On powers of Lindelöf spaces

Isaac Gorelic

Abstract. We present a forcing construction of a Hausdorff zero-dimensional Lindelöf space X whose square X^2 is again Lindelöf but its cube X^3 has a closed discrete subspace of size $\mathfrak{c}^+$, hence the Lindelöf degree $L(X^3) = \mathfrak{c}^+$. In our model the Continuum Hypothesis holds true.

After that we give a description of a forcing notion to get a space X such that $L(X^n) = \aleph_0$ for all positive integers n, but $L(X^{\aleph_0}) = \mathfrak{c}^+ = \aleph_2$.

Keywords: forcing, topology, products, Lindelöf

Classification: 54D20,54B10, 03E35

Introduction

It is well-known that a product of two Lindelöf spaces need not be Lindelöf. Indeed, the product of two Sorgenfrey lines has a closed discrete subspace of size $2^{\aleph_0} = \mathfrak{c}$. The general problem of the degree of non-productivity of the Lindelöf property is discussed in [2] and [5].

In 1978, Shelah, Hajnal and Juhasz proved that it is consistent that there is a Lindelöf space whose square has a closed discrete subspace of size $\mathfrak{c}^+ = \aleph_2$ (see [1], [2], [3]).

In 1990 we gave a consistent example of a Lindelöf space whose square has a closed discrete subspace of size $2^{\aleph_1}$, cardinal $2^{\aleph_1}$ arbitrarily large and does not depend on the size of the Continuum, see [4] and [5]. This is the best result up to this point. We conjecture that $2^{\aleph_1}$ is the true upper bound on the sizes of closed discrete subspaces of squares of Lindelöf spaces.

Definition. For a topological space X, $L(X)$ is the smallest cardinal κ such that every open cover of X has a subcover of size at most κ.

It is known about the higher powers of Lindelöf spaces that for each positive integer n, there is a space X such that X^n is Lindelöf, but $L(X^{n+1}) = \mathfrak{c}$, see [6].

The aim of the present paper is to show a consistent example of a space whose square is Lindelöf, but $L(X^3) = \mathfrak{c}^+$. It is an open problem whether $L(X^3) = 2^{\aleph_1} > \aleph_2$ is possible.

Our reference for forcing and basics is Kunen's book [7].

Theorem. Con$(ZF) \Longrightarrow$ Con$(ZFC+CH+$ *"There is a Lindelöf Hausdorff zero-dimensional space X with all points G_δ sets, $|X| = \omega_2 = \mathfrak{c}^+$, $L(X) = L(X^2) = \omega$, and $L(X^3) = \omega_2 = \mathfrak{c}^+$"*$)$.

The proof will consist of the following:

A) Definitions,
B) Main Lemma, and
C) Facts, of which the last Corollary furnishes the space X mentioned in the theorem.

A) Definitions

0. Let $F\colon \omega_2 \times \omega_2 \longrightarrow \{0,1,2\}$ be fixed.

1. $D(f)$ denotes the domain of the function f and if $D(f) \subset \omega_2$, then $\mu(f) :=$ min $D(f)$ or ω_2, if $f = \phi$.

2. For $x \in \omega_2$ and $i \in 3$, let $A^i_x := \{y \in \omega_2 : \ y \neq x \text{ and } F(x,y) = i\}$.

3. $\forall s \in Fn(\omega_2, 3)\ U_s := \bigcap\limits_{x \in D(s)} (A^{s(x)}_x \cup \{x\})$.

4. $\mathcal{U}_F := \{U_s : \ s \in Fn(\omega_2, 3)\}$.

5. F is *flexible* if $(\forall y \neq z$ in $\omega_2)$ $(\forall i,j \in 3)$ $(\exists x \in \omega_2 \backslash \{y,z\})$ $F(x,y) = i$ and $F(x,z) = j$.

6. Define $\varphi\colon \omega_2 \times \omega_2 \longrightarrow \omega_2 + 1$ by letting
$$\begin{aligned}\varphi(y,y) &:= \omega_2, \text{ and for } y \neq z\\ \varphi(y,z) &:= \min\left(\{\delta \in y \cap z : \ F(\delta,y) \neq F(\delta,z)\} \cup \{y \cap z\}\right),\end{aligned}$$
i.e. the least $\delta \in \omega_2$ s.t. $(F(\delta,y) \neq F(\delta,z)$, or $\delta = y$ or $\delta = z)$.

7. We say that $\mathcal{U}_F \times \mathcal{U}_F$ is *sort-of-Lindelöf* if every "cover" $c\colon \omega_2^2 \longrightarrow \big(Fn(\omega_2,3)\big)^2$ satisfying $(\forall \langle y,z\rangle \in \omega_2^2)$ $c(y,z) = \langle s,t\rangle \Longrightarrow$
(i) $\langle y,z\rangle \in U_s \times U_t$, and
(ii) $s{\restriction}\varphi(y,z) = t{\restriction}\varphi(y,z)$
has a countable "subcover", i.e. $\exists$ countable $A \subset \omega_2$ s.t.
$$\forall \langle y,z\rangle \in \omega_2^2\ \exists \langle a,b\rangle \in A^2$$
with $\langle y,z\rangle \in U_{c_1(a,b)} \times U_{c_2(a,b)}$ (where $c_1(a,b)$ is the left coordinate of c(a,b), and $c_2(a,b)$ is the right one).
We remark that (ii) simply means that $\forall y \in \omega_2([c(y,y) = \langle s,t\rangle$ and $y \in D(s) \cap D(t)] \Longrightarrow s(y) = t(y))$.

8. For an $S \subset \omega_2$, let $(S)^0 = S$ and $(S)^1 = \omega_2 \backslash S$.

9. For $k \in Fn(\omega_2, 2)$, let
$$\begin{aligned} V^0_k &= \bigcap_{x \in D(k)} (A^0_x)^{k(x)}\,,\\ V^1_k &= \bigcap_{x \in D(k)} (A^1_x)^{k(x)}\,,\\ V^2_k &= \bigcap_{x \in D(k)} (A^2_x)^{k(x)}\,.\end{aligned}$$

10. Let τ^0, τ^1, τ^2 be topologies on ω_2 generated, respectively, by the following bases:

$$\{V_k^0 :\ k \in Fn(\omega_2, 2)\},$$
$$\{V_k^1 :\ k \in Fn(\omega_2, 2)\},$$
$$\{V_k^2 :\ k \in Fn(\omega_2, 2)\}.$$

So, e.g., τ^0 is generated on ω_2 by a subbasis $\{A_x^0, \omega_2 \backslash A_x^0 : x \in \omega_2\}$.

11. The definition of the forcing notion $(\mathbb{P}, \leq)$. $p \in \mathbb{P}$ iff $p = \langle A, f, T \rangle$, and
   (i) $A \subset \omega_2$ and $|A| \leq \omega$.
   (ii) $f\colon\ A^2 \longrightarrow 3$.
   (iii) $|T| \leq \omega$ and $(\forall B \in T)\ B \subset \big(Fn(A,3)\big)^2$ and $A^2 = \bigcup\{U_s \times U_t :\ \langle s,t \rangle \in B\} \cap A^2$.

(iv) $\forall B \in T$
$\forall \delta, \delta' \in A$
$\forall h \in Fn(A \backslash \delta, 3)$
$\forall h' \in Fn(A \backslash \delta', 3)$
$\forall y \in A \backslash \delta$
$\forall z \in A \backslash \delta'$

(a) $(\exists \langle s,t \rangle \in B)(\exists \langle s', t' \rangle \in B)$
 (α) $\langle y, z \rangle \in U_s \times U_{t \restriction \delta'}$ and $t \not\perp h'$,
 (β) $\langle y, z \rangle \in U_{s \restriction \delta} \times U_t$ and $s \not\perp h$

(b) $(\exists \langle s,t \rangle \in B)$
$$\langle y, z \rangle \in U_{s \restriction \delta} \times U_{t \restriction \delta'} \text{ and } s \not\perp h \text{ and } t \not\perp h'\,.$$

(c) If $y = z$, then $(\exists \langle s,t \rangle \in B)$ and $(\exists \langle s', t' \rangle \in B)$ s.t.
 (α) If $\delta \leq \mu(h')$, then
 $\langle z, z \rangle \in U_{s \restriction \delta} \times U_{t \restriction \delta}$
 and $h \cup s \cup \big(t \restriction \mu(h')\big) \in Fn$ and $t \not\perp h'$, and
 (β) if $\delta' \leq \mu(h)$, then
 $\langle z, z \rangle \in U_{s' \restriction \delta'} \times U_{t' \restriction \delta'}$ and
 $h' \cup t' \cup \big(s' \restriction \mu(h)\big) \in Fn$ and $s' \not\perp h$. □

Let $E^p(\delta, y, z) \overset{df}{\iff} \delta, y, z \in A$ and $\delta \leq y, z$ and $(\forall x \in A \cap \delta)\ f^p(x,y) = f^p(x,z)$.
Let $q \leq p$ if, by definition, $A^q \supset A^p$, $f^q \supset f^p$, $T^q \supset T^p$ and $E^q \supset E^p$. □

B) Main Lemma

Let $V \models ZFC + CH$ and let $\mathbb{P}$ be defined in V by Definition 11. *Then $\mathbb{P}$ is ω_1-complete and has $\omega_2 - cc$.*

Let G be $\mathbb{P}$-generic over V, and let $F = \bigcup\{f^p : p \in G\}$. Then $F\colon\ \omega_2 \times \omega_2 \longrightarrow 3$ is a flexible total function and $\mathcal{U}_F \times \mathcal{U}_F$ is sort-of-Lindelöf.

Proof: The fact that $\mathbb{P}$ is ω_1-complete (i.e. that the naturally defined infimum of a countable descending sequence of conditions belongs to $\mathbb{P}$) is obvious, because "$p \in \mathbb{P}$" is a finitary property (i.e. if $p \notin \mathbb{P}$, then there is a finite collection of finite parts of p (as a structure) witnessing this). $\square$

We will prove 3 lemmas, of which Lemma 1 implies the totality, Lemma 2 implies the flexibility of F, and Lemma 3 establishes the ω_2-chain condition of $\mathbb{P}$. The final statement of the Main Lemma is proved last.

Lemma 1. *Let $p = \langle A, f, T\rangle \in \mathbb{P}$. Then*

$$\begin{aligned}&\forall \tilde{z} \in \omega_2 \backslash A\\ &\exists \tilde{g} : \ (A \cup \{\tilde{z}\})^2 \longrightarrow 3 \textit{ extending } f, \textit{ s.t.}\\ &q \ := \ \langle A \cup \{\tilde{z}\}, \tilde{g}, T\rangle \in \mathbb{P} \textit{ and } q \leq p\,.\end{aligned}$$

Proof. Assume $A\backslash\tilde{z} \neq \phi$ (otherwise, use Lemma 2). So choose the least $a \in A\backslash\tilde{z}$. We will define by induction a partial function g: $A \longrightarrow 3$ such that, if $(\forall x \in A)\ \tilde{g}(x, \tilde{z}) = g(x)$, then $q \in \mathbb{P}$. g will be an increasing union $g = \bigcup\limits_{i<\omega} g_i$. Let g_0: $A \cap \tilde{z} \longrightarrow 3$ be defined by $g_0(x) := f(x, a)$, for every $x \in A \cap \tilde{z} = A \cap a$.

Let

$$\begin{aligned}\mathcal{S} \ &= \ \{\langle B_i, \delta_i, \delta_i', h_i, h_i', y_i, z_i\rangle : \ i < \omega\}\\ &= \ \{B, \delta, \delta', h, h', y, z\rangle : \ B \in T, \ \delta, \delta' \in A\\ &\qquad h \in Fn(A\backslash\delta, 3), \ h' \in Fn(A\backslash\delta', 3),\\ &\qquad y \in (A \cup \{\tilde{z}\})\backslash\delta, \ z \in (A \cup \{\tilde{z}\}\backslash\delta', \ \textit{and}\ (y = \tilde{z} \text{ or } z = \tilde{z})\}\,.\end{aligned}$$

Step $i \geq 1$. Consider $\langle\cdots\rangle_i \in \mathcal{S}$.

Case 1. $y_i = z_i = \tilde{z}$.

0. By (iv)-c-α applied to $\langle a, a\rangle$, $\delta = \delta' = a$, $h = g_i\restriction_{A\backslash a}$ and $h' = \phi$, $\exists\langle s, t\rangle \in B$ s.t. $\langle a, a\rangle \in U_{s\restriction a} \times U_{t\restriction a}$ and $g_i \cup s \cup t \in F_n$. Let $g_{i+1}^0 \ := \ g_i \cup s \cup t$. This will guarantee that

$$\langle\tilde{z}, \tilde{z}\rangle \in U_s \times U_t \qquad \text{for iii}_q.$$

1. Apply (iv)-b to $\langle a, a\rangle$ with $\delta = a$, $h = g_{i+1}^0\restriction_{A\backslash a}$, $\delta' = \delta_i'$, $h' = h_i'$. Then $\exists\langle s, t\rangle \in B$, s.t. $\langle a, a\rangle \in U_{s\restriction a} \times U_{t\restriction\delta_i'}$ and $s \not\perp g_{i+1}^0\restriction_{A\backslash a}$ and $t \not\perp h_i'$.

Let $g_{i+1}^1 = g_{i+1}^0 \cup s$. This will guarantee that

$$\langle\tilde{z}, \tilde{z}\rangle \in U_s \times U_{t\restriction\delta_i'} \quad \text{and} \quad t \not\perp h_i'\,.$$

2. (iv)-b of q is obtained similarly from (iv)-b. We have g_{i+1}^2 at this stage. Note that (b) is automatic because of $E^q(\tilde{z}, \tilde{z}, a)$. And the same applies to (c). Let $g_{i+1} \ := \ g_{i+1}^2$. $\square$

Case 2. $y_i = \tilde{z}$ and $z_i \neq \tilde{z}$, i.e. $z_i \in A$.

0. By (iv)-a-β applied to $\langle a, z_i\rangle$ with $\delta' = a$ and $h' = g_i\restriction_{A\backslash a}$, $\exists\langle s,t\rangle \in B$, s.t. $\langle a, z_i\rangle \in U_{s\restriction a} \times U_t$ and $s \not\perp g_i\restriction_{A\backslash a}$. Let $g^0_{i+1} := g_i \cup s$. This guarantees that

$$\langle \tilde{z}, z_i\rangle \in U_s \times U_t\,.$$

1. Apply (iv)-b to $\langle a, z_i\rangle$ and $\delta = a$, $h = g^1_{i+1}\restriction A\backslash a$, $\delta' = \delta'_i$, $h' = h'_i$. Get $\langle s,t\rangle \in B$ s.t. $\langle a, z_i\rangle \in U_{s\restriction a} \times U_{t\restriction\delta'}$ and $t \not\perp h'_i$ and $s \not\perp g^1_{i+1}\restriction_{A\backslash a}$. Set $g^2_{i+1} := g^1_{i+1} \cup s$, thus guaranteeing

$$\langle \tilde{z}, z_i\rangle \in U_s \times U_{t\restriction\delta'} \quad \text{and} \quad t \not\perp h'_i\,.$$

Note: (a)(β) of q for $\langle \tilde{z}, z_i\rangle$ is automatic because of $E^q(\tilde{z}, \tilde{z}, a)$. (b)$_q$ is automatic for the same reason and (c)$_q$ does not apply here. Set $g_{i+1} = g^1_{i+1}$.

□

Case 3. $y_i \in A$ and $z_i \in \tilde{z}$.

0. By (iv)-a-α, $\exists\langle s,t\rangle \in B$ s.t. $\langle y_i, a\rangle \in U_s \times U_{t\restriction a}$ and $t \not\perp g_i\restriction_{A\backslash a}$. Let $g^0_{i+1} = g_i \cup t$, guaranteeing

$$\langle y_i, \tilde{z}\rangle \in U_s \times U_t\,,$$

i.e. (iii)$_q$ at $\langle y_i, \tilde{z}\rangle$.

1. Note that (a)(α) is automatic. Let $\langle s,t\rangle \in B$ be s.t. $\langle y_i, a\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction a}$ and $s \not\perp h_i$ and $t \not\perp g^0_{i+1}\restriction_{A\backslash a}$ (by (iv)-b). Set $g^1_{i+1} := g^0_{i+1} \cup t$, thus guaranteeing that

$$\langle y_i, \tilde{z}\rangle \in U_{s\restriction\delta_i} \times U_t \quad \text{and} \quad s \not\perp h_i\,.$$

2. Note again that (b) is automatic and (c) does not apply. Let $g_{i+1} := g^1_{i+1}$.

□

Lemma 2. *Let $p = \langle A, f, T\rangle \in \mathbb{P}$. Let $\gamma \in A$, $r \in Fn(A\backslash\gamma, 3)$, $\bar{z} \in A\backslash\gamma$ and $\tilde{z} \in \omega_2\backslash \sup^+ A$. Then $\exists\tilde{g}$: $(A\cup\{\tilde{z}\})^2 \longrightarrow 3$ extending f s.t. $q := \langle A\cup\{\tilde{z}\}, \tilde{g}, T\rangle \in \mathbb{P}$, $q \le p$, $\tilde{z} \in U_r$ and $E^q(\gamma, \bar{z}, \tilde{z})$.*

Proof: Let

$$\begin{aligned}\mathcal{S} &= \{\langle B_i, \delta_i, \delta'_i, h_i, h'_i, y_i, z_i\rangle :\ i < \omega\}\\ &= \{\langle B, \delta, \delta', h, h', y, z\rangle :\ B \in T,\ \delta, \delta' \in A,\ h \in Fn(A\backslash\delta, 3),\\ &\qquad h' \in Fn(A\backslash\delta', 3),\ y \in (A \cup \{\tilde{z}\})\backslash\delta,\ z \in (A \cup \{\tilde{z}\})\backslash\delta' \text{ and}\\ &\qquad (y = \tilde{z} \text{ or } z = \tilde{z})\}\,.\end{aligned}$$

As in the proof of Lemma 1, let $\forall x \in A \cap \gamma$ $g_0(x) = f(x, z)$ and $\forall x \in D(r)$ $g_0(x) = r(x)$, so g_0: $(A \cap \gamma) \cup D(r) \longrightarrow 3$. This guarantees at once that $\tilde{z} \in U_r$.

Step $i > 0$. Consider $\langle \cdots\rangle_i \in \mathcal{S}$.

Case 1. $y_i = z_i = \tilde{z}$.

0. By (iv)-c applied to $\langle \bar{z}, \bar{z} \rangle$ with $h = g_i{\restriction}A\backslash\gamma$, $h' = \phi$, $\delta = \delta' = \gamma$, $\exists\langle s, t\rangle \in B$ s.t. $\langle \bar{z}, \bar{z} \rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $g_i{\restriction}_{A\backslash\gamma} \cup s \cup t \in Fn$.

Let $g^0_{i+1} = g_i \cup s \cup t$. Thus

$$\langle \tilde{z}, \tilde{z} \rangle \in U_s \times U_t \,,$$

and so is covered by B_i.

1. If $\delta'_i \le \gamma$, by (iv)-b, $\exists\langle s, t\rangle \in B_i$, s.t. $\langle \bar{z}, \bar{z} \rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp g^0_{i+1}{\restriction}_{A\backslash\gamma}$ and $t \not\perp h'_i$.

Let $g^1_{i+1} = g^0_{i+1} \cup s$, guaranteeing

$$\langle \tilde{z}, \tilde{z} \rangle \in U_s \times U_{t\restriction\delta'_i} \text{ and } t \not\perp h'_i \,.$$

If $\delta'_i > \gamma$, by (iv)-c-α, $\exists\langle s, t\rangle \in B_i$ s.t. $\langle \bar{z}, \bar{z} \rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g^0_{i+1}{\restriction}_{A\backslash\gamma}) \cup s \cup t{\restriction}\delta'_i \in Fn$ and $t \not\perp h'_i$.

Set $g^0_{i+1} = g^0_{i+1} \cup s \cup t{\restriction}\delta'_i$, implying

$$\langle \tilde{z}, \tilde{z} \rangle \in U_s \times U_{t\restriction\delta'_i} \text{ and } t \not\perp h'_i \,.$$

2. Symmetrically we obtain g^2_{i+1} guaranteeing (a)(β) (i.e. that $\langle \tilde{z}, \tilde{z} \rangle \in U_{s\restriction\delta_i} \times U_t$ and $s \not\perp h_i$, for some $\langle s, t\rangle$ in B_i).

3. Note that if $\delta_i, \delta'_i \le \gamma$, then (b) for $\langle \tilde{z}, \tilde{z} \rangle$ follows automatically from (b) for $\langle \bar{z}, \bar{z} \rangle$. If one of δ_i, δ'_i is $\le \gamma$, then e.g. in $\delta'_i \le \gamma < \delta_i$ case, by (b), $\exists\langle s, t\rangle \in B_i$, s.t. $\langle \bar{z}, \bar{z} \rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp (g^2_{i+1}{\restriction}\delta_i\backslash\gamma) \cup h_i$ and $t \not\perp h'_i$. Let $g^3_{i+1} = g^2_{i+1} \cup s{\restriction}\delta_i$, guaranteeing

$$\langle \tilde{z}, \tilde{z} \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta'_i} \text{ and } s \not\perp h_i \text{ and } t \not\perp h'_i \,.$$

Similarly for the symmetric case of $\delta_i \le \gamma < \delta'_i$.

If $\gamma < \delta_i, \delta'_i$, then if $\delta_i \le \delta'_i$ use $(c)(b)$, and if $\delta_i > \delta'_i$, use (c)(α), e.g. if $\delta_i \le \delta'_i$, then $\gamma \le \mu(h_i) \ge \delta_i$. So by $(c)(\beta)$, $\exists\langle s, t\rangle \in B_i$, s.t. $\langle \bar{z}, \bar{z} \rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g^2_{i+1}{\restriction}\delta'_i\backslash\gamma) \cup h'_i \cup t \cup s{\restriction}\delta_i \in Fn$ and $s \not\perp h_i$. Let $g^3_{i+1} = g^2_{i+1} \cup (t{\restriction}\delta'_i) \cup (s{\restriction}\delta_i)$, guaranteeing

$$\langle \tilde{z}, \tilde{z} \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta'_i} \text{ and } s \not\perp h_i \text{ and } t \not\perp h'_i \,.$$

4. Suppose $\gamma < \delta_i \le \mu(h'_i)$. By (c)($\alpha$), $\exists\langle s, t\rangle \in B$ s.t. $\langle \bar{z}, \bar{z} \rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $s \cup [(g^3_{i+1}{\restriction}\delta_i\backslash\gamma) \cup h_i] \cup (t{\restriction}\mu(h'_i)) \in Fn$ and $t \not\perp h'_i$. Set $g^4_{i+1} = g^3_{i+1} \cup (s{\restriction}\delta_i) \cup (t{\restriction}\delta_i)$, guaranteeing

$$\langle \tilde{z}, \tilde{z} \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i} \,,$$
$$h_i \cup s \cup t{\restriction}\mu(h'_i) \in Fn \text{ and } t \not\perp h'_i \,.$$

5. Suppose now that also $\gamma < \delta'_i \le \mu(h_i)$. Then, by (c)($\beta$) $\exists\langle s,t\rangle \in B$ s.t. $\langle \bar{z},\bar{z}\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $[(g^4_{i+1}\restriction\delta'_i\backslash\gamma) \cup h'_i] \cup t \cup (s\restriction\mu(h_i)) \in Fn$ and $s \not\perp h_i$. Let $g^5_{i+1} = g^4_{i+1} \cup (t\restriction\delta'_i) \cup (s\restriction\delta'_i)$, guaranteeing

$$\langle \tilde{z},\tilde{z}\rangle \in U_{s\restriction\delta'_i} \times U_{t\restriction\delta'_i}\,,$$
$$h'_i \cup t \cup (s\restriction\mu(h_i)) \in Fn \text{ and } \ s \not\perp h_i\,.$$

Finally, let $g_{i+1} = g^5_{i+1}$.

Case 2. $y_i = \tilde{z}$ and $z_i \in A$.

0. By (a)(β), $\langle \bar{z}, z_i\rangle \in U_{s\restriction\gamma} \times U_t$ with $s \not\perp g_i\restriction_{A\backslash\gamma}$. Let $g^0_{i+1} = g_i \cup s$, guaranteeing

$$\langle \tilde{z}, z_i\rangle \in U_s \times U_t\,.$$

1. By (b), $\langle \bar{z}, z_i\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp g^0_{i+1}\restriction_{A\backslash\gamma}$ and $t \not\perp h'_i$. Let $g'_{i+1} = g^0_{i+1} \cup s$. Then

$$\langle \tilde{z}, z_i\rangle \in U_s \times U_{t\restriction\delta_i} \quad \text{and} \quad t \not\perp h'_i\,.$$

2. W.l.o.g., $\gamma < \delta_i$. $\langle z, z_i\rangle \in U_{s\restriction\gamma} \times U_t$ and $s \not\perp (g'_{i+1}\restriction\delta_i\backslash\gamma) \cup h_i$. Let $g^2_{i+1} = g'_{i+1} \cup (s\restriction\delta_i)$. Then

$$\langle \tilde{z}, z_i\rangle \in U_{s\restriction\delta_i} \times U_t \quad \text{and} \quad s \not\perp h_i\,.$$

3. $\langle z, z_i\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp (g^2_{i+1}\restriction\delta_i\backslash\gamma) \cup h_i$ and $t \not\perp h'_i$. Let $g^3_{i+1} = g^2_{i+1} \cup (s\restriction\delta_i)$. Then

$$\langle \tilde{z}, z_i\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta'_i}\,, s \not\perp h_i \quad \text{and} \quad t \not\perp h'_i\,.$$

Finally, let $g_{i+1} = g^3_{i+1}$.

Case 3. $y_i \in A$ and $z_i = \tilde{z}$. This is symmetric to Case 2.

End of the i-th induction step. □

Lemma 3. *$\mathbb{P}$ has $\omega_2 - cc$.*

PROOF: Let $\mathbb{Q} \subset \mathbb{P}$ with $|\mathbb{Q}| \ge \omega_2$. By CH and the Δ-system lemma, we may assume that there are $p \ne p'$ in $\mathbb{Q}$, $p = \langle A, f, T\rangle$, $p' = \langle A', f', T'\rangle$ such that

$$A \cap A' \;=: \;\Delta < A\backslash\Delta < A'\backslash\Delta\,,$$

$tp\, A = tp\, A'$ and f, f' are "typewise the same", so $f\restriction\Delta^2 = f'\restriction\Delta^2$, and $z \in A$ and $z' \in A'$ with $tp(A \cap z) = tp(A \cap z')$ implies $(\forall x \in A)\ f(x,z) = f(x,z')$.

Let $\gamma :=$ the least ordinal in $A\backslash\Delta$, and let z' denote the member of A' corresponding to $z \in A$. (So $tp(A \cap z) = tp(A' \cap z)$ and $A' \cap \gamma' = \Delta$).

We want to extend $(f \cup f')$ to $g\colon (A \cup A')^2 \longrightarrow 3$ so that $q := \langle A \cup A', g, T \cup T' \rangle \in \mathbb{P}$ and $q \leq p, p'$. First we will define g on $(A \backslash \Delta) \times (A' \backslash \Delta)$. For every $z \in A \backslash \Delta$, let

$$g_{-1}^{z} \;=\; \phi\,.$$

By induction in ω steps, we will extend every $g_{-1}^{z} (z \in A \backslash \Delta)$ to a partial function $g^z\colon A \backslash \Delta \longrightarrow 3$ s.t., in the process,

(i) $(\forall i)\ g_i^z$ will all be finite.

(ii) $E^p(\gamma, y, z) \Longrightarrow (\forall i)\ g_i^z = g_i^y$.

Let

$$\begin{aligned} \mathcal{S} = \quad & \{\langle B_i, \delta_i, \delta_i', h_i, h_i', y_i, z_i \rangle :\ i < \omega\} \\ r = \quad & \{\langle B, \delta, \delta', h, h', \tilde{y}, \tilde{z} \rangle :\ B \in T,\ \delta, \delta' \in A,\ h \in Fn(A \backslash \delta, 3), \\ & h' \in Fn(A \backslash \delta', 3),\ \tilde{y}, \tilde{z} \in (A \cup A') \backslash \Delta,\ \delta \leq \tilde{y},\ \delta' \leq \tilde{z}\}\,. \end{aligned}$$

Step $i \geq 0$. Consider $\langle \cdots \rangle_i \in \mathcal{S}$.

There are 3 relevant cases (for the future pairs involving $y, z \in A \backslash \Delta$):

(1) $\tilde{y}_i \in A$ and $\tilde{z}_i = z' \in A'$

(2) $\tilde{y}_i = y' \in A'$ and $\tilde{z}_i \in A$

(3) $\tilde{y}_i = y' \in A'$ and $\tilde{z}_i = z' \in A'$.

Case 1. $\tilde{y}_i = y \in A$ and $\tilde{z}_i = z' \in A'$.

0. By (a)(α), $\exists \langle s, t \rangle \in B_i$ s.t. $\langle y, z \rangle \in U_s \times U_{t \restriction \gamma}$ and $t \not\perp g_{i-1}^z$.

Let $g_i^{z^0} := g_{i-1}^z \cup t\restriction_{A \backslash \gamma}$. This will guarantee that

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_s \times U_t\,.$$

(∗) Let also $g_i^{x^0} := g_i^{z^0}$ for every $x \in A \backslash \gamma$ with $E^p(\gamma, x, z)$.

Let $g_i^{x^0} := g_{i-1}^x$ for all other $x \in A \backslash \gamma$.

1. By (a)(β) of p, $\exists \langle s, t \rangle \in B_i$ s.t. $\langle y, z \rangle \in U_s \times U_{t \restriction \gamma}$ and $t \not\perp (g_i^{z^0} \restriction \delta_i') \cup h_i'$. Let $g_i^{z^1} := g_i^{z^0} \cup (t \restriction (\delta_i' \backslash \gamma))$, guaranteeing

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_s \times U_{t \restriction \delta_i'} \quad \text{and} \quad t \not\perp h_i'\,.$$

Then $(*)$-update, i.e. let $g_i^{x^1} = g_i^{z^1}$ for every $x \in A \backslash \gamma$ with $E^p(\gamma, x, z)$, and $g_i^{x^1} = g_i^{x^0}$ for all other $x \in A \backslash \gamma$.

2. Concerning (a)(β): Similarly, by (b) of p get $\langle s, t \rangle \in B_i$ s.t. $\langle y, z \rangle \in U_{s \restriction \delta_i} \times U_{t \restriction \gamma}$ and $s \not\perp h_i$ and $t \not\perp g_i^{z^1}$.

Let $g_i^{z^2} = g_i^{z^1} \cup (t \restriction_{(A \backslash \gamma)})$, guaranteeing that

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_{s \restriction \delta_i} \times U_t \quad \text{and} \quad s \not\perp h_i\,.$$

Then $(*)$-update.

3. (b) Assume w.l.o.g. that $\gamma < \delta'_i$. Here $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\gamma}$ and $s \not\perp h_i$ and $t \not\perp g_i^{z^2}\restriction\delta'_i \cup h'_i$. Let $g_i^{z^3} := g_i^{z^2} \cup t\restriction(\delta'_i\backslash\gamma)$. This will guarantee that

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta'_i}\,, \quad s \not\perp h_i \quad \text{and} \quad t \not\perp h'_i\,.$$

Then $(*)$-update all γ-twins of z.

Finally, let $\forall x \in A\backslash\Delta$, $g_i^x := g_i^{x^3}$ $\square$

Case 2. Is entirely symmetric.

Case 3. $\tilde{y}_i = y' \in A'$ and $\tilde{z}_i = z' \in A'$
Subcase 3a. $E^p(\gamma,y,z)$. So $g_{i-1}^z = g_{i-1}^y$.

0. By (c)(α) of p, $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $g_{i-1}^y \cup s \cup t \in Fn$ (i.e. $h' = \phi$ and $h = g_{i-1}^y$ here).

Let $g_i^{y^0} := g_{i-1}^y \cup (s\cup t)\restriction(A\backslash\gamma)$. Also $(*)$-update g_{i-1}^x's, i.e. for every $x \in A\backslash\gamma$ s.t. $E^p(\gamma,y,x)$, set $g_i^{x^0} := g_i^{y^0}$, and for every other $x \in A\backslash\Delta$, set $g_i^{x^0} = g_{i-1}^x$. This will guarantee that

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_s \times U_t\,.$$

1. (a)(α) If $\delta'_i \le \gamma$, then by (b), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp g_i^{y^0}$ and $t \not\perp h'_i$. Let $g_i^{y^1} := g_i^{y^0} \cup s\restriction(A\backslash\gamma)$, and $(*)$-update. Then

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_s \times U_{t\restriction\delta'_i} \quad \text{and} \quad t \not\perp h'_i\,.$$

If $\delta'_i > \gamma$, then by (c)(α), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $g_i^{y^0} \cup s \cup t\restriction\delta'_i \in Fn$ and $t \not\perp h'_i$. Let $g_i^{y^1} := g_i^{y^0} \cup (s\restriction(A\backslash\gamma)) \cup (t\restriction\delta'_i)$, and $(*)$-update. Then

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_s \times U_{t\restriction\delta'_i} \quad \text{and} \quad t \not\perp h'_i\,.$$

2. Concerning $(a)(\beta)$: Similarly to **1**, get $\langle s,t\rangle \in B_i$ and update to $g_i^{y^2}$, guaranteeing

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_{s\restriction\gamma} \times U_t \quad \text{and} \quad s \not\perp h_i\,.$$

3. Concerning (b). There are 4 possibilities here:

(1) $\delta_i \le \gamma$ and $\delta'_i \le \gamma$
(2) $\delta_i \le \gamma$ and $\delta'_i > \gamma$
(3) $\delta_i > \gamma$ and $\delta'_i \le \gamma$
(4) $\delta_i > \gamma$ and $\delta'_i > \gamma$, 4(a) $\delta_i \le \delta'_i$, 4(b) $\delta'_i < \delta_i$.

If (1) — there is nothing to do: make $g_i^{x^3} = g_i^{x^2}$ for all $x \in A\backslash A$.

If (2), then by (b), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\gamma}$ and $s \not\perp h_i$ and $t \not\perp (g_i^{y^2}\restriction\delta'_i) \cup h'_i$.

Let $g_i^{y^3} = g_i^{y^2} \cup t{\restriction}(\delta_i' \backslash \gamma)$ and $(*)$-update. Then

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i'}\,, \quad s \not\leq h_i \quad \text{and} \quad t \not\leq h_i'\,.$$

If (3), act similarly.

If 4(a), then by (c)(α), $\exists \langle s, t\rangle \in B_i$ s.t. $\langle y, y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g_i^{y^2}{\restriction}\delta_i) \cup h_i \cup s \cup t{\restriction}\delta_i' \in Fn$.

Let $g_i^{y^3} = g_i^{y^2} \cup s{\restriction}(\delta_i \backslash \gamma) \cup t{\restriction}(\delta_i' \backslash \gamma)$ and $(*)$-update. Then

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i'}\,, \quad s \not\leq h_i \quad \text{and} \quad t \not\leq h'\,.$$

If 4(b), then by (c)(β), $\exists \langle s, t\rangle \in B_i$ s.t. $\langle y, y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g_i^{y^2}{\restriction}\delta_i') \cup h_i' \cup t \cup s{\restriction}\delta_i \in Fn$.

Let $g_i^{y^3} = g_i^{y^2} \cup t{\restriction}(\delta_i' \backslash \gamma) \cup s{\restriction}(\delta_i \backslash \gamma)$ and $(*)$-update. Then the same formula as in 4(a) holds.

4. Concerning (c)(α): If $y = z$ and $\delta_i \leq \mu(h_i)$, then w.l.o.g. $\gamma < \delta_i$ and, by (c)(α) of p, $\exists \langle s, t\rangle \in B_i$ s.t. $\langle y, y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g_i^{y^3}{\restriction}\delta_i) \cup h_i \cup s \cup (t{\restriction}\mu(h_i')) \in Fn$, and $t \not\leq h_i'$.

Let $g_i^{y^4} = g_i^{y^3} \cup s{\restriction}(\delta_i \backslash \gamma) \cup t{\restriction}(\delta_i \backslash \gamma)$, and $(*)$-update. Then

$$\langle \tilde{z}_i, \tilde{z}_i \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i}\,, \quad h_i \cup s \cup (t{\restriction}\mu(h_i')) \in Fn \quad \text{and} \quad t \not\leq h_i'\,.$$

5. Concerning (c)(β): If $y = z$ and $\delta_i' \leq \mu(h_i)$, then w.l.o.g. $\gamma < \delta_i'$, and by (c)(β) of p, $\exists \langle s, t\rangle \in B_i$ s.t. $\langle y, y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g_i^{y^4}{\restriction}\delta_i') \cup h_i' \cup t \cup (s{\restriction}\mu(h_i)) \in Fn$ and $s \not\leq h_i$.

Let $g_i^{y^5} = g_2^{y^4} \cup t{\restriction}(\delta_i' \backslash \gamma) \cup s{\restriction}(\delta_i' \backslash \gamma)$ and $(*)$-update. Then

$$\langle \tilde{z}_i, \tilde{z}_i \rangle \in U_{s\restriction\delta_i'} \times U_{t\restriction\delta_i'}\,,$$
$$h_i' \cup t \cup (s{\restriction}\mu(h_i)) \in Fn \quad \text{and} \quad s \not\leq h_i\,.$$

□

Subcase 3b. Not — $E^p(\gamma, y, z)$.

0. By (b) of p, $\exists \langle s, t\rangle \in B_i$ s.t. $\langle y, z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $s \not\leq g_{i-1}^y$ and $t \not\leq g_{i-1}^z$.

Let $g_i^{y^0} = g_{i-1}^y \cup s{\restriction}(A \backslash \gamma)$ and $g_i^{z^0} = g_{i-1}^z \cup t{\restriction}(A \backslash \gamma)$. Then

$$\langle \tilde{y}_i, \tilde{y}_i \rangle \in U_s \times U_t\,.$$

Then **$(*)$**-update, i.e.

(a) for every $x \in A \backslash \Delta$ s.t. $E^p(\gamma, x, y)$, set $g_i^{x^0} = g_i^{y^0}$,
(b) for every $x \in A \backslash \Delta$ s.t. $E^p(\gamma, x, z)$, set $g_i^{x^0} = g_i^{z^0}$ and
(c) for every other $x \in A \backslash \Delta$, set $g_i^{x^0} = g_{i-1}^x$.

1. Concerning (a)(α): Again, if $\delta_i' \le \gamma$, then, by (b) of p, $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta_i'}$, $s \not\perp g_i^{y^0}$ and $t \not\perp h_i'$.

Let $g_i^{y^1} = g_i^{y^0} \cup s\restriction(A\backslash\gamma)$. Then

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_s \times U_{t\restriction\delta_i'} \quad \text{and} \quad t \not\perp h_i' \,.$$

Then ($*$)-update, i.e. all $x \in A\backslash\delta$ with $E^p(\gamma,x,y)$ will get $g_i^{x^1} = g_i^{y^0}$.

If $\gamma < \delta_i'$, then $\exists\langle s,t\rangle \in B_i$, s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$, $s \not\perp g_i^{y^0}$ and $t \not\perp (g_i^{z^0}\restriction\delta_i') \cup h_i'$.

Let $g_i^{y^1} = g_i^{y^0} \cup s\restriction(A\backslash\gamma)$ and $g_i^{z^1} = g_i^{z^0} \cup t\restriction(\delta_i'\backslash\gamma)$, and ($**$)-update, as (mutatis mutandis) in **0.**

2. Re (a)(β). If $\delta_i \le \gamma$, then $\exists\langle s,t\rangle \in B$, s.t. $\langle y,z\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\gamma}$, $s \not\perp h_i$ and $t \not\perp g_i^{z^1}$.

Let $g_i^{z^2} = g_i^{z^1} \cup (t\restriction(A\backslash\gamma))$ and $g_i^{y^2} = g_i^{y^1}$ and ($**$)-update, as in **0**. Then

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_{s\restriction\delta_i} \times U_t \quad \text{and} \quad s \not\perp h_i \,.$$

If $\gamma < \delta_i$, then $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$, $s \not\perp (g_i^{y^1}\restriction\delta_i) \cup h_i$ and $t \not\perp g_i^{z^1}$.

Let $g_i^{y^2} = g_i^{y^1} \cup (s\restriction(\delta_i\backslash\gamma))$ and $g_i^{z^2} = g_i^{z^1} \cup (t\restriction(A\backslash\gamma))$ and ($**$)-update. Then the formula above holds.

3. (b) Again, there are 4 possibilities here:

(1) $\delta_i \le \gamma$ and $\delta_i' \le \gamma$,
(2) $\delta_i \le \gamma$ and $\delta_i' > \gamma$,
(3) $\delta_i > \gamma$ and $\delta_i' \le \gamma$,
(4) $\delta_i > \gamma$ and $\delta_i' > \gamma$.

If (1), do nothing.

If (2), then by (b), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\gamma}$, $s \not\perp h_i$ and $t \not\perp (g_i^{z^2}\restriction\delta_i') \cup h_i'$.

Let $g_i^{y^3} = g_i^{y^2}$ and $g_i^{z^3} = g_i^{z^2} \cup (t\restriction(\delta_i'\backslash\gamma))$. Then ($**$)-update. Then

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i'} \,, \quad s \not\perp h_i \quad \text{and} \quad t \not\perp h_i' \,.$$

If (3), then, by (b), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta_i'}$, $s \not\perp g_i^{y^2}\restriction\delta_i \cup h_i$ and $t \not\perp h_i'$.

Let $g_i^{y^3} = g_i^{y^2} \cup (s\restriction(\delta_i\backslash\gamma))$ and $g_i^{z^3} = g_i^{z^2}$. Then ($**$)-update.

If (4), then $\exists\langle s,t\rangle \in B$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$, $s \not\perp g_i^{y^2}\restriction\delta_i \cup h_i$ and $t \not\perp g_i^{z^2}\restriction\delta_i' \cup h_i'$.

Let $g_i^{y^3} = g_i^{y^2} \cup (s \restriction (\delta_i \backslash \gamma))$ and $g_i^{z^3} = g_i^{z^2} \cup (t \restriction (\delta_i' \backslash \gamma))$. Then $(**)$-update.
End of the Subcase 3b and of Case 3. $\square$

For every $z \in A \backslash \Delta$, g_i^z is defined as the most recent value.
End of the i-th induction step. $\square$

At the end of induction, let for every $z \in A \backslash \Delta$

$$g^z = \bigcup_{i<\omega} g_i^z .$$

Finally, define g on $(A \backslash \Delta) \times (A' \backslash \Delta)$ by the following rule:

$$g(y, z') = \begin{cases} g^z(y), & \text{if } y \in \operatorname{dom}(g^z) \\ 0 & \text{otherwise.} \end{cases}$$

The extension procedure for g on $(A' \backslash \Delta) \times (A \backslash \Delta)$ and the condition p' is the same. (We do not have to take care there of γ'-twins, but we may).

Since the construction has, as in side remarks, a verification of the conditions (iii) and (iv) of q, we are done. $\square$

The proof that in $V[G]$ $\mathcal{U}_F \times \mathcal{U}_F$ is sort-of-Lindelöf:

1. Let $c \in V[G]$ be as in the definition (7), and let σ be a $\mathbb{P}$-name for it.

2. It is enough to show that, for every $p \in \mathbb{P}$ with

$$(*) \qquad p \Vdash \text{“definition (7) for } \sigma\text{”} ,$$

there are $p^* \leq p$ and a countable $A^* \subset \omega_2$ s.t.

$$p^* \Vdash \check{\omega}_2^2 = \bigcup \{\dot{U}_{\sigma 1(y,z)} \times \dot{U}_{\sigma 2(y,z)} : \ \langle y, z \rangle \in \check{A}^* \times \check{A}^* \}.$$

3. Note that $\forall q \in \mathbb{P}$ with $q \Vdash$ “$(*)$”, $\forall \langle y, z \rangle \in \omega_2^2$ $\exists r = r(q, y, z) \leq q$ and $\exists \langle s, t \rangle \in (Fn(\omega_2, 3))^2$ s.t. $r \Vdash \sigma(\check{y}, \check{z}) = \langle \check{s}, \check{t} \rangle$ and $D(s) \cup D(t) \subset A^r$.

4. Let φ: $\omega \longrightarrow \omega \times \omega$ be a bijection s.t. $(\forall n \in \omega)$ $(\varphi(n) = \langle i, j \rangle \Rightarrow n \geq i)$.

5. We will construct an ω-sequence of conditions $p_0 \geq p_1 \geq \cdots \geq p_n \geq \cdots$, $n < \omega$ by induction, starting with $p_0 = p$.

6. If $p_i = \langle A_i, f_i, T_i \rangle$ has been already constructed, we fix an ω-enumeration

$$\begin{aligned} \mathcal{S}^i &= \{\langle \delta_j^i, y_j^i, z_j^i, h_j^i \rangle : \ j < \omega\} \\ &= \{\langle \delta, y, z, h \rangle : \ \delta, y, z \in A_i, \ \delta \leq z, \ h \in Fn(A_i \backslash \delta, 3)\}. \end{aligned}$$

7. *Step $n+1$, for $n \geq 0$.* How to choose p_{n+1}?

(1) Find $\varphi(n) = \langle i, j\rangle$, $i \leq n$.
(2) Consider $\langle \delta^i_j, y^i_j, z^i_j, h^i_j\rangle \in \mathcal{S}^i$ and pick $z_n \in \omega_2 \backslash \sup^+ A_n$.
(3) Apply Lemma 2 to p_n and z_n, to get $q_n \leq p_n$ such that

$$z_n \in A^{q_n}$$
$$E^{q_n}(\delta^i_j, z^i_j, z_n) \text{ holds, and}$$
$$z_n \in U_{h^i_j}.$$

(4) Apply note in **3.** to get $p_{n+1} = r(q_n, y^i_j, z^i_j)$.

8. So $p_{n+1} || \!- \sigma(\check{y}^i_j, \check{z}^i_j) = \langle \check{s}_n, \check{t}_n\rangle$, for some s_n, t_n in $Fn(A_{n+1}, 3)$. Also, for every μ,

$$E^{p_{n+1}}(\mu, y^i_j, z^i_j) \longrightarrow s_n \restriction \mu \ = \ t_n \restriction \mu\,.$$

(Because here $p_{n+1} || \!- \dot{\varphi}(\check{y}, \check{z}) \geq \check{\mu}$).

9. Let $q^* := \langle A^*, f^*, T^*\rangle$, where $A^* = \bigcup_i A_i$, $f^* = \bigcup_i f_i$, $T^* = \bigcup_i T_i$. Then $q^* \in \mathbb{P}$, because $\mathbb{P}$ is ω_1-complete.

10. Let

$$\begin{aligned} B^* &:= \{\langle s_n, t_n\rangle : \ n < \omega\} \\ &= \{\langle s, t\rangle \in (Fn(A^*, 3))^2 : \ (\exists \langle y, z\rangle \in A^* \times A^*) \\ &\quad\ (q^* || \!- \sigma(\check{y}, \check{z}) = \langle \check{s}, \check{t}\rangle)\}\,. \end{aligned}$$

Let $p^* := \langle A^*, f^*, T^* \cup \{B^*\}\rangle$.

11. Claim $p^* \in \mathbb{P}$.

Regarding (iii) of p^* at B^*.

$< y, z > \in \ A^* \times A^* \Rightarrow (\exists n \in \omega)$ s.t. $\varphi(n) =< i, j >$ and $< y, z > = < y^i_j, z^i_j >$. Then $p_{n+1} || \!- \sigma(\check{y}, \check{z}) = \langle \check{s}_n, \check{t}_n\rangle$, as remarked in **8**. Then $q^* || \!- (\langle \check{y}, \check{z}\rangle \in \dot{U}_{\check{s}_n} \times \dot{U}_{\check{t}_n})$, because $q^* \leq p$ and $\leq p_{n+1}$. Then $\langle y, z\rangle \in U^{q^*}_{s_n} \times U^{q^*}_{t_n}$, by absoluteness (because $D(s_n) \cup D(t_n) \subset A^*$). □

Regarding (iv) of p^* at B^*

Suppose $\delta, \delta' \in A$, $h \in Fn(A\backslash\delta, 3)$, $h' \in Fn(A\backslash\delta', 3)$, $y \in A\backslash\delta$, $z \in A\backslash\delta'$.

(a)(α). Find $n \in \omega$ s.t. $\varphi(u) = \langle i, j\rangle$ and $z = z^i_j$, $h' = h^i_j$ and $\delta' = \delta^i_j$.

By (iii) p^* already checked, $\exists k \in \omega$ s.t. $\langle y, z_n\rangle \in U_{s_k} \times U_{t_k}$ and, by choice in **7**, $E^{q^*}(\delta', z, z_n)$ and $z_n \in U_{h'} \backslash D(h')$, so $t_k \not\perp h'$. Then

$$\langle y, z\rangle \in U_{s_k} \times U_{t_k \restriction \delta'} \quad \text{and} \quad t_k \not\perp h'\,.$$

(a)(β). Similarly, find $n \in \omega$ s.t. $\varphi(n) = \langle i, j\rangle$, $y = z^i_j$, $h = h^i_j$, $\delta = \delta^i_j$. Then, by (iii) of q^*, $\exists k \in \omega$ s.t. $\langle z_n, z\rangle \in U_{s_k} \times U_{t_k}$ and $E^{q^*}(\delta, y, z_n)$ and $z_n \in U_h \backslash D(h)$. Then

$$\langle y, z\rangle \in U_{s_k\restriction\delta} \times U_t \quad \text{and} \quad s_k \not\leq h\,.$$

(b). Find $n_1, n_2 \in \omega$ s.t. $\varphi(n_1) = \langle i_1, j_1\rangle$, $\varphi(n_2) = \langle i_2, j_2\rangle$, and $y = z^{i_1}_{j_1}$, $\delta = \delta^{i_1}_{j_1}$, $h = h^{i_1}_{j_1}$ and $z = z^{i_2}_{j_2}$, $\delta' = \delta^{i_2}_{j_2}$, $h' = h^{i_2}_{j_2}$. Then $E^{q^*}(\delta, y, z_{n_1})$, $z_{n_1} \in U_h \backslash D(h)$, $E^{q^*}(\delta', z, z_{n_2})$ and $z_{n_2} \in U_{h'} \backslash D(h')$, by construction.

By (iii) of p^*, $\exists k \in \omega$ s.t. $\langle z_{n_1}, z_{n_2}\rangle \in U_{s_k} \times U_{t_k}$, implying that

$$\langle y, z\rangle \in U_{s_k\restriction\delta} \times U_{t_k\restriction\delta'} \quad \text{and} \quad s_k \not\leq h \text{ and } t_k \not\leq h'\,.$$

(c)(α). Suppose $y = z$ and $\delta \leq \mu(h')$. Find $n_1, n_2 \in \omega$ s.t. $\varphi(n_1) = \langle i_2, j_1\rangle$, $\varphi(n_2) = \langle i_2, j_2\rangle$, $z = z^{i_1}_{j_1}$, $\delta = \delta^{i_1}_{j_1}$, $h = h^{i_1}_{j_1}$ and $z_{n_1} = z^{i_2}_{j_2}$, $\mu(h') = \delta^{i_2}_{j_2}$, $h' = h^{i_2}_{j_2}$. Then $E^{q^*}(\delta, z, z_{n_1})$, $z_{n_1} \in U_h \backslash D(h)$ and $E^{q^*}(\mu(h'), z_{n_1}, z_{n_2})$, $z_{n_2} \in U_{h'} \backslash D(h')$.

By (iii) of p^*, pick $k \in \omega$ s.t. $\langle z_{n_1}, z_{n_2}\rangle \in U_{s_k} \times U_{t_k}$. This implies that

$$\langle z, z\rangle \in U_{s_k\restriction\delta} \times U_{t_k\restriction\delta}\,, h \cup s_k \cup (t_k\restriction\mu(h')) \in Fn \quad \textit{and} \quad t \not\leq h'$$

because $s_k\restriction\mu(h') = t_k\restriction\mu(h')$, by **8.**

(c)(β). Similarly, assuming $y = z$ and $\delta' \leq \mu(h)$, find $n_1, n_2 \in \omega$, $\varphi(n_1) = \langle i_1, j_1\rangle$, $\varphi(n_2) = \langle i_2, j_2\rangle$ s.t. $z = z^{i_1}_{j_1}$, $\delta' = \delta^{i_1}_{j_1}$, $h' = h^{i_1}_{j_1}$, $z_{n_1} = z^{i_2}_{j_2}$, $\mu(h) = \delta^{i_2}_{j_2}$, $h = h^{i_2}_{j_2}$. Then $E^{q^*}(\delta', z, z_{n_1})$, $z_{n_1} \in U_{h'} \backslash D(h')$ and $E^{q^*}(\mu(h), z_{n_1}, z_{n_2})$, $z_{n_2} \in U_h \backslash D(h)$. Let $\langle z_{n_2}, z_{n_1}\rangle \in U_{s_k} \times U_{t_k}$ for some $k \in \omega$. Then

$$\langle z, z\rangle \in U_{s_k\restriction\delta'} \times U_{t_k\restriction\delta'},\ h' \cup t_k \cup (s_k\restriction\mu(h)) \in Fn \quad \text{and} \quad s_k \not\leq h$$

because $s_k\restriction\mu(h) = t_k\restriction\mu(h)$, by **8.** $\square$

12. Finally, $p^* \in \mathbb{P} \Rightarrow p^* \leq p$ and

$$\begin{aligned} p^* \Vdash \text{“}\omega_2^2 &= \bigcup\{\dot{U}_s \times \dot{U}_t :\ \langle s, t\rangle \in B^*\} \\ &= \bigcup\{\dot{U}_{\sigma_1(y,z)} \times \dot{U}_{\sigma_2(y,z)} :\ \langle y, z\rangle \in \check{A}^* \times \check{A}^*\}\text{”}\,. \end{aligned}$$

(The first line is a consequence of Lemma 1 and $p^* \Vdash$ “$\check{A}^* \times \check{A}^* = \bigcup\{\dot{U}_s \times U_t :\ \langle s, t\rangle \in B^*\}$”.) As required. $\square$

This concludes the proof of the Main Lemma.

C) Facts about F in V[G]

Fact 1. $\mathcal{U}_F$ is a Lindelöf family, i.e. every $\mathcal{U}_F$-cover of ω_2 has a countable subcover.

Proof: Let c: $\omega_2 \longrightarrow Fn(\omega_2, 3)$ be a $\mathcal{U}_F$-cover of ω_2, i.e. $\forall y \in \omega_2\ y \in U_{c(y)}$. If $z \in \omega_2$ and $\varphi(y,z) = \delta$, then $z \in U_{c(y)\restriction\delta}$. Define d: $\omega_2^2 \longrightarrow (Fn(\omega_2,3))^2$ by

$$d(y,z) \;=\; \langle c(y), c(y)\restriction\varphi(y,z)\rangle\,.$$

Then d is as in Definition (7). By Main Lemma, $\exists$ countable $A \subset \omega_2$ s.t. $d''A^2$ covers ω_2^2. But then $c''A$ covers ω_2. [$y \in \omega_2 \Rightarrow \langle y, 0\rangle \in U_s \times U_t$, where $\langle s,t\rangle = d(a,b)$ for some $\langle a,b\rangle \in A^2 \Rightarrow y \in U_s = U_{c(a)}$ by definition of d]. □

Fact 2. Each of τ^0, τ^1, τ^2 is a Lindelöf topology.

Proof: Let $\mathcal{C}$ be a cover of ω_2 by τ^0-basic open sets, i.e.

$$\omega_2 \;=\; \bigcup\{V_k^0 :\; k \in \mathcal{C}\}\,, \quad \mathcal{C} \subset Fn(\omega_2, 2)\,.$$

$\forall z \in \omega_2$ pick $k_z \in \mathcal{C}$ s.t. $z \in V^0_{k_z}$. Let s_z: $D(k_z) \longrightarrow 3$ be defined by

$$s_z(x) \;=\; \begin{cases} i \in 3 \text{ s.t. } s \in A^i_x\,, & \text{if } z \neq x \\ 1 \text{ (or 2)} & \text{if } z = x \end{cases}$$

Then $z \in U_{s_z}$ and $\exists F_z \subset D(k_z)$ s.t.

$$z \in U_{s_z} \backslash F_z \;\subset V^0_{k_z}.$$

$$[\text{Indeed, } \forall x \in D(k_z) \quad A_x^{s_z(x)} \subset (A^0_x)^{k_z(x)}$$

$$\Downarrow$$

$$\bigcap_{x\in D(k_z)} A_x^{s_z(x)} \subset V^0_{k_z}\,.$$

Let $F_z = (\bigcap\limits_{x \in D(k_z)} (A_x^{s_z(x)} \cup \{x\}))\backslash \bigcap_{x\in D(k_z)} A_x^{s_z(x)} \subset D(k_z)$.

Then $U_{s_z} := \bigcap_{x\in D(k_z)} (A_x^{s_z(x)} \cup \{x\}) \subset V^0_{k_z} \cup F_z$.]

So $\{s_z :\ z \in \omega_2\}$ is a $\mathcal{U}_F$-cover of ω_2, hence, by Fact 1, there is a countable subcover $\{s_{z_i} :\ i \in \omega\}$. But then $\bigcup\{V^0_{k_{z_i}} :\ i < \omega\}$ is co-countable in ω_2, and hence τ^0 is a Lindelöf topology. □

Fact 3. Each of τ^0, τ^1, τ^2 is a points G_δ topology.

Proof: For τ^0. Fix $z \in \omega_2$. By flexibility of F, $\forall y \neq z\ \exists x \in \omega_2 \backslash \{y, z\}$ s.t.

$$F(x, y) = 0 \quad \text{and} \quad F(x, z) = 1 \text{ (2 is equally possible).}$$

Let $K = \{x \in \omega_2 \backslash \{z\} : F(x, z) = 1\}$. Then $\omega_2 = \bigcup_{x \in K} (A^0_x \cup \{x\}) \cup (A^0_z \cup \{z\})$.

By Lindelöfness of $\mathcal{U}_F$, $\exists$ countable $K_0 \subset K$ s.t.

$$\omega_2 = \bigcup_{x \in K_0} (A^0_x \cup \{x\}) \cup (A^0_z \cup \{z\}).$$

Consequently, we have

$$\omega_2 \backslash \{z\} = \bigcup_{x \in K_0} (A^0_x \cup \{x\}) \cup A^0_z,$$

so $\omega \backslash \{z\}$ is a countable union of τ^0-closed (points are closed by flexibility of F) sets, and so $\{z\}$ is a G_δ of τ^0. □

Fact 4. Each of $\tau^i \times \tau^j$, $i, j \in 3$, is a Lindelöf topology on ω_2^2.

Proof: For $\tau^0 \times \tau^1$.

A. Suppose $\langle y, z \rangle \in V^0_k \times V^1_\ell$. Then, as in Fact 2, define 2 functions s, t: $D(k) \cup D(\ell) \longrightarrow 3$ as follows:

$$s(x) = \begin{cases} i \in 3 \text{ s.t. } y \in A^i_x, & \text{if } y \neq x \\ 2, & \text{if } y = x. \end{cases}$$
$$t(x) = \begin{cases} i \in 3 \text{ s.t. } z \in A^i_x, & \text{if } z \neq x \\ 2, & \text{if } z = x. \end{cases}$$

Then $s \restriction \varphi(y, z) = t \restriction \varphi(y, z)$. (Indeed, if $y = z$, then by observation that definitions of s and t coincide. If $y \neq z$ and $x < \varphi(y, z)$, then $F(x, y) = F(x, z)$, and $y \in A^i_x \leftrightarrow z \in A^i_x$, $(y \neq x \neq z)$.) Also, as in Fact 3, $\exists$ finite $F, G \subset D(k) \cup D(\ell)$ s.t. $y \in U_s \backslash F \subset V^0_k$ and $z \in U_t \backslash G \subset V^1_\ell$, so $\langle y, z \rangle \in U_s \backslash F \times U_t \backslash G \subset V^0_k \times V^1_\ell$, and $U_s \times U_t \subset (V^0_k \cup F) \times (V^1_\ell \cup G)$.

B. Let $\mathcal{C}$ be a $\tau^0 \times \tau^1$ cover of ω_2^2, and let $\mathcal{D} \subset \mathcal{U}_F \times \mathcal{U}_F$ be its refinement, obtained, for each point as in **A**, point by point. Since, by the Main Lemma, $\mathcal{U}_F \times \mathcal{U}_F$ is sort-of-Lindelöf and $\mathcal{D}$ satisfies Definition (7), there is a countable subcover of $\mathcal{D}$, say $\{U_{S_i} \times U_{t_i} : i < \omega\} \subset \mathcal{D}$. Then

$$\begin{aligned} \omega_2^2 &= \bigcup_{i<\omega} (U_{S_i} \times U_{t_i}) = \bigcup_{i<\omega} [(V^0_{k_i} \cup F_i) \times (V^1_{\ell_i} \cup G_i)] \\ &= \bigcup_{i<\omega} [(V^0_{k_i} \times V^1_{\ell_i}) \cup (V^0_{k_i} \times G_i) \cup (F_i \times V^1_{\ell_i}) \cup (F_i \times G_i)]. \end{aligned}$$

Since $V^0_{k_i}$ is Lindelöf in τ^0, $V^1_{l_i}$ in τ^1 by Fact 2, and F_i and G_i are finite, $\mathcal{D}$ has a countable subco $\tau^0 \times \tau^1$ is Lindelöf. □

(Other cases of $\langle i, j \rangle \in 3 \times 3$ are similar.)

Fact 5. In $\tau^0 \times \tau^1 \times \tau^2$, ω_2^3 has a closed discrete diagonal.

PROOF: Closed by the flexibility of F, and $\langle x, x, x\rangle \in (A_x^0)^c \times (A_x^1)^c \times (A_x^2)^c$ witnesses the discreteness. $\square$

Corollary. *Let, in $V[G]$, $S := (\omega_2, \tau^0) \oplus (\omega_2, \tau^1) \oplus (\omega_2, \tau^2)$. Then S and S^2 are Lindelöf points G_δ 0-dimensional spaces, and $L(S^3) = \mathfrak{c}^+ = \omega_2$.* $\square$

This finishes the proof of our theorem. $\square$

We conclude with a sketch of the forcing notion $\mathbb{P}$ to get a zero-dimensional space X such that,for all finite n, $L(X^n) = \aleph_0$, but $L(X^{\aleph_0}) = \mathfrak{c}^+ = \aleph_2$. $p \in \mathbb{P}$ iff $p = \langle A, f, \vec{\mathcal{B}}\rangle$, where

(i) $A \in [\omega_2]^{\leq\omega}$

(ii) $f : A^2 \to \omega$

(iii) $\vec{\mathcal{B}} = \langle \mathcal{B}_n : n \in \omega\backslash 1\rangle$ and $\forall n\ |\mathcal{B}_n| \leq \omega$ and $\forall B \in \mathcal{B}_n\ B \subset (Fn(A,\omega))^n$ (and $A^n = \bigcup\{U_{s_0} \times ... \times U_{s_{n-1}} : \vec{s} \in B\} \cap A^n$; this follows from (iv)).

(iv) $\forall n \in \omega\backslash 1$
$\forall B \in \mathcal{B}_n$
$\forall \vec{z} \in A^n$
$\forall$ partition of n, $N \cup \tilde{N} = n$
$\forall \vec{\delta} \in A^{\tilde{N}}$
$\forall \vec{h} \in (Fn(A,\omega))^{\tilde{N}}$ s.t. $\vec{h} \geq \vec{\delta}$ (i.e. $D(h_i) \geq \delta_i,\ \forall i$).
$\forall$ assignment $\Bigg[\!\!\Bigg[$ for $\forall z \in ran(\vec{z} \restriction \tilde{N})$,
of •**1** a finite tree $(T^z, \preceq)$ s.t.

$$(t \in T^z \Rightarrow t = \langle \delta_t, h_t\rangle),$$
$$\delta_t \in A\ \&\ h_t \in Fn(A\backslash\delta_t, \omega),$$
$$\&\ (s \prec t \Rightarrow \delta_s < \delta_t)$$
$$\&\ (t \in Lev_0(T) \Rightarrow \delta_t \leq z);$$

and
of •**2** an identification map $m^Z : N^z \to T$, where $N^z := \{i \in \tilde{N} : z_i = z\}$, s.t.
$(m^z(i) = t \Rightarrow (\delta_i = \delta_t \& h_i = h_t))\Bigg]\!\!\Bigg]$
$\exists \vec{s} \in B$ such that
(a) $(\forall i, j \in N)\ [z_i \in \mathcal{U}_{s_i}\ \&\ (z_i = z_j \Rightarrow s_i = s_j)]$
and
(b) $\forall z \in ran(\vec{z} \restriction \tilde{N})$
$\forall i, j \in N^z$
(1) $m^z(i) = m^z(j) \Rightarrow s_i = s_j$;
(2) if $m^z(i) \prec m^z(j)$ and $t \in T^z$ is the immediate successor of $m^z(i)$ in the chain of

T^z leading to $m^z(j)$, then $s_i \restriction \delta_t = s_j \restriction \delta_t$;
(3) $s_i \not\perp h_i$;
(4) if $t \in Lev_0(T)$ & $t \preceq m^z(i)$,
then

$$z \in \mathcal{U}_{s_i} \restriction \delta_t.$$

Let E^p be defined by

$$E^p(\delta, y, z) \Leftrightarrow \begin{cases} \delta, y, z \in A^p \\ \delta \leq y, z \\ \forall x \in (A^p \cap \delta)\ f^p(x, y) = f^p(x, z). \end{cases}$$

We define $q \leq p$ iff $A^q \supset A^p,\ f^q \supset f^p$, $(\forall n \in \omega - 1)\ \mathcal{B}^q_n \supset \mathcal{B}^p_n$, and $E^q \supset E^p$. End of definition.

It is worth observing that if T^z contains a chain of the form

$$\delta_0 \quad h_0 \quad \delta_1 \quad h_1 \quad \delta_2 \quad h_2 \quad \delta_3 \quad h_3$$

then

$$h_0 \cup s_0 \cup (s_1 \restriction \delta_1) \cup (s_2 \restriction \delta_1) \cup (s_3 \restriction \delta_1) \in Fn$$

&

$$h_1 \cup s_1 \cup (s_2 \restriction \delta_2) \cup (s_3 \restriction \delta_2) \in Fn$$

&

$$h_2 \cup s_2 \cup (s_3 \restriction \delta_3) \in Fn$$

&

$$h_3 \cup s_3 \in Fn.$$

$\mathbb{P}$ preserves cardinals and CH, (it is ω_1-complete & ω_2-cc). If G is $\mathbb{P}$-generic over V and$V \models CH$, then $\exists X \in V[G]$, s.t.

X is a Lindelöf Hausdorff 0-dimensional space of size $\aleph_2$, and

$$\forall n < \omega\ L(X^n) = \aleph_0$$
$$\text{and } L(X^\omega) = \mathfrak{c}^+ = \aleph_2.$$

With slightly simpler partial orders, we can set for every $n < \omega$ a Hausdorff Y_n s.t.

$$L(Y_n^n) = \aleph_0 \ \&\ L(Y_n^{n+1}) = \mathfrak{c}^+ = \aleph_2.$$

Acknowledgements. The contents of this paper is a part of the doctoral thesis written at the University of Toronto under the supervision of Professor William Weiss. He originally posed the exact question answered here. The author is very grateful to him for advice and encouragement during numerous discussions of the subject, and to Professor Bohuslav Balcar for helping to write this paper.

References

[1] Shelah S., *On some problems in general topology*, preprint, 1978.

[2] Juhász I., *Cardinal Functions II*, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984.

[3] Hajnal A., Juhàsz I., *Lindelöf spaces à la Shelah*, Coll. Mat. Soc. Bolyai, Budapest, 1978.

[4] Gorelic I., *The Baire Category and forcing large Lindelöf spaces with points G_δ*, Proceedings Amer. Math. Soc. **118** (1993), 603–607.

[5] Juhász I., *Cardinal Functions*, in: M. Hušek and J. van Mill, eds., Recent Progress in General Topology, North-Holland, 1992.

[6] Przymusinski T.C., *Normality and paracompactness in finite and countable cartesian products*, Fund. Math. **105** (1980), 87–104.

[7] Kunen K., *Set Theory*, North-Holland, Amsterdam, 1980.

The Center for Theoretical Study, Charles University, Ovocný trh 3,
116 36 Praha 1, Czech Republic

(*Received* April 6, 1992, *revised* October 26, 1993)