Note on Petrie and Hamiltonian cycles in cubic polyhedral graphs

J. Ivančo, S. Jendroľ, M. Tkáč

Abstract. In this note we show that deciding the existence of a Hamiltonian cycle in a cubic plane graph is equivalent to the problem of the existence of an associated cubic plane multi-3-gonal graph with a Hamiltonian cycle which takes alternately left and right edges at each successive vertex, i.e. it is also a Petrie cycle. The Petrie Hamiltonian cycle in an n-vertex plane cubic graph can be recognized by an $O(n)$-algorithm.

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Throughout this note we shall consider cubic polyhedral graphs, i.e. 3-valent plane 3-connected graphs (see Grünbaum [4], Malkevitch [6]).

Many papers are devoted to the study of the existence of Hamiltonian cycles in cubic plane graphs, see e.g. Holton and McKay [5] or Malkevitch [6] for recent surveys. In Fleischner [2, Chapter VI] there is proved that the problem of finding a Hamiltonian cycle in a cubic plane graph is equivalent to the problem of finding an A-*trail*, that is an Eulerian trail whose consecutive edges (including the last and the first) lie on a common face, in an associated Eulerian plane graph.

In this note we show that the cubic hamiltonian problem is equivalent to the problem of finding a cubic multi-3-gonal plane graph M (i.e. having sizes of all faces $\equiv 0 \pmod 3$) with a Petrie cycle which passes through all vertices of M. A cycle C in a cubic graph is said to be a *Petrie cycle* if every two, but no three, consecutive edges of C (including the last and the first) lie on a common face. A path with this property is known to be a Petrie path (a Petrie arc), cf. Coxeter [1], Grünbaum [4, p. 258].

Petrie cycles do not always exist in cubic plane graphs. For example, a graph of a k-side prisma, $k \geq 3$, has a Petrie cycle if and only if $k \equiv 0 \pmod 4$. Because every Petrie cycle is uniquely determined by arbitrary two of its consecutive edges, the existence of an $O(n)$-algorithm which decides if there is a Petrie cycle crossing all vertices of an n-vertex cubic plane graph is easily seen. Such cycle is called a Petrie Hamiltonian cycle (a *PH-cycle* in the sequel).

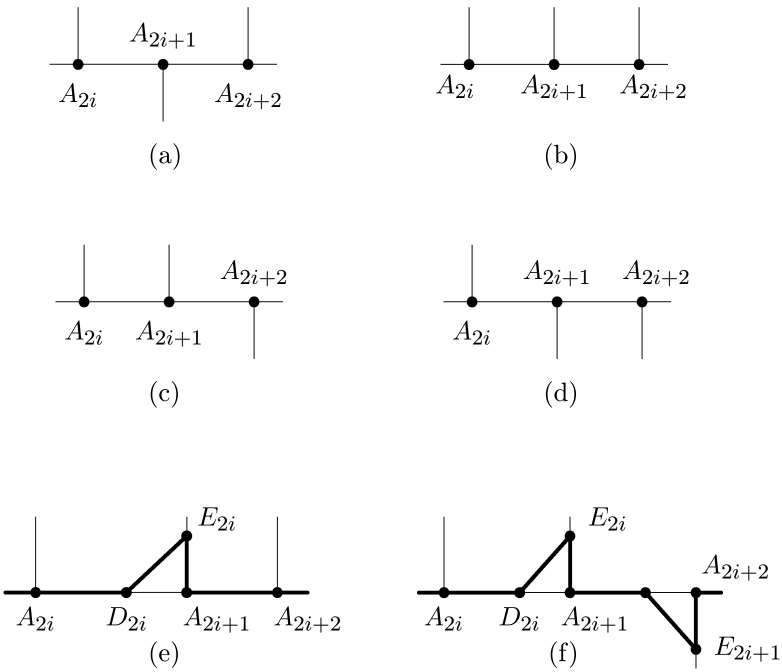
Let G be a cubic plane graph and A be its vertex adjacent to the vertices B_1, B_2, B_3 and incident with faces α_1, α_2, α_3. By a *cutting off* the vertex A of G we mean the placing new vertices A_1 and A_2 on the edges AB_1 and AB_2 of G, respectively, and joining them by a new edge A_1A_2 (i.e. a replacing of the vertex

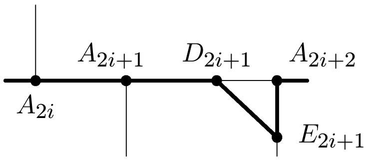
A by a triangle AA_1A_2). This changes the graph G into a cubic plane graph G^* with a new triangle AA_1A_2 and new faces α'_1, α'_2, α'_3 instead of the faces α_1, α_2, α_3 of G. If the face α_i, $i = 1, 2, 3$ is an r_i-gon, the face α'_i is an $(r_i + 1)$-gon. The change G into G^* is denoted by $G^* = G\nabla A$. Let $\mathcal{S} = \{A_i | 1 \le i \le t\}$ be a set of vertices of G. Let $G_0 = G$, $G_i = G_{i-1}\nabla A_i$ for all $i = 1, 2, \ldots, t$. We put

$$G\nabla\mathcal{S} := G_t.$$

Lemma 1. *Let C be a cycle of the length $2k$ in a cubic plane graph G. Then there is a set $\mathcal{S}$ of, say t, vertices of C such that $G^* = G\nabla\mathcal{S}$ has a Petrie cycle C^* of the length $2(k + t)$.*

Proof: Denote the vertices of cycle C successively $A_0, A_1, \ldots, A_{2k-1}$. Let h be an edge incident with the vertex A_0 lying outside of C. We will construct G^* together with its Petrie cycle C^*. Let $G_0 = G$. Suppose we have a graph G_i, $i = 0, 1, \ldots, k-2$ with a Petrie path P_i starting in A_0 and ending in A_{2i} and such that for continuation of P_i the right edge in the vertex A_{2i} must be chosen. In the graph G_i one of the four situations (a), (b), (c), (d) depicted in Fig. 1 appears.





(g)

Figure 1

In the situation (a) of Fig. 1 we put $G_{i+1} := G_i$ and $P_{i=1} := P_i \cup A_{2i+1}A_{2i+2}$. In the situation (b) of Fig. 1 we cut off the vertex A_{2i+1} as it is shown in Fig. 1 (e) and put $G_{i+1} = G_i \nabla A_{2i+1}$ and $P_{i+1} := P_i \cup D_{2i}E_{2i}A_{2i+1}A_{2i+2}$. Changes for the situation (c) and (d) are depicted in Fig. 1 (f) and (g), respectively.

In the graph G_{k-1} we have the Petrie path from A_0 to A_{2k-2} and, because of h, only the situation of Figure (a) or (b) appears. In the first case we put $G^* := G_{k-1}$ and $C^* := P_{k-1} \cup A_{2k-1}A_0$. In the second case $G^* := G_{k-1}\nabla A_{2k-1}$ and $C^* := P_{k-1} \cup D_{2k-2}E_{2k-2}A_{2k-1}A_0$.

The proposition concerning the length of C^* is clear from the above. □

Corollary 2. *If C is a Hamiltonian cycle in a cubic plane graph, then there is a set $\mathcal{S}$ of vertices of G such that $G\nabla\mathcal{S}$ has a Hamiltonian cycle C^* which is also a Petrie cycle.* □

Theorem 3. *A cubic plane graph G is Hamiltonian if and only if there exists a set $\mathcal{S}$ of vertices of G such that the graph $G\nabla\mathcal{S}$ has a PH-cycle.*

PROOF: Since G is cubic it has even number of vertices and the necessity follows from Lemma 1 and Corollary 2.

Sufficiency. Let H^* be a PH-cycle in $G\nabla\mathcal{S}$. It is easy to see that each triangle of $G\nabla\mathcal{S}$ has two of its edges on H^*. Let $\tau_1, \tau_2, \dots, \tau_s$, $s \geq 1$, be triangles obtained by cutting off the vertices from $\mathcal{S}$ in G. If we delete from $G\nabla\mathcal{S}$ the edge of τ_j, for any $1 \leq j \leq s$, not lying on H^* and then forget the vertices of degree two, we get a Hamiltonian cycle H in G. □

The problem of deciding the existence of Hamiltonian cycles in cubic, planar, 3-connected graphs, is an NP-complete problem, see Garey et al [3]. Therefore one could think, to find a Hamiltonian cycle by using Theorem 3, it is necessary to consider as set $\mathcal{S}$ all of 2^n subsets of the vertex set of an n-vertex cubic plane graph. But the following theorem provides some restrictions on $\mathcal{S}$.

Theorem 4. *If a cubic polyhedral n-vertex graph G has a PH-cycle then*

(i) *all faces of G are multi-3-gonal,*
(ii) $4 \leq n \equiv 0 \pmod 4$, $n \neq 8$.

PROOF: Suppose C is a PH-cycle in G. Then it is easy to see that each third edge of any face in G is a chord of C. Further there is the same number, say

t, of chords in the interior and in the exterior of C. Every chord makes two non-adjacent vertices of C trivalent. Hence C must have $4t$ vertices.

Let G be a cubic polyhedral graph on 8 vertices and with a PH-cycle C. Let the vertices of C be successively $A_1, A_2, \dots, A_8$. Without loss of generality we can assume that the edges A_1A_3 and A_5A_7 lie inside of C. Because of planarity of G, the edges A_2A_6 and A_4A_8 cannot exist in G. The existence of an edge A_2A_4 or A_2A_8 leads to the contradiction with the 3-connectivity of G. $\square$

Note that for any n, $4 \le n \equiv 0 \pmod 4$, $n \ne 8$, there exists an n-vertex cubic polyhedral graph with PH-cycle. The proof of this statement is left to the reader.

As the cutting off a vertex A of a graph G leads to the increasing of the number in $G\nabla A$ by two, Theorem 4 yields

Corollary 5. *Let G be an n-vertex plane cubic graph having PH-cycle, then*

$$|\mathcal{S}| \equiv \frac{n}{2} \pmod 2.$$

$\square$

Here and in the sequel, $\mathcal{S}$ is as in Theorem 3.

Many other restrictions on $\mathcal{S}$ are given by (i) of Theorem 4. To obtain a multi-3-gonal face from an m-gonal face α, $m \equiv j \pmod 3$, $j = 1, 2, 3$, $3t - j$ vertices must be cut off for some $t = 1, 2, \dots, \lfloor \frac{m}{3} \rfloor$. By this we have

Corollary 6. *Let $p_k(G)$ denote the number of k-gonal faces of an n-vertex cubic plane graph G having PH-cycle and $K = 2\sum_{k\ge1} p_{3k+1}(G) + \sum_{k\ge1} p_{3k+2}(G)$. Then*

$$\frac{K}{3} \le |\mathcal{S}| \le n - \frac{K}{3}.$$

$\square$

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J. Ivančo, S. Jendroľ

Department of Geometry and Algebra, Šafárik University, Jesenná 5, 041 54 Košice, Slovakia

M. Tkáč
Department of Mathematics, Technical University, Letná 9, 040 01 Košice, Slovakia

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