On ω-limit sets of nonautonomous differential equations

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Abstract. In this paper the ω-limit behaviour of trajectories of solutions of ordinary differential equations is studied by methods of an axiomatic theory of solution spaces. We prove, under very general assumptions, semi-invariance of ω-limit sets and a Poincaré-Bendixon type theorem.

Keywords: ω-limit sets, stationary points, the Poincaré-Bendixon theorem

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1. In the present paper we operate in the framework of an axiomatic theory of spaces of solutions of ordinary differential equations introduced by V.V. Filippov (see, e.g. [8]–[13]). The theory deals with families of continuous functions satisfying one or another set of axioms describing fundamental properties of solutions of ordinary differential equations. These properties include, for example, the solvability of the Cauchy problem, the uniqueness of its solution, and the compactness of certain families of solutions. A very small number of basic axioms suffices to develop a theory which provides substantial results. The methods of verifying the axiomatic conditions for specific types of differential equations have been worked out in a series of papers by V.V. Filippov.

Topological structures introduced on the sets of solutions of ordinary differential equations play a key role in the axiomatic theory. By exploiting these structures it is often possible to express in the language of convergence information which would be classically stated in terms of properties of functions occurring in the equation.

Results obtained in the framework of the axiomatic theory are applicable to a wide range of ordinary differential equations with discontinuous terms or other types of singularities. The theory also enables one to show that many "non-traditional" types of ordinary differential equations have "traditional" properties. We add also that the theory's scope embraces naturally the differential inclusions and ordinary differential equations with a control.

It should be noted that well before the studies by V.V. Filippov there had been papers in mathematical literature in which similar axiomatic approaches to the treatment of ordinary differential equations were used (see [24], [15], [23]). However, the theories presented in those papers did not demonstrate the full potential of the axiomatic approach, since they were not extended to cover significantly

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broader types of equations than were already covered by the classical theory. In particular, a technique for dealing with equations having singularities was not worked out there. This perhaps explains why these earlier abstract approaches failed to flourish.

The axiomatic theory considered here leads to new results not only for ordinary differential equations with singularities but also in the most classical realms. The following simple example illustrates this. Consider the perturbed autonomous differential equation $y' = g(y)(1 + t \cdot \exp(-t\|y\|))$, where $y' = dy/dt$, $(t, y) \in \mathbf{R} \times \mathbf{R}^2$, the function $g : \mathbf{R}^2 \to \mathbf{R}^2$ is continuous, and $g(0, 0) \neq (0, 0)$. One may expect that there is a close connection between the solutions of this equation and those of the autonomous equation $y' = g(y)$, because the perturbation becomes small as $t \to \infty$ if $y \neq (0, 0)$. However, when $y = (0, 0)$ the norm of the perturbation term increases unboundedly as $t \to \infty$ and has no majorant, depending on t, which vanishes as $t \to \infty$. This is a serious obstacle to applying traditional techniques based only upon estimates of functions involved in the equation. On the other hand, the above perturbed equation can be analyzed without difficulties in the framework of the theory under review [8, Chapter IX, § 8].

2. The paper is concerned with studying the asymptotic behaviour of solutions to nonautonomous ordinary differential equations by methods of the axiomatic theory. We examine relations between the solutions of an equation and the solutions of its limiting equations. There is an appreciable difference between the approach to this employed in the paper and the traditional one. We compare them below.

For a differential equation $y' = f(t, y)$ consider the translated equations $y' = f^\tau(t, y)$, where for any real number τ the translate function f^τ is defined by $f^\tau(t, y) = f(t + \tau, y)$. Under the traditional approach, the functions f^τ are embedded in a function space endowed with a certain convergence structure, and then the limit points (functions) g_α of the set of translates are identified as $\tau \to \infty$. The equations $y' = g_\alpha(t, y)$ are the limiting equations of the original equation $y' = f(t, y)$. They govern the asymptotic behaviour of its solutions as $t \to \infty$ (see, e.g. [20] for details, and also [1]–[4] where limiting equations of a more general type than ordinary differential equations are considered). Under the approach used here, the asymptotic behaviour of solutions is studied by passing to a limit not in a function space containing the translates but directly in the space of sets of solutions. In this connection a question may arise about how to establish convergence in the space of sets of solutions; if one had to deal each time with the right-hand sides of the corresponding differential equations, there would be no essential advantages in this approach. The answer is as follows: although for relatively simple equations the convergence of solutions is verified by working explicitly with the right-hand sides, for more complicated equations (with singularities, etc.) the necessary convergence can be *deduced* from the convergence of solutions of related equations with a simpler structure. Indeed the possibility of applying results iteratively constitutes one of the most valuable features of the axiomatic approach.

The paper is organized as follows. In Section 1 we provide a background to the axiomatic theory and present concepts and ideas necessary for the development of the theory in subsequent sections. In Section 2 we study invariance properties of ω-limit sets for trajectories of solutions. The classical Poincaré-Bendixon theorem for planar autonomous equations is generalized in Section 3 in the case when the ω-limit set of a trajectory has singular (critical) points.

1. Preliminaries from an axiomatic theory of ordinary differential equations

The notions and results of this section are preparatory for our main theorems given in Sections 2 and 3. To make the exposition more or less self-contained we also present here relevant material from V.V. Filippov's theory.

1. Let U be an open set in the product $\mathbf{R} \times L$ of the real line $\mathbf{R}$ and a finite-dimensional Euclidean space L. Consider the set $C_s(U)$ of continuous functions defined on finite closed intervals (which could degenerate to a point) of $\mathbf{R}$ with values in the space L, whose graphs are in U. Each function from $C_s(U)$ is represented by its graph which is a compact subset of U. The Vietoris topology [7] on the space of closed subsets of U thus induces a topology on $C_s(U)$. This topology on $C_s(U)$ is metrizable and generated by the Hausdorff metric [7]. It induces the topology of uniform convergence on any set of functions with a fixed domain of definition.

For any function z, we denote by $\mathrm{dom}\,(z)$ the domain of definition of z.

Denote by $R(U)$ the set of all subspaces Z of $C_s(U)$ satisfying the following conditions (1.1) and (1.2):

(1.1) if $z \in Z$ and the closed interval I lies in $\mathrm{dom}\,(z)$, then $z|_I \in Z$.

(1.2) if the functions $z_n \in Z$ $(n = 1, 2)$ are defined on closed intervals I_n with nonempty intersection and coincide on $I_1 \cap I_2$, then the function z defined on $I_1 \cup I_2$ by $z(t) = z_n(t)$, $t \in I_n$, also belongs to Z.

If $Z \in R(U)$ satisfies the condition:

(1.3) for any compact set $K \subseteq U$ the set of all elements of Z with graphs in K is compact,

we write $Z \in R_c(U)$.

The set of all $Z \in R(U)$ satisfying the condition:

(1.4) for any point $(t, y) \in U$ there exists a function $z \in Z$ defined on an interval containing t in its interior such that $z(t) = y$,

is denoted by $R_e(U)$. Denote $R_{ce}(U) = R_c(U) \cap R_e(U)$.

2. Consider a differential equation $y' = f(t, y)$ generated by a mapping $f : U \to L$ (or, more generally, a differential inclusion $y' \in f(t, y)$ if f is multi-valued). A solution of this equation/inclusion is understood to be a generalized absolutely continuous function [18] which belongs to $C_s(U)$ and has almost everywhere an approximate derivative [18] satisfying the equation/inclusion. The concept of a solution is thus essentially widened, since the solutions are usually assumed to be continuously differentiable or at least absolutely continuous functions. It should be noted that this definition does not provide additional solutions in situations covered by the classical Cauchy-Peano and Carathéodory theorems [5, Theorems 1.1.2 and 2.1.1].

Clearly the set of thus defined solutions to the equation/inclusion satisfies conditions (1.1) and (1.2), i.e. belongs to $R(U)$. Condition (1.4) corresponds to the existence theorem for a solution of the Cauchy problem. Condition (1.3) together with the condition of uniqueness of the solution to the Cauchy problem is equivalent to the theorem on the continuous dependence of the solution on the initial values. Conditions (1.3) and (1.4) hold, for example, if the function f is continuous or satisfies the Carathéodory conditions, or — in the case of differential inclusions — satisfies the assumptions of Davy's theorem [6].

3. For $Z \subseteq C_s(U)$ denote by Z^+ (respectively, by Z^-) the set of all continuous mappings z of arbitrary half-open intervals $[a,b)$, $a < b \leq \infty$ (respectively, intervals $(a,b]$, $-\infty \leq a < b$), to L such that: (1) $z|_I \in Z$ for every closed interval $I \subseteq \operatorname{dom}(z)$, (2) there are no elements in Z that extend z.

For any point $p \in U$ with components $t \in \mathbf{R}$, $y \in L$, denote by $\Phi_Z(p)$ the set of all mappings z which map arbitrary open intervals (a,b) to L, $-\infty \leq a < b \leq \infty$, such that $a < t < b$, $z(t) = y$, $z|_{(a,\,t]} \in Z^-$, and $z|_{[t,\,b)} \in Z^+$. Define $Z^{-+} = \cup\{\Phi_Z(p) : p \in U\}$. Clearly, for $Z \in R(U)$ a function z belongs to Z^{-+} if and only if it belongs to $\Phi_Z(p)$ for every p on the graph of z. For $Z \in R(U)$, Z^{-+} can be defined as the set of all mappings $z : (a,b) \to L$ having the property: $z|_{[s,\,t]} \in Z$ for all s,t with $a < s \leq t < b$, and such that z cannot be extended to a larger open interval with this property still holding.

In the following two propositions we slightly modify results of [8, Chapter IX, § 2].

Proposition 1.1. *Suppose a subspace $Z \subseteq C_s(U)$ satisfies* (1.3)*. Let $z \in Z^+$, $\operatorname{dom}(z) = [a,b)$. Then for every compact set $K \subseteq U$ there exists a point $c \in \operatorname{dom}(z)$ such that for all $t \in [c,b)$ the point $(t, z(t))$ belongs to $U \setminus K$.*

Proposition 1.2. *If $Z \in R_e(U)$ and $z \in Z \cup Z^- \cup Z^+$, then there exists a function in Z^{-+} extending z.*

4. As shown in [8, Chapter IX, § 3], the following concept of convergence of subspaces of $C_s(\mathbf{R} \times L)$ is adequate for the continuous dependence of a solution to the Cauchy problem on parameters in the right-hand side of an ordinary differential equation.

Definition 1.1 (see [8, IX.3.8]). A sequence $\{Z_i : i \in \mathbf{N}\}$ of subspaces of $C_s(\mathbf{R} \times L)$ converges in U to a space $Z \subseteq C_s(U)$ if every sequence of functions $z_j \in Z_{i_j}$ $(i_1 < i_2 < \dots)$ with graphs lying in an arbitrary compact set $K \subseteq U$ has a subsequence convergent to a function belonging to Z.

We present now an important theorem due to V.V. Filippov ([8, Theorem IX.3.12]) on convergent sequences of subspaces of $C_s(U)$ (leaving aside superfluous assumptions made in the original formulation).

Theorem 1.1. *Suppose a sequence $\{Z_i : i \in \mathbf{N}\} \subseteq R_c(U)$ converges in U to a space $Z \subseteq C_s(U)$ and a sequence $\{p_i : i \in \mathbf{N}\} \subseteq U$ converges to a point $p \in U$. Assume that, for each $i \in \mathbf{N}$, p_i belongs to the graph of some function $z_i \in Z_i^{-+}$. Then there exists a function $z^* \in Z^{-+}$ whose graph contains p and a subsequence $\{z_{i_k} : k \in \mathbf{N}\}$ such that, if I is any finite interval in $\operatorname{dom}(z^*)$, then*

(1) *$I \subseteq \operatorname{dom}(z_{i_k})$ for all k large enough;*
(2) *if $I \subseteq \operatorname{dom}(z_{i_k})$ for all $k \geq k_0$, then $\{z_{i_k}|_I : k \geq k_0\}$ converges to $z^*|_I \in Z$ uniformly as $k \to \infty$.*

5. Let V be an open set in L. For any $z \in C_s(\mathbf{R} \times V)$ with $\operatorname{dom}(z) = [a, b]$ and any $\tau \in \mathbf{R}$ let z^τ be a function defined on $[a - \tau,\ b - \tau]$ by the formula $z^\tau(t) = z(t + \tau)$. For any $Z \subseteq C_s(\mathbf{R} \times V)$ we denote by Z^τ the set $\{z^\tau : z \in Z\}$. If Z is the space of solutions of the equation $y' = f(t, y)$, then Z^τ is the space of solutions of the equation $y' = f(t + \tau, y)$.

A space $Z \subseteq C_s(\mathbf{R} \times V)$ is said to be *autonomous*, if $Z^\tau = Z$ for all $\tau \in \mathbf{R}$. Clearly, the space of solutions of an autonomous differential equation $y' = f(y)$ (more generally, of an inclusion $y' \in f(y)$) in $\mathbf{R} \times V$ is autonomous in the sense of the above definition.

Let $a_0 \in \mathbf{R} \cup \{-\infty\}$ and $U_0 = (a_0, \infty) \times V$.

Definition 1.2 ([13])**.** A space $X \subseteq C_s(U_0)$ converges in U_0 to a family ζ of subspaces of $C_s(U_0)$ as $t \to \infty$ if for each sequence $\tau_i \to \infty$ the sequence of spaces $\{X^{\tau_i} : i \in \mathbf{N}\}$ has a subsequence convergent in U_0 to a space $Z \in \zeta$.

The following two examples related to the above convergence are important for applications.

Example 1.1. Let $f(t, y)$ be a continuous function defined on $\mathbf{R} \times V$ such that for every y, $f(t, y)$ is almost periodic in t, uniformly for y in compact sets (that is, for any compact set $K \subseteq V$ and any $\varepsilon > 0$ the set $\{s \in \mathbf{R} : \|f(t + s, y) - f(t, y)\| \leq \varepsilon$ for all $t \in \mathbf{R}$ and all $y \in K\}$ is relatively dense in $\mathbf{R}$).

The function f has the following property: for every sequence $\alpha = \{\tau_i : i \in \mathbf{N}\}$ of real numbers there is a subsequence $\{\tau_{i_k} : k \in \mathbf{N}\}$ and a continuous function $f_\alpha(t, y)$ which is almost periodic in t such that

$$f(t + \tau_{i_k}, y) \to f_\alpha(t, y) \quad \text{as} \quad k \to \infty \tag{1.5}$$

uniformly for $t \in \mathbf{R}$ and y in compact subsets of V (see [17, p. 401], [20, p. 259]).

Let X be the space of solutions of the differential equation $y' = f(t, y)$ in $\mathbf{R} \times V$ and ζ be the family of solution spaces of the equations $y' = f_\alpha(t, y)$, where the functions f_α are described above. It can be easily verified, by employing (1.5), that for any sequence $\{\tau_i : i \in \mathbf{N}\} \subseteq \mathbf{R}$ the sequence of spaces $\{X^{\tau_i} : i \in \mathbf{N}\}$ has a subsequence convergent in $\mathbf{R} \times V$ to some $Z \in \zeta$.

The family ζ can be indicated more explicitly, for example, in the following particular case. Let $\varphi(t,y) = \varphi_1(t,y) + \ldots + \varphi_k(t,y)$, where for each $j = 1, \ldots, k$ the function φ_j is continuous on $\mathbf{R} \times V$ and periodic in t of period p_j (clearly, $\varphi(t,y)$ is almost periodic in t, uniformly for y in compact sets). Denote by X_0 the space of solutions of the equation $y' = \varphi(t,y)$, and for any set $\{r_1, \ldots, r_k\} \subseteq \mathbf{R}$ denote by $Z(r_1, \ldots, r_k)$ the space of solutions of the equation $y' = \varphi_1(t+r_1,y) + \ldots + \varphi_k(t+r_k,y)$. Let $\zeta_0 = \{Z(r_1, \ldots, r_k) : r_j \in [0,p_j],\ 1 \leq j \leq k\}$. For every sequence $\{\tau_i : i \in \mathbf{N}\} \subseteq \mathbf{R}$ there is a subsequence of $\{X_0^{\tau_i} : i \in \mathbf{N}\}$ which converges in $\mathbf{R} \times V$ to some space from ζ_0. Indeed, let $a_{ij} \in [0,p_j)$ be such that $\tau_i - a_{ij}$ is an integral multiple of p_j. If r_j^* is a limit point of $\{a_{ij} : i \in \mathbf{N}\}$ for $1 \leq j \leq k$, then $\{X_0^{\tau_i} : i \in \mathbf{N}\}$ converges in $\mathbf{R} \times V$ to the space $Z(r_1^*, \ldots, r_k^*) \in \zeta_0$.

Example 1.2. Let $g : \mathbf{R} \times V \to L$ be a function satisfying the Carathéodory conditions, i.e. $g(t,y)$ is measurable in t, continuous in y and for every compact set $B \subseteq V$ there exists a locally Lebesgue integrable function $m_B(t)$ such that $\|g(t,y)\| \leq m_B(t)$ for all $y \in B$. Suppose in addition that $m_B(t)$ has a uniformly continuous primitive or, more generally, the function g satisfies Assumption (A) from [1], [3], [4]. Assume that as $t \to \infty$ g becomes small in the following sense:

(1.6) for every closed interval $[a,b] \subseteq \mathbf{R}$, whenever the sequence $\{u_k : k \in \mathbf{N}\}$ of continuous functions $u_k : [a,b] \to V$ converges uniformly to a function u and $t_k \to \infty$ as $k \to \infty$, then $\int_a^b g(t_k + t,\, u_k(t))dt \to 0$.

(This type of convergence was considered in [1], [3], [4]). Condition (1.6) holds, for example, if for every compact set $B \subseteq V$ there exists a real-valued function $\gamma_B(t)$ defined on $[t_0,\infty)$ such that $\|g(t,y)\| \leq \gamma_B(t)$ for all $y \in B$ and $t \geq t_0$, and either $\gamma_B(t) \to 0$ as $t \to \infty$, or $\int_{t_0}^{\infty} \gamma_B(t)dt < \infty$.

Let X^* be the space of solutions of the differential equation $y' = f(t,y) + g(t,y)$ in $\mathbf{R} \times V$ where $f(t,y)$ is as in Example 1.1. The space X^* belongs to $R_{ce}(\mathbf{R} \times V)$ and converges in $\mathbf{R} \times V$ to the family ζ from the previous example as $t \to \infty$ (see [10]–[13]).

Remark. The convergence of X^* to ζ in Example 1.2 holds under more general assumptions. It follows from the treatment in [10], [11] that $f(t,y)$ may be assumed, for instance, to be discontinuous on a countable closed set, and $g(t,y)$ need not have a majorant (or satisfy the Assumption (A) mentioned above) on all compact sets $B \subseteq V$.

6. We denote the set of values of a function z by $\mathrm{Im}\,(z)$. The set $\mathrm{Im}\,(z)$ will be called the trajectory, or orbit of z.

The following definitions generalize corresponding concepts related to solutions of differential equations (see [21], [3]).

Definition 1.3. A set $M \subseteq L$ is semi-invariant with respect to a space $Z \subseteq C_s(U)$ if for every $y \in M$ there exists a function $z \in Z^{-+}$ such that $y \in \mathrm{Im}\,(z)$ and $\mathrm{Im}\,(z) \subseteq M$.

Definition 1.4. Suppose that $F \subseteq C_s(U)^+ \cup C_s(U)^{-+}$. Let $\sup\{t : t \in \operatorname{dom}(z)\} = T \leq \infty$ for all $z \in F$. The generalized ω-limit set $\Omega(F)$ of F consists of those points $y \in L$ for which there exist sequences $z_j \in F$ and $t_j \in \operatorname{dom}(z_j)$ such that $t_j \to T$ and $z_j(t_j) \to y$ as $j \to \infty$.

It is clear that $\Omega(F)$ is closed in L. If F consists of a single function z, we write $\Omega(z)$ for $\Omega(\{z\})$.

Remark. Let $U_0 = (a_0, \infty) \times V$, where $a_0 \in \mathbf{R} \cup \{-\infty\}$ and V is an open subset of L. It follows from Proposition 1.1 that if a subspace $Z \subseteq C_s(U)$ satisfies (1.3), $z \in Z^+$ and $\Omega(z) \cap V$ is nonempty, then $\sup\{t : t \in \operatorname{dom}(z)\} = \infty$.

It is not difficult to verify the following proposition (cf. [14, Theorem VII.1.1]).

Proposition 1.3 ([11]). *If $z \in C_s(U)^+ \cup C_s(U)^{-+}$ and $\Omega(z)$ is compact, then $\Omega(z)$ is connected.*

2. Semi-invariance of generalized ω-limit sets

Invariance properties of ω-limit sets for trajectories of solutions of ordinary differential equations are of major significance and were studied in numerous papers (for a survey see [21], [1] and the references therein). The following theorem demonstrates the semi-invariance of generalized ω-limit sets in a very general setting.

Let V be an open set in L, let $a_0 \in \mathbf{R} \cup \{-\infty\}$ and $U_0 = (a_0, \infty) \times V$.

Theorem 2.1. *Suppose $X \in R_{ce}(U_0)$, ζ is a family of subspaces of $C_s(U_0)$ and X converges in U_0 to ζ as $t \to \infty$. Let $F \subseteq X^+$, $\operatorname{dom}(z) = [\alpha, \infty)$ for all $z \in F$, and $y \in V \cap \Omega(F)$. Then for every $a > a_0$ there exists a space $Z \in \zeta$ and a function $z^* \in Z^{-+}$ such that $z^*(a) = y$ and $\operatorname{Im}(z^*) \subseteq \Omega(F)$. If, moreover, $\Omega(F)$ is a compact subset of V, then $\operatorname{dom}(z^*) = (a_0, \infty)$.*

Remark. Theorem 2.1 is a generalization of a theorem due to V.V. Filippov [8, Theorem IX.6.10]. In his theorem ζ consists of one autonomous space and F consists of a single function.

Proof: Since $y \in \Omega(F)$, there exist sequences $\{x_i : i \in \mathbf{N}\} \subseteq F$ and $\{t_i : i \in \mathbf{N}\} \subseteq (a_0, \infty)$ such that $t_i \to \infty$ and $x_i(t_i) \to y$ as $i \to \infty$. By virtue of Proposition 1.2, for each i there exists a function $u_i \in X^{-+}$ extending x_i.

Let $a > a_0$ be given. Denote $t_i - a$ by τ_i and $X^{\tau_i} \cap C_s(U_0)$ by X_i. The assumption $X \in R_{ce}(U_0)$ implies that $X_i \in R_{ce}(U_0)$ for all $i \in \mathbf{N}$. Since X converges in U_0 to ζ as $t \to \infty$, without loss of generality we may assume (by passing to a subsequence) that the sequence $\{X_i : i \in \mathbf{N}\}$ converges in U_0 to some space $Z \in \zeta$. Denote by z_i the restriction of $u_i^{\tau_i}$ to (a_0, ∞) (we recall that $u^\tau(t) = u(t + \tau)$). One easily verifies that $z_i \in X_i^{-+}$. Since $z_i(a) = u_i(t_i) = x_i(t_i)$, $z_i(a) \to y$ as $i \to \infty$. The sequence of points $p_i \in U_0$ with components a, $z_i(a)$ converges in U_0 to the point p with components a, y.

We can now apply Theorem 1.1 to the sequence of spaces $\{X_i : i \in \mathbf{N}\}$ convergent to $Z \in \zeta$, the points $p_i \to p$ and the functions $z_i \in X_i^{-+}$. Thus we obtain a function $z^* \in Z^{-+}$ and a subsequence $\{z_{i_k} : k \in \mathbf{N}\}$ such that $z^*(a) = y$, any point $t \in \mathrm{dom}\,(z^*)$ belongs to $\mathrm{dom}\,(z_{i_k})$ for k large enough, and

$$z_{i_k}(t) \to z^*(t) \quad \text{as} \quad k \to \infty. \tag{2.1}$$

Fix any $t \in \mathrm{dom}\,(z^*)$. Take $k_0 \in \mathbf{N}$ sufficiently large so that, whenever $k \geq k_0$, $t \in \mathrm{dom}\,(z_{i_k})$ and $s_k := t + t_k - a \geq \alpha$. Then for $k \geq k_0$ we have

$$z_{i_k}(t) = u_{i_k}(s_k) = x_{i_k}(s_k). \tag{2.2}$$

It follows from (2.1) and (2.2) that $x_{i_k}(s_k) \to z^*(t)$ as $k \to \infty$. Since $s_k \to \infty$ as $k \to \infty$, we conclude that $z^*(t) \in \Omega(F)$. Thus, $\mathrm{Im}\,(z^*) \subseteq \Omega(F)$, and the first statement of the theorem is proved.

Suppose now that $\Omega(F)$ is a compact subset of V. Let $\mathrm{dom}\,(z^*) = (c, d)$, $a_0 \leq c < d \leq \infty$. Let us prove that $d = \infty$.

Assume on the contrary that d is finite. Since $z^* \in Z^{-+}$, the function $w = z^*|_{[m,d)}$ belongs to Z^+ for a point $m \in (c, d)$. Let $\{d_l : l \in \mathbf{N}\}$ be a sequence of points in (m, d) which tends to d as $l \to \infty$. Let a compact set $C \subseteq V$ be the closure of some ε-neighbourhood of $\Omega(F)$. Recall that the functions z_{i_k} were chosen according to Theorem 1.1. Using the fact that $\mathrm{Im}\,(z^*)$ lies in the interior of C, one can deduce from Theorem 1.1 that there is a subsequence $\{\psi_l : l \in \mathbf{N}\}$ of $\{z_{i_k} : k \in \mathbf{N}\}$ such that $[m, d_l] \subseteq \mathrm{dom}\,(\psi_l)$ and $\psi_l(t) \in C$ for all $t \in [m, d_l]$. Denote by w_l the restriction of ψ_l to the closed interval $[m, d_l]$. The graph of any function w_l lies in the compact set $C \times [m, d]$. Since $\{X_i : i \in \mathbf{N}\}$ converges in U_0 to the space Z, this implies that there is a subsequence of $\{w_l : l \in \mathbf{N}\}$ convergent to some function $v \in Z$ with $\mathrm{dom}\,(v) = [m, d]$. It follows from (2.1) that $v(t) = z^*(t) = w(t)$ for all $t \in [m, d)$, so v is an extension of the function w. But this is impossible, since $w \in Z^+$. The contradiction obtained implies that $d = \infty$.

One can prove similarly that $c = a_0$. Hence $\mathrm{dom}\,(z^*) = (a_0, \infty)$. The proof is completed. □

We now present a definition of a stationary point of a subspace of $C_s(U)$ which is analogous to that related to differential equations.

Definition 2.1 (see [8, IX.6.6])**.** A point $y \in L$ is called a stationary point of a space $Z \subseteq C_s(U)$ if there is a function $z \in Z^{-+}$ with $\mathrm{Im}\,(z) = \{y\}$.

Theorem 2.1 implies the following corollary which is a generalization of [14, Corollary VII.1.1].

Corollary 2.1. *Suppose a space $X \in R_{ce}(U_0)$ converges in U_0 to a family ζ of subspaces of $C_s(U_0)$ as $t \to \infty$. Let $x \in X^+$ and $\Omega(x)$ consist of a single point $y \in V$. Then y is a stationary point of some space $Z \in \zeta$ and $x(t) \to y$ as $t \to \infty$.*

Theorem 2.2. *Let $X \in R_{ce}(U_0)$ and $Z \subseteq C_s(U_0)$. Suppose that there exists a sequence $\{t_i : i \in \mathbf{N}\} \to \infty$ such that $\{X^{t_i} : i \in \mathbf{N}\}$ converges in U_0 to Z. Assume further that $x \in X^+$ and $\Omega(x)$ is a nonempty compact subset of V. Then there exists a function $z \in Z^{-+}$ such that* $\mathrm{Im}\,(z) \subseteq \Omega(x)$ *and* $\mathrm{dom}\,(z) = (a_0, \infty)$.

PROOF: Since $\Omega(x)$ is nonempty and compact, the set $\mathrm{Im}\,(x)$ is precompact. Therefore the sequence $\{x(t_i) : i \in \mathbf{N}\}$ has a subsequence convergent to a point $y \in \Omega(x)$. The rest of the proof follows along the lines of the proof of Theorem 2.1 (for $F = \{x\}$) and is omitted. □

Remarks. 1. Theorems 2.1 and 2.2 are similar to results due to Artstein ([1, Theorems 7.2 and 7.3]) applicable to ordinary differential equations $y' = f(t, y)$ with f satisfying the Carathéodory conditions and a certain equicontinuity assumption. Our theorems are of a more general form and obtained by using an approach different from that in [1]. It should be noted that a generalization in another direction, namely over ordinary integral-like operator equations, was given in [1, Section 13].

2. As Example 1.2 shows, the theorems of this section may be applied to perturbed almost periodic differential equations. In relation to these equations Theorem 2.1 generalizes a result of Miller [17].

3. On the structure of ω-limit sets containing singular points

It cannot be overemphasized how important the Poincaré-Bendixon theory is to the study of the geometry of solutions of autonomous differential equations on the plane. We recall briefly two key results of this theory which describe ω-limit sets of trajectories of solutions of these equations ([14, Theorems VII.4.1 and VII.4.2]).

Consider the autonomous differential equation

$$y' = f(y), \tag{A}$$

where $f(y)$ is a continuous function on an open plane set V. Suppose that z is a maximally extended solution of (A), the orbit of z has no self-intersections, and the ω-limit set Ω of z is a nonempty compact subset of V. According to classical results due to Poincaré and Bendixon:

(*) if Ω contains no stationary points of (A), then it is an orbit of a periodic solution of (A);

(**) if Ω is not a point and contains a finite set F of stationary points of (A), then Ω is the union of F and at most a denumerable set of orbits of solutions to (A) joining points from F.

Markus proved in [16] a theorem similar to (*) for asymptotically autonomous differential equations on a plane. We present below a stronger version of Markus' theorem given in [1].

Consider in $\mathbf{R} \times V$ the equation

$$y' = f(y) + g(t, y), \tag{E}$$

where $f(y)$ is as in (A) and $g(t, y)$ as in Example 1.2 (for $L = \mathbf{R}^2$).

Theorem (Markus)**.** *Let z be a maximally extended solution of* (E)*. Suppose that the ω-limit set Ω of z does not contain stationary points of* (A)*, is compact and lies in V. Assume that the Cauchy problem for* (A) *has a unique solution for each initial value. Then Ω is the union of closed orbits of solutions to* (A)*.*

Note that under the assumptions of Markus' theorem the orbit of z may self-intersect (which is not permitted in (*)). This mildness of suppositions has as a result that Markus' theorem does not cover (*) even if the uniqueness of a solution for the Cauchy problem for (A) is assumed.

An extension of (**) for asymptotically autonomous planar ordinary differential equations was given recently by Thieme [22, Theorem 1.6]. As Markus did, Thieme does not forbid the trajectory of a solution to have self-intersections but this again, as in case of Markus' theorem, makes the description of the ω-limit set less precise than in (**).

In the framework of an axiomatic theory of ordinary differential equations, V.V. Filippov proved a generalization of (*) (see [8, Theorem IX.8.4]) which is applicable to perturbed autonomous planar ODEs and differential inclusions (as in (*), it is assumed in his theorem that the trajectory should not self-intersect). V.V. Filippov's approach to the Poincaré-Bendixon theorem (*) was developed in [19]. In recent papers [12], [13] V.V. Filippov presented a further generalization of (*) involving the convergence indicated in Definition 1.2. We use in this section V.V. Filippov's technique to prove (see Theorem 3.1 below) a generalization of (**). Our theorem also employs the concept of convergence described in Definition 1.2. The theorem has applications, for example, to planar ordinary differential equations of the type indicated in Example 1.2. Before formulating the theorem we will give first auxiliary definitions and results.

Let $M \subseteq L$, $Z \subseteq C_s(U)$ and $\{z \in Z : \mathrm{Im}\,(z) \subseteq M\} \neq \emptyset$.

Definition 3.1 (see [8, IX.6.2])**.** The diameter $\mathrm{diam}_Z\, M$ of the set M with respect to the space Z is a $d \in \mathbf{R} \cup \{\infty\}$ defined by $d = \sup\{b - a : [a, b] = \mathrm{dom}\,(z),\ \mathrm{Im}\,(z) \subseteq M\}$.

The following definition deals with some geometric properties of trajectories of functions belonging to a subspace of $C_s(U)$. It is motivated by the assumptions of Theorem 10 in [12].

Definition 3.2. We say that a point $y \in L$ is an *R-point* for a space $Z \subseteq C_s(U)$ if there exists a neighbourhood Ny of y such that the following properties (R1)–(R3) hold:

(R1) $\mathrm{diam}_Z\, Ny$ is finite;

(R2) in Ny there are no closed orbits of functions from Z;

(R3) there exits a neighbourhood Gy of y such that the closure K of Gy lies in Ny and is homeomorphic to a disk, the boundary ∂K of K contains two disjoint closed arcs l_1 and l_2 such that if $z \in Z$, $\operatorname{Im}(z) \subseteq K$, $c \in [a,b] = \operatorname{dom}(z)$, $z(c) = y$ and $z(a)$, $z(b) \in \partial K$, then $z(a) \in l_1$, $z(b) \in l_2$.

The behaviour of trajectories of functions from $Z \subseteq C_s(U)$ in the neighbourhood Ny of y which is an R-point for Z is similar to that of solutions of a planar autonomous ordinary differential equation near its regular (i.e. non-stationary) point. Indeed, it follows from the treatment carried out in [8, Chapter IX, § 6, § 8] that the regular points of a planar equation $y' = f(y)$ with a continuous function f are exactly the R-points for the space of its solutions.

A point $y \in L$ is said to be an *S-point* for $Z \subseteq C_s(U)$ if it is not an R-point for it. S-points are analogues to stationary points of ordinary differential equations.

Consider the following assumptions:

(3.1) V is an open set in the plane, $a_0 \in \mathbf{R} \cup \{-\infty\}$ and $U_0 = (a_0, \infty) \times V$;

(3.2) a space $X \in R_{ce}(U_0)$ converges in U_0 to a family ζ of subspaces of $C_s(U_0)$ as $t \to \infty$;

(3.3) a function z belongs to X^+, and the orbit of z does not intersect itself, i.e. $z(s) \neq z(t)$ for all $s, t \in \operatorname{dom}(z)$ with $s \neq t$.

Let $\cup\zeta$ denote the set $\cup\{Z : Z \in \zeta\}$.

Lemma 3.1. *Suppose* (3.1)–(3.3) *hold, and let a point $y \in \Omega(z) \cap V$ be an R-point for $\cup\zeta$. Then there exists a neighbourhood Oy of y and a function $z^* \in \cup\zeta$ such that $Oy \cap \Omega(z) = Oy \cap \operatorname{Im}(z^*)$ and this intersection is homeomorphic to an open interval.*

This lemma is analogous to [8, Proposition IX.8.3] and can be proved in a similar way. We will outline its proof omitting some details which are not difficult to verify. The proof employs the following auxiliary result (cf. [8, Lemma IX.8.2]) which can be proved using (3.2).

Lemma 3.2. *Under the assumptions of Lemma* 3.1 *there exists an ε-neighborhood Oy of y and a number $T \in \operatorname{dom}(z)$ such that*

(1) *$Oy \subseteq Gy$, where Gy is a neighbourhood of y satisfying Definition* 3.2 *for $Z = \cup\zeta$;*

(2) *there exist two disjoint closed arcs s_1 and s_2 on ∂K (K is the closure of Gy) having the property: for any closed interval $[a,b] \subseteq \operatorname{dom}(z)$ with $a > T$, if $z([a,b]) \cap Oy \neq \emptyset$, $z([a,b]) \subseteq K$ and $z(a)$, $z(b) \in \partial K$, then $z(a) \in s_1$, $z(b) \in s_2$.*

We now proceed to *the proof* of Lemma 3.1.

Choose Oy, T, s_1 and s_2 according to Lemma 3.2. The set $\{t \in \mathbf{R} : t > T,\ z(t) \in Gy\}$ can be represented as a countable union of its maximal connected subsets (which are disjoint open intervals). Let γ be the family consisting of those of the intervals that intersect the set $z^{-1}(Oy)$. One can verify that no subfamily of γ can accumulate to a point in $\mathbf{R}$. Let us enumerate the intervals in γ according to their location in $\mathbf{R}$ in increasing order: $I_1, I_2, \dots$. Let $I_n = \{t \in \mathbf{R} : a_n < t < b_n\}$. Clearly, $z(a_n)$ and $z(b_n)$ lie in ∂K, so that, by Lemma 3.2, $z(a_n) \in s_1$ and $z(b_n) \in s_2$.

Making use of the assumption that the orbit of z does not intersect itself, one can prove that the points $z(a_1), z(a_2), \dots$ are situated on s_1 monotonically: any point $z(a_{n+1})$ is between $z(a_n)$ and $z(a_{n+2})$. The points $z(b_1), z(b_2), \dots$ are also situated on s_2 monotonically, and if $z(a_1), z(a_2), \dots$ are located on s_1, say, clockwise, then $z(b_1), z(b_2), \dots$ are located on s_2 anticlockwise. For any $n \in \mathbf{N}$ denote by L_n the arc of ∂K with $z(a_n)$ and $z(b_n)$ as its endpoints, which does not contain the points $z(a_k)$ and $z(b_k)$ for $k > n$. It is clear that, for every n, the curve $z(t)$, $a_n < t < b_n$, lies in the interior of the region bounded by L_{n+1} and the curve $z(t)$, $a_{n+1} \leq t \leq b_{n+1}$.

Let Ny be a neighbourhood of y satisfying Definition 3.2 for $Z = \cup\zeta$. Since $K \subseteq Ny$, it follows from (R2) that $\operatorname{diam}_{\cup\zeta} K$ is finite. We can therefore apply (3.2) to find for an arbitrary $a \in \operatorname{dom}(z)$ a subsequence of the sequence of functions $\{(z|_{[a_n,b_n]})^{a_n - a} : n \in \mathbf{N}\}$ (we recall that $z^\tau(t) = z(t+\tau)$) converging to some function $z^* \in \cup\zeta$. One easily verifies, using results of the analysis carried out in the previous paragraph, that the set $\operatorname{Im}(z^*)$ coincides with the set of limit points of all sequences of the form $\{z(t_n) : n \in \mathbf{N}\}$, where t_n is an arbitrary point from $[a_n, b_n]$. Hence $Oy \cap \Omega(z) = Oy \cap \operatorname{Im}(z^*)$. Since $\operatorname{Im}(z^*) \subseteq K \subseteq Ny$ and Ny satisfies (R2), the orbit of z^* has no self-intersections, so $Oy \cap \operatorname{Im}(z^*)$ is homeomorphic to an open interval. The lemma is proved. □

Remark. The proof of Lemma 3.1 shows that the orbit of z has no points in common with $\Omega(z)$ in Oy.

Theorem 3.1. *Suppose* (3.1)–(3.3) *hold. Assume that the set $\Omega(z) \subseteq V$ consists of more than one point, is compact and that the S-points for $\cup\zeta$ contained in $\Omega(z)$ are a non-empty finite set, F. Then for every point $y_0 \in \Omega(z) \setminus F$ there exists a function $z_0 \in \cup\{Z^{-+} : Z \in \zeta\}$ such that*

(i) $\operatorname{dom}(z_0) = (a_0, \infty)$ *and* $y_0 = z_0(a)$ *for some* $a \in \operatorname{dom}(z_0)$;
(ii) $\operatorname{Im}(z_0) \subseteq \Omega(z)$;
(iii) *if $z_0(t)$ does not belong to F for any $t \in [a, \infty)$* (*respectively, $t \in (a_0, a]$*), *then the limit of $z_0(t)$ as $t \to \infty$* (*respectively, as $t \to a_0$*) *exists and lies in F, and $z_0(t_1) \neq z_0(t_2)$ for all distinct t_1, t_2 from $[a, \infty)$* (*respectively, from $(a_0, a]$*).

Proof: 1. By Proposition 1.3 the compact set $\Omega(z)$ is connected and by assumption it is not a point. Therefore $\Omega(z)$ cannot consist of a finite number of points, and so $\Omega(z) \setminus F \neq \emptyset$. Take any $y_0 \in \Omega(z) \setminus F$. By Theorem 2.1 there exists a space $Z_0 \in \zeta$ and a function $z_0 \in Z_0^{-+}$ with $\operatorname{dom}(z_0) = (a_0, \infty)$ such that $y_0 = z_0(a)$ for some $a \in \operatorname{dom}(z_0)$ and $\operatorname{Im}(z_0) \subseteq \Omega(z)$. Thus (i) and (ii) are verified.

We will prove (iii) for the case when $z_0(t) \notin F$ for all $t \in [a, \infty)$. The other case is treated similarly.

Denote by z^0 the restriction of z_0 to $[a, \infty)$. We may assume that $z^0 \in Z_0^+$ (in fact, for any $b \in \mathrm{dom}\,(z_0)$ the restriction of z_0 to $[b, \infty)$ belongs to Z_0^+). By assumption, $z^0(t) \notin F$ for all $t \in \mathrm{dom}\,(z^0)$, so that $z^0(t)$ is an R-point for $\cup\zeta$ for all $t \in [a, \infty)$. 2. Assume that $\Omega(z^0)$ contains an R-point for $\cup\zeta$. We will show that this assumption leads to a contradiction.

We will prove first that the assumption implies that there is a closed interval $I \subseteq \mathrm{dom}\,(z^0)$ such that the trajectory of the function $u = z^0|_I$ is a Jordan curve.

Let $y \in \Omega(z^0)$ be an R-point for $\cup\zeta$. Since the orbit of z^0 lies in the compact set $\Omega(z)$, $\Omega(z^0) \subseteq \Omega(z)$, so that $y \in \Omega(z)$. We apply now Lemma 3.1 to the function z and the point y to obtain a neighbourhood Oy and a function z^* as described in the lemma. The point y being an R-point for $\cup\zeta$, one may assume without loss of generality that $\mathrm{diam}_{\cup\zeta}\, Oy$ is finite.

Since $y \in \Omega(z^0)$, the orbit of z^0 must enter Oy. As this orbit lies in $\Omega(z)$ and $Oy \cap \Omega(z) = Oy \cap \mathrm{Im}\,(z^*)$, we have $Oy \cap \mathrm{Im}\,(z^0) \subseteq Oy \cap \mathrm{Im}\,(z^*)$; the latter intersection is homeomorphic to an open interval. One readily sees that the behaviour of the orbit of z^0 is as follows: it enters Oy and goes there along a curve lying in the orbit of z^*, leaves Oy (because Oy is of finite diameter with respect to $\cup\zeta$), then enters Oy again (since $y \in \Omega(z^0)$), goes along a subset of $\mathrm{Im}\,(z^*)$, leaves Oy, and so on. But then the orbit of z^0 should self-intersect in Oy, so that $z^0(\alpha) = z^0(\beta)$ for some $\beta > \alpha \geq a$.

We claim that the real number $\delta = \inf\{t-s : s,\, t \in [\alpha, \beta],\ s < t,\ z^0(s) = z^0(t)\}$ is positive. Otherwise, for each $n \in \mathbf{N}$ there are numbers $\sigma_n,\, \tau_n \in [\alpha, \beta],\ \sigma_n < \tau_n$, such that $z^0(\sigma_n) = z^0(\tau_n)$ and $\tau_n - \sigma_n < 1/n$. Without loss of generality one may assume that the sequences σ_n and τ_n converge to some $t_0 \in [\alpha, \beta]$ as $n \to \infty$. By (iii), the point $x = z^0(t_0)$ is an R-point for $\cup\zeta$, so x has a neighbourhood Ox satisfying (R2) for $Z = \cup\zeta$. The continuity of z^0 implies that the orbit of $z_n = z^0|_{[\sigma_n, \tau_n]}$ lies in Ox for n large enough. As $z_n(\sigma_n) = z_n(\tau_n)$, $z_n \in Z_0$ (since $z^0 \in Z_0^+$), and $Z_0 \in \zeta$, this contradicts (R2). The claim is proved.

Thus, for each $n \in \mathbf{N}$ there are t_n^* and t_n^{**} in $[\alpha, \beta]$, $t_n^* < t_n^{**}$, such that $z^0(t_n^*) = z^0(t_n^{**})$ and $t_n^{**} - t_n^*$ tends to δ from above as $n \to \infty$. Without loss of generality we may assume that the sequences $\{t_n^* : n \in \mathbf{N}\}$ and $\{t_n^{**} : n \in \mathbf{N}\}$ converge to some points c and d, respectively ($d = c + \delta$). Clearly $z^0(c) = z^0(d)$ and $z^0(s) \neq z^0(t)$ for $c \leq s < t \leq d$. Therefore the trajectory of $u = z^0|_{[c,d]}$ is a Jordan curve.

3. We claim that $\mathrm{Im}\,(u)$ contains an S-point for $\cup\zeta$.

Assume the contrary. Then, since $\Omega(z)$ contains S-points for $\cup\zeta$ and $\mathrm{Im}\,(u)$ does not, the set $A = \Omega(z) \setminus \mathrm{Im}\,(u)$ is nonempty. Since $\Omega(z)$ is connected, A cannot be closed, so that there is a $y^* \in \mathrm{Im}\,(u)$ which is a limit point of A. By the assumption of this paragraph y^* is an R-point for $\cup\zeta$.

Apply now Lemma 3.1 to the point y^* and the function z to obtain a neighbourhood Oy^* of y^* and a function $\tilde{z} \in \cup\zeta$ such that the set $Oy^* \cap \Omega(z) = Oy^* \cap \mathrm{Im}\,(\tilde{z})$ is homeomorphic to an open interval. Shrinking Oy^* if necessary, one may assume that Oy^* satisfies (R1)–(R2) for $Z = \cup\zeta$. We have

$$Oy^* \cap \mathrm{Im}\,(u) \subseteq Oy^* \cap \Omega(z) = Oy^* \cap \mathrm{Im}\,(\tilde{z}), \tag{3.4}$$

so that the orbit of u goes in Oy^* along the orbit of $\tilde{z}$. Taking into account that Oy^* satisfies (R1)–(R2), that $\mathrm{Im}\,(u)$ is a Jordan curve and that $Oy^* \cap \mathrm{Im}\,(\tilde{z})$ is homeomorphic to an open interval, one deduces from (3.4) that $Oy^* \cap \mathrm{Im}\,(u) = Oy^* \cap \mathrm{Im}\,(\tilde{z})$. It now follows from (3.4) that $Oy^* \cap \Omega(z) = Oy^* \cap \mathrm{Im}\,(u)$. Hence y^* cannot be a limit point of A. The contradiction obtained proves that in $\mathrm{Im}\,(u)$ there is an S-point for $\cup\zeta$.

The claim proved yields that $\mathrm{Im}\,(u) \cap F \neq \emptyset$. Since $\mathrm{Im}\,(u) \subseteq \mathrm{Im}\,(z^0)$, this implies that $\mathrm{Im}\,(z^0) \cap F$ is nonempty, i.e. that $z_0(t) \in F$ for some $t \geq a$. This contradicts what was assumed in (iii).

Thus the assumption that in $\Omega(z^0)$ there is an R-point for $\cup\zeta$ (see the beginning of Part 2 of the proof) leads to a contradiction. Therefore the set $\Omega(z_0) = \Omega(z^0)$ contains only S-points for $\cup\zeta$.

4. As the set $\Omega(z_0)$ lies in $\Omega(z)$ and consists only of S-points for $\cup\zeta$, $\Omega(z_0)$ is contained in the finite set F. But $\Omega(z_0)$ is connected, so it is a point from F. Since the ω-limit set of a trajectory of z_0 consists of a single point, the trajectory tends to this point as $t \to \infty$.

The assertion that the orbit of the restriction of z_0 to $[a, \infty)$ has no self-intersections now easily follows from Lemma 3.1 (by (iii), any point on this orbit is an R-point for $\cup\zeta$). The theorem is proved. □

Remark. Let M be the set of orbits of functions z_0 related to *all* $y_0 \in \Omega(z) \setminus F$. Making use of Lemma 3.1 and the Remark following its proof, one can prove, employing arguments analogous to those in [14, Theorem VII.4.2], that M is countable.

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