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A contribution to the equivalence results for the product of distributions

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Abstract. Products $[S] \cdot [T]$ and $[S] \cdot T$, defined by model delta-nets, are equivalent.

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Introduction

Let S and T be distributions on $\mathbb{R}^d$. Kamiński in [3] considers the following definitions for their product by regularization using model delta-sequences:

$$[S \cdot T] = \lim_{n\to\infty} (S * \varphi_n) \cdot (T * \varphi_n), \tag{1}$$

$$[S] \cdot [T] = \lim_{n\to\infty} (S * \varphi_n) \cdot (T * \psi_n), \tag{2}$$

$$[S] \cdot T = \lim_{n\to\infty} (S * \varphi_n) \cdot T. \tag{3}$$

The model delta-sequence $\{\varphi_n\} \subset \mathcal{D}(\mathbb{R}^d)$ is defined to be a sequence of testing functions

$$\varphi_n(x) = \beta_n^d \varphi(\beta_n x) \qquad \left(x \in \mathbb{R}^d\right) \tag{4}$$

where $\varphi \in \mathcal{D}(\mathbb{R}^d)$ $\int \varphi = 1$, $\beta_n \in \mathbb{R}$, $\beta_n \to \infty$. For each of the definitions of the product above it is required that the limit in the second member exists and does not depend on the choice of delta-sequences $\{\varphi_n\}$, $\{\psi_n\}$.

Oberguggenberger [4], Wawak [8] and others use nets of testing functions instead of sequences. The model delta-net $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is defined by

$$\varphi_\varepsilon(x) = \varepsilon^{-d} \varphi_1\left(\tfrac{x}{\varepsilon}\right), \tag{5}$$

where $\varphi_1 \in \mathcal{D}(\mathbb{R}^d)$, $\int \varphi_1 = 1$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. It is natural to define the product of distributions using delta-nets to be

$$[S \cdot T] = \lim_{\varepsilon\searrow 0} (S * \varphi_\varepsilon) \cdot (T * \varphi_\varepsilon), \tag{6}$$

$$[S] \cdot [T] = \lim_{\varepsilon\searrow 0} (S * \varphi_\varepsilon) \cdot (T * \psi_\varepsilon), \tag{7}$$

$$[S] \cdot T = \lim_{\varepsilon\searrow 0} (S * \varphi_\varepsilon) \cdot T \tag{8}$$

whenever, for each definition, the limit in $\mathcal{D}'$ exists and does not depend on the choice of model delta-nets $\{\varphi_\varepsilon\}$, $\{\psi_\varepsilon\}$.

It is well known (see Kamiński [3]) that the definitions (2) and (3) are equivalent, while the definition (1) is strictly more general. It is easy to see that the definitions (3) and (8) are equivalent, too. In other words, it does not matter if we use model delta-nets or model delta-sequences for defining the product $[S]\cdot T$. However, the equivalence between (7) and (8) is not so evident and for proving this equivalence, we cannot refer to the results contained in [3] concerning the equivalence between (2) and (3). The definition (7) looks to be more general than (2). The matter is as follows. The choice of the number sequence $\{\beta_n\}$ in (4) influences the speed of convergence of the sequence $\{\varphi_n\}$ to the Dirac measure δ. Hence, from the existence of the product $[S]\cdot[T]$ by (2) we can easily deduce that the product $[S]\cdot T$ by (3) is the same, if we let the sequence $\{\psi_n\}$ converge to δ "much more quickly" than $\{\varphi_n\}$. On the other hand, for the definition (7) this method fails, because the speed of convergence of both nets $\{\varphi_\varepsilon\}$, $\{\psi_\varepsilon\}$ is the same. The aim of the paper is to remove this gap showing the equivalence of the definitions (7) and (8). Thanks to what is said above, it suffices to prove the following theorem.

Theorem. *Let K be the closed unit ball in $\mathbb{R}^d$ and suppose that for any nets $\{\varphi_\varepsilon\}_{\varepsilon>0}$, $\{\psi_\varepsilon\}_{\varepsilon>0}$ satisfying* (5) *with $\varphi_1\in\mathcal{D}(K)$, $\varphi_1\geq 0$ $\int\varphi_1=1$ and the same for ψ_1, the relation*

$$\lim_{\varepsilon\searrow 0}\langle (S*\varphi_\varepsilon)(T*\psi_\varepsilon)\,,\,\omega\rangle=\langle W,\omega\rangle$$

holds for any testing function $\omega\in\mathcal{D}(\mathbb{R}^d)$. Then

$$\lim_{\varepsilon\searrow 0}\langle (S*\varphi_\varepsilon)T,\omega\rangle=\langle W,\omega\rangle\,.$$

For proving the theorem, we use Lemma 5 of Itano [2], p. 166, as follows.

Lemma. *Let K_1,K_2 be compact subsets of $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$ resp. and let $W_\varepsilon\in\mathcal{D}'(K_1\times K_2)$ for $\varepsilon>0$. The sufficient* (*and necessary*) *condition for the net $\{W_\varepsilon\}_{\varepsilon>0}$ to be convergent to a distribution $W\in\mathcal{D}'(K_1\times K_2)$ is that for any two testing functions $\varphi\in\mathcal{D}(K_1)$, $\psi\in\mathcal{D}(K_2)$ the relation*

$$\lim_{\varepsilon\searrow 0}\langle W_\varepsilon(x,y),\,\varphi(x)\psi(y)\rangle=\langle W(x,y),\,\varphi(x)\psi(y)\rangle \tag{9}$$

holds.

Proof of the theorem: Let us calculate

$$\begin{aligned}\langle (S*\varphi_\varepsilon)(T*\psi_\varepsilon)\,,\,\omega\rangle&=\int\langle S(u),\,\varphi_\varepsilon(z-u)\rangle_u\,\langle T(v),\,\psi_\varepsilon(z-v)\rangle_v\,\omega(z)\,\mathrm{d}z\\ &=\langle S(u)\times T(v),\,\textstyle\int\varphi_\varepsilon(z-u)\,\psi_\varepsilon(z-v)\,\omega(z)\,\mathrm{d}z\rangle=\\ &\left\langle S(u)\times T(v),\,\varepsilon^{-2d}\int\varphi_1\left(\frac{z-u}{\varepsilon}\right)\psi_1\left(\frac{z-v}{\varepsilon}\right)\omega(z)\,\mathrm{d}z\right\rangle.\end{aligned}\tag{10}$$

For $\varepsilon > 0$, we can define a distribution W_ε on $\mathbb{R}^{3d}$ by

$$\langle W_\varepsilon, \Phi\rangle = \left\langle S(u)\times T(v),\, \varepsilon^{-2d}\int \Phi\left(\frac{z-u}{\varepsilon}, \frac{z-v}{\varepsilon}, z\right)\mathrm{d}z\right\rangle . \tag{11}$$

($\Phi \in \mathcal{D}(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d)$) and for $\Phi(x,y,z) = \varphi_1(x)\psi_1(y)\omega(z)$, we have by (10) $\langle W_\varepsilon, \Phi\rangle = \langle (S*\varphi_\varepsilon)(T*\psi_\varepsilon),\, \omega\rangle$. Hence by the hypothesis of the theorem, if the functions φ_1, ψ_1 satisfy

$$\varphi_1 \geq 0\,,\ \psi_1 \geq 0,\ \textstyle\int\varphi_1 = 1,\ \int\psi_1 = 1, \tag{12}$$

and

$$\Phi(x,y,z) = \varphi_1(x)\psi_1(y)\omega(z),$$

we have

$$\lim_{\varepsilon\searrow 0}\langle W_\varepsilon, \Phi\rangle = \textstyle\int\varphi_1\cdot\int\psi_1\cdot\langle W,\omega\rangle\,.$$

Evidently this equality remains true even without the conditions (12). Generalizing the lemma above for 3 variables, we obtain

$$\lim_{\varepsilon\searrow 0} W_\varepsilon(x,y,z) = \mathbf{1}(x)\times\mathbf{1}(y)\times W(z).$$

For a given φ_1 satisfying (12), the set of testing functions

$$\Phi_\varepsilon(x,y,z) := \varphi_1(x-y)\,\omega(z-\varepsilon y)\,\varphi_1(x) \qquad (0<\varepsilon\leq 1)$$

is evidently bounded. By the well known result that a convergent sequence of distributions converges uniformly on a bounded set of testing functions, we have

$$\lim_{\varepsilon\searrow 0}\langle W_\varepsilon, \Phi_\varepsilon\rangle = \left\langle W(z),\, \int\varphi_1(x-y)\,\omega(z)\,\varphi_1(x)\,\mathrm{d}x\,\mathrm{d}y\right\rangle = \langle W,\omega\rangle\,.$$

The first member can be calculated by (11)

$$\begin{aligned}\langle W_\varepsilon, \Phi_\varepsilon\rangle &= \left\langle S(u)\times T(v),\ \varepsilon^{-2d}\int\varphi_1\left(\frac{v-u}{\varepsilon}\right)\omega(v)\varphi_1\left(\frac{z-u}{\varepsilon}\right)\,\mathrm{d}z\right\rangle\\ &= \left\langle S(u)\times T(v),\ \varepsilon^{-d}\varphi_1\left(\frac{v-u}{\varepsilon}\right)\omega(v)\right\rangle = \langle (S*\varphi_\varepsilon)\,T,\,\omega\rangle\,,\end{aligned}$$

which proves the theorem. $\square$

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