $f \in C(X)$, so $\Sigma(C(X)) \le 1$ by Proposition 4, while there is an essentially unbounded set $X \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinal $\mathfrak{b}$, and this X is not an s_1-space.

If we look at the family $\mathcal{S}$ of s_1-subsets of $\mathbb{R}$, we see that $\mathcal{S}$ is closed under continuous images, countable unions and intersection with F_σ sets ((a), (b) and (d) of Corollary 13). I believe that I have an example, subject to the continuum hypothesis, of an $X \in \mathcal{S}$ such that $X \setminus \mathbb{Q} \notin \mathcal{S}$ (see [6, § 1]); in particular, G_δ subsets of s_1-spaces need not be s_1-spaces.

It is natural to think of s_1-spaces as 'thin'. Among the familiar classes of 'thin' sets, the most immediately relevant is the class of 'γ-spaces' of [7]; these are all s_1-spaces because if X is a γ-space then $C(X)$, with the pointwise topology, is a Fréchet-Urysohn space ([7, § 2, Theorem 2]). A Sierpiński set in $\mathbb{R}$ cannot be a γ-space, while a Lusin set cannot be an s_1-space; so (under the continuum hypothesis) there is an s_1-space which is not a γ-space, and there is a set with Rothberger's property (that is, all its continuous images in $\mathbb{R}$ have strong measure 0) which is not an s_1-space.

Again using the continuum hypothesis, it is easy to construct two Sierpiński sets $X, Y \subseteq \mathbb{R}$ such that $X + Y = \mathbb{R}$; so that X and Y are s_1-spaces while $X \times Y$ is not (because $X + Y$ is a continuous image of $X \times Y$).

It is perhaps worth remarking that (at least if the continuum hypothesis is true) there is an s_1-space X with a double sequence $\langle f_{mn} \rangle_{m,n\in\mathbb{N}}$ in $C(X)$ such that $\lim_{n\to\infty} f_{mn} = 0$ for every m, but for every sequence $\langle k(m) \rangle_{m\in\mathbb{N}}$ in $\mathbb{N}$ and every infinite $J \subseteq \mathbb{N}$ there are $\langle n(m) \rangle_{m\in\mathbb{N}}$, $x \in X$ such that $n(m) \ge k(m)$ for every m and $\limsup_{m\in J, m\to\infty} f_{m,n(m)} > 0$ ([6, 1C]).

15. Problems

(a) The problem arises: if X is a topological space such that neither $[0,1]$ nor any essentially unbounded subset of $\mathbb{N}^{\mathbb{N}}$ is a continuous image of X, must X be an s_1-space? For compact spaces, this is true, by 13 (g). Of course it is enough to consider subspaces of $\mathbb{R}$. Note that if E is a non-meager subset of $\mathbb{R}$, then either E includes an interval and $[0,1]$ is a continuous image of E, or $\mathbb{R} \setminus E$ is dense and E is homeomorphic to a non-meager subset of $\mathbb{R} \setminus \mathbb{Q}$, which is in turn homeomorphic to a non-meager subset of $\mathbb{N}^{\mathbb{N}}$, which must be essentially unbounded; so if neither $[0,1]$ nor any essentially unbounded subset of $\mathbb{N}^{\mathbb{N}}$ is a continuous image of X, then nor is any non-meager subset of $\mathbb{R}$. It is consistent to suppose that every subset of $\mathbb{R}$ of cardinal $\mathfrak{b}$ is meager (add ω_2 random reals to a model of ZFC + CH); in these circumstances there will be an X, not an s_1-space, such that every continuous image of X in $\mathbb{R}$ is meager.

(b) Another problem arises if we look at uniformly bounded sets. Writing $B_1(C(X))$ for the unit ball of $C(X)$, I do not know whether $\Sigma(B_1(C(X)))$ is always equal to $\Sigma(C(X))$, even though $\Sigma(B_1(C(X))) \le 1$ iff $\Sigma(C(X)) \le 1$ (Proposition 12). The methods of Theorem 11 may be relevant; they show, in particular, that for compact X we do have $\Sigma(B_1(C(X))) = \Sigma(C(X))$. I believe that I can prove the same equality for metrizable X ([6, § 2]).

(c) In 13 (b) we saw that a countable union of s_1-spaces is an s_1-space. Of course the union of $\mathfrak{b}$ s_1-spaces need not be an s_1-space. But is the union of fewer than $\mathfrak{b}$ spaces necessarily an s_1-space, even when $\mathfrak{b} > \omega_1$?

16. Weak topologies on Banach spaces

Some of the interest of the pointwise topology on $C(X)$ for compact Hausdorff spaces X arises from the study of weak topologies on Banach spaces. If E is a normed space with dual E^*, and X is the unit ball of E^* with the w*-topology $\mathfrak{T}_s(E^*, E)$, then X is a compact Hausdorff space and E, with its weak topology $\mathfrak{T}_s(E, E^*)$, can be identified with a subspace of $C(X)$, which if E is a Banach space is $\mathfrak{T}_p$-closed, by Grothendieck's theorem ([10, 21.9.(4)]).

If we now examine the possible values of $\Sigma(E)$, we get a sharp dichotomy just as in Theorem 9.

17. Theorem. *Let E be a normed space, with its weak topology $\mathfrak{T}_s(E, E^*)$.*

(a) *If every weakly convergent sequence in E is norm-convergent, then $\Sigma(E) \leq 1$.*

(b) *If there is a weakly convergent sequence in E which is not norm-convergent, then $\Sigma(E) = \omega_1$.*

PROOF: **(a)** If weakly convergent sequences in E are norm-convergent, then $\sigma(A)$, for the weak topology, is always equal to $\sigma(A)$ for the norm topology; but the latter is metrizable, so $\sigma(A)$ is never greater than 1, for any $A \subseteq E$.

(b) Otherwise, there is a sequence which converges to 0 for the weak topology, but is bounded away from 0 for the norm; dividing each term of the sequence by its norm, we obtain a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ of vectors of norm 1 which is weakly convergent to 0. Now enumerate Seq as $\langle u_n \rangle_{n\in\mathbb{N}}$. For $t \in$ Seq set

$$z_t = \sum\{4^m x_n : m,\, n \in \mathbb{N},\, u_m < u_n \leq t\}.$$

Recalling that any $\mathfrak{T}_s(E, E^*)$-convergent sequence must be norm-bounded ([2, § II.3, Theorem 1]), it is easy to see that the map $t \mapsto z_t : \mathrm{Seq} \to E$ satisfies the conditions (i) and (ii) of § 8. Now, just as in the proof of Theorem 9, we can take any non-zero $e \in E$ and find a family $\langle \delta_t \rangle_{t\in\mathrm{Seq}}$ in $[0, 1]$ such that $t \mapsto z_t + \delta_t e$ is a sequentially regular embedding. So Lemma 8 gives the result. □

18. Remarks

(a) Alternative (a) of the dichotomy above is the 'Schur property'. The simplest non-trivial example is $E = \ell^1(I)$ for any set I ([10, 22.4.(2)]; [8, 27.13]). For further examples see [1, Chapter V].

(b) Note that Theorem 17 really seems to differ from Theorem 9 because $[0, 1]$ is a continuous image of the unit ball of E^* for any non-trivial normed space E; moreover, if E^* is norm-separable, then bounded subsets of E are metrizable for $\mathfrak{T}_s(E, E^*)$, so that the sets A of Theorem 17 certainly cannot be taken to be bounded. Again, if E is separable, the unit ball of E^* will be w*-metrizable, so that $\sigma(A) < \omega_1$ for every $A \subseteq E$, by §§ 2–3 above.

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References

[1] Bourgain J., *New classes of $\mathcal{L}_p$ spaces*, Springer, 1981 (Lecture Notes in Mathematics 889).
[2] Day M.M., *Normed Spaces*, Springer, 1962.
[3] van Douwen E.K., *The integers and topology*, pp. 111–167 in [11].
[4] Dugundji J., *An extension of Tietze's theorem*, Pacific J. Math. **1** (1951), 353–367.
[5] Engelking R., *General Topology*, Heldermann, 1989.
[6] Fremlin D.H., *Supplement to "Convergent sequences in $C_p(X)$"*, University of Essex Mathematics Department Research Report 92-14.
[7] Gerlits J., Nagy Z., *Some properties of $C(X)$*, Topology Appl. **14** (1982), 151–161.
[8] Jameson G.J.O., *Topology and Normed Spaces*, Chapman & Hall, 1974.
[9] Kechris A.S., Louveau A., *Descriptive Set Theory and Sets of Uniqueness*, Cambridge U.P., 1987.
[10] Köthe G., *Topologische Lineare Räume*, Springer, 1960.
[11] Kunen K., Vaughan J.E., *Handbook of Set-Theoretic Topology*, North-Holland, 1984.
[12] Kuratowski K., *Topology*, vol I., Academic, 1966.
[13] Miller A.W., *On the length of Borel hierarchies*, Ann. Math. Logic **16** (1979), 233–267.

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