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Opial's property and James' quasi-reflexive spaces

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Abstract. Two of James' three quasi-reflexive spaces, as well as the James Tree, have the uniform w^*-Opial property.

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Introduction

Let $(X, \|\cdot\|)$ be a Banach space with a Schauder finite dimensional decomposition (FDD) [1], [20]. Define $\beta_p((X, \|\cdot\|))$ for $p \in [1, \infty)$ to be the infimum of the set of numbers λ such that

$$(\|x\|^p + \|y\|^p)^{1/p} \le \lambda \|x + y\| \tag{1}$$

for every x and y in X with $\operatorname{supp}(x) < \operatorname{supp}(y)$ (we use here the notation in [1], [15]). In [15] M.A. Khamsi proved the following result.

Theorem A. *Let $(X, \|\cdot\|)$ be a Banach space with a finite codimensional subspace Y such that $\beta_p((Y, \|\cdot\|)) < 2^{1/p}$ for some $p \in [0, \infty)$. Then X has weak normal structure.*

He then used this theorem to deduce that James quasi-reflexive space, which consists of all null sequences $x = \{x^i\} = \sum_{i=1}^{\infty} e_i$ ($\{e_i\}$ is the standard basis in c_0) for which the squared variation

$$\sup_{\substack{m \\ p_1 < \dots < p_m}} \left[\sum_{j=2}^{m} |x^{p_j} - x^{p_{j-1}}|^2 \right]^{1/2} \tag{2}$$

is finite, with the norm $\|\cdot\|$ given by (2), has weak normal structure by claiming that $\beta_2((J\|\cdot\|_1)) = 1$. As a matter of fact, $\beta_2((J, \|\cdot\|_1)) \ge 2^{1/2}$, which can be easily seen by taking $x = e_2$ and $y = e_3$. Fortunately, Theorem A remains true with a slight modification of the definition of $\beta_p((X, \|\cdot\|))$, namely

$$\widetilde{\beta}((X, \|\cdot\|)) =$$
$$= \inf_{k=0,1,2,\dots} \left\{ \inf[\lambda : (1) \text{ is valid for } x, y \in X \text{ with } \operatorname{supp}(x) + k < \operatorname{supp}(y)] \right\},$$

and $\widetilde{\beta}_2((J, \|\cdot\|)) = 1$. Thus $(J, \|\cdot\|_1)$ does indeed possess weak normal structure. In the present paper we will prove that $(J, \|\cdot\|_1)$ has, in fact, the uniform w^*-Opial property [23], which, of course, also implies weak normal structure [1], [3], [4], [8].

Definitions and notations

We recall that a (dual) Banach space $(X, \|\cdot\|)$ has the (w^*-) Opial property if whenever a sequence $\{x_n\}$ in X converges weakly (weakly*) to x_0, then for $x \neq x_0$

$$\liminf_n \|x_0 - x_n\| < \liminf \|x - x_n\|$$

[21], [22]. Opial's property plays an important role in the study of weak convergence of iterates and random products of nonexpansive mappings and of the asymptotic behavior of nonlinear semigroups [4], [5], [7], [13], [18], [21], [22]. Moreover, it can be introduced in the open unit ball of a complex Hilbert space, equipped with the hyperbolic metric, where it is useful in proving the existence of fixed points of holomorphic self-mappings of B [5], [6].

The (dual) Banach space $(X, \|\cdot\|)$ is said to have the uniform (w^*-) Opial property [23] if for every $c > 0$ there exists an $r > 0$ such that

$$1 + r \leq \liminf_n \|x - x_n\| \tag{3}$$

for each $x \in X$ with $\|x\| \geq c$ and every sequence $\{x_n\}$ with $w\text{-}\lim_n x_n = 0$ ($w^*\text{-}\lim_n x_n = 0$) and $\liminf_n \|x_n\| \geq 1$.

In the linear space J defined by (2) one uses three different, but equivalent norms, $\|\cdot\|_1$ (defined by (2)), $\|\cdot\|_2$, and $\|\cdot\|_3$, introduced by R.C. James [9], [10], [11]:

$$\|x\|_2 = \sup_{\substack{k \\ p_1 \cdots < p_{2k}}} \left[\sum_{j=1}^{k} |x^{p_{2j}} - x^{p_{2j-1}}|^2 \right]^{1/2},$$

$$\|x\|_3 = \sup_{\substack{m \\ p_1 \cdots < p_m}} \left[\sum_{j=2}^{m} |x^{p_j} - x^{p_{j-1}}|^2 + |x^{p_m} - x^{p_1}|^2 \right]^{1/2}.$$

The choice of norms depends on one's goals [2], [9], [10], [11], [20].

In [14] M.A. Khamsi used the ultraproduct method to prove that $(J, \|\cdot\|_3)$ has the fixed point property for nonexpansive mappings (FPP), i.e. for every nonempty weakly compact convex subset C of $(J, \|\cdot\|_3)$ any nonexpansive self-mapping $T : C \to C$ has a fixed point. D. Tingley [24] has recently shown that $(J, \|\cdot\|_3)$ has, in fact, weak normal structure ([3]): every nonempty weakly compact convex subset C of $(J, \|\cdot\|_3)$ with $\operatorname{diam} C > 0$ has a nondiametral point y, i.e.

$$\sup_{x \in C} \|y - x\|_3 < \operatorname{diam} C.$$

This property immediately guarantees the FPP [17]. The proof of weak normal structure is based on the following property of weakly convergent sequences in $(J, \|\cdot\|_3)$: if $\{x_n\}$ converges weakly to 0 and $\operatorname{diam}\{x_n\} > 0$, then

$$\sup_m \left(\limsup_n \|x_m - x_n\|_3 \right) > \liminf_n \|x_n\|_3 .$$

But it is easy to observe that the sequence $\{-e_n + e_{n+1}\}$ tends weakly to 0 in $(J, \|\cdot\|_3)$ and

$$\|\frac{1}{3}e_1 + e_n - e_{n+1}\|_3 = \| - e_n + e_{n+1}\|_3 = \sqrt{8}$$

for $n \geq 3$. Therefore $(J, \|\cdot\|_3)$ does not have Opial's property.

Main result

In this section we are concerned with the spaces $(J, \|\cdot\|_1)$ and $(J, \|\cdot\|_2)$.

The predual Banach space I to $(J, \|\cdot\|_j)$, $j = 1, 2$, is generated by the biorthogonal functionals $\{f_n\}$ to the basis $\{u_n\} = \{e_1 + \cdots + e_n\}$ [12], [19]. Throughout this paper we will always treat J as I^*.

Theorem. *For $j = 1, 2$ the space $(J, \|\cdot\|_j)$ has the uniform w^*-Opial property.*

Proof: Let $k \in \mathbb{N}$ and let P_k and Q_k be the natural projections in J associated with the basis $\{u_n\}$:

$$P_k x = \sum_{n=1}^{k} \xi^n u_n$$

and

$$Q_k x = \sum_{n=k+1}^{\infty} \xi^n u_n$$

for each $x = \sum_{n=1}^{\infty} \xi^n u_n \in J$. Note that if $x = \sum_{n=1}^{\infty} \xi^n u_n$, then

$$\|x\|_1 = \sup_{\substack{m \\ p_1 < \cdots < p_m}} \left\{ \sum_{j=2}^{m} \left[\sum_{n=p_{j-1}}^{p_j - 1} \xi^n \right]^2 \right\}^{1/2}$$

and

$$\|x\|_2 = \sup_{\substack{k \\ p_1 < \cdots < p_{2k}}} \left\{ \sum_{j=1}^{k} \left[\sum_{n=p_{2j-1}}^{p_{2j} - 1} \xi^n \right]^2 \right\}^{1/2}$$

[11], [12]. Directly from these formulas we obtain

$$\|x\|_j = \lim_k \|P_k x\|_j$$

and

$$\lim_k \|Q_k x\|_j = 0$$

for all $x \in J$ and $j = 1, 2$. Assume that a sequence $\{x_n\}$ in $(J, \|\cdot\|_j)$ converges weakly* to 0 and let $x \in J$. Then we have

$$\lim_n \|P_k x_n\|_j = 0,$$
$$\liminf_n \|Q_k x_n\|_j = \liminf_n \|x_n\|_j,$$

and

$$\begin{aligned}
\liminf_n \|x - x_n\| &\geq \liminf_n \Big[\|P_k x - Q_{k+1} x_n\|_j - \|Q_k x\|_j - \|P_{k+1} x_n\|_j \Big] \\
&= \liminf_n \Big[\|P_k x - Q_{k+1} x_n\|_j - \|Q_k x\|_j \Big] \\
&\geq \liminf_n \Big[\|P_k x\|_j^2 + \|Q_{k+1} x_n\|_j^2 \Big]^{1/2} - \|Q_k x\|_j \\
&= \Big[\|P_k x\|_j^2 + \liminf_n \|Q_{k+1} x_n\|_j^2 \Big]^{1/2} - \|Q_k x\|_j \\
&= \Big[\|P_k x\|_j^2 + \liminf_n \|x_n\|_j^2 \Big]^{1/2} - \|Q_k x\|_j
\end{aligned}$$

for $k = 1, 2, \ldots$. Hence we obtain the following inequality

$$(*) \qquad \begin{aligned}
\liminf_n \|x - x_n\|_j &\geq \lim_k \left\{ \Big[\|P_k x\|_j^2 + \liminf_n \|x_n\|_j^2 \Big]^{1/2} - \|Q_k x\|_j \right\} \\
&= \Big[\|x\|_j^2 + \liminf_n \|x_n\|_j^2 \Big]^{1/2}
\end{aligned}$$

which leads to (3). In other words, $(J, \|\cdot\|_j)$ has the uniform w^*-Opial property for $j = 1, 2$. □

Corollary 1. *For $j = 1, 2$ the space $(J, \|\cdot\|_j)$ has the uniform Opial property.*

Remark 1. The uniform w^*-Opial property of $(J, \|\cdot\|_j)$, $j = 1, 2$, implies the following important property of these spaces. The (w^*-) modulus of noncompact convexity of a (dual) Banach space $(X, \|\cdot\|)$ is the function $\Delta_x : [0, 1] \to [0, 1]$ $(\Delta_x^* : [0, 1] \to [0, 1])$ defined by

$$\Delta_x(\varepsilon) = \inf\{1 - \operatorname{dist}(0, A)\}$$
$$(\Delta_x^*(\varepsilon) = \inf\{1 - \operatorname{dist}(0, A)\}),$$

where the infimum is taken over all convex (weak* compact convex) subsets A of the closed unit ball with $\chi(A) \geq \varepsilon$, and χ is the Hausdorff measure of noncompactness [4]. In the case of $(J, \|\cdot\|_j)$, $j = 1, 2$, the inequality $(*)$ implies $\Delta_j^*(\varepsilon) > 0$

for all $\varepsilon > 0$. This means that these spaces are Δ^*-uniformly convex and every weakly* compact convex subset C of $(J, \|\cdot\|_j)$ $(j = 1, 2)$ has a compact asymptotic center [4]. Taking $A = \operatorname{conv}\{u_n\}$, where $u_n = \sum_{i=1}^{n}$, we see that

$$\chi(A) = 1$$

and

$$\underset{x \in A}{\forall} \|x_j\| = 1$$

$(j = 1, 2)$. Therefore $\Delta_x \equiv 0$ for $X = (J, \|\cdot\|_j)$, $j = 1, 2$.

Here we have to mention that generally the uniform Opial property does not imply the Δ-uniform convexity as the following example shows.

Example ([23]). For $\lambda > 1$ let X be the space l_2 with the norm

$$\|(\alpha_n)\| = \max\{\lambda|\alpha_1|,\ \|(\alpha_n)\|_2\}$$

where $\|\cdot\|_2$ is the norm in l_2. Then

$$\begin{aligned}\liminf_n \|x_n - x\| &= \max\left\{\lambda|\alpha_1|,\ \left(\liminf_n \|x_n\|_2^2 + \|x\|_2^2\right)^{1/2}\right\} \\ &\geq \left(1 + \|x\|_2^2\right)^{1/2} \geq \left(1 + \lambda^{-2}\|x\|_2^2\right)^{1/2}\end{aligned}$$

for each $x \in X$ and each sequence $\{x_n\}$ with $w\text{-}\lim_n x_n = 0$ and $\liminf \|x_n\| \geq 1$. This inequality guarantees the uniform Opial property of X, but $\Delta_x(\varepsilon) = 0$ for all $\varepsilon \leq (1 - \lambda^{-2})^{1/2}$.

Remark 2. It is easy to observe that James Tree JT constructed by R.C. James [11] also has the w^*-uniform Opial property, where JT is the dual space to the Banach space B generated by the biorthogonal functionals $\{f_{n,i}\}$ to the basis $\{e_{n,i}\}$ (this basis is analogous to the basis $\{u_n\}$ in J) given in [19]. The proof of this fact is a slight modification of the proof of the Theorem. Corollary 1 and Remark 1 are also valid for JT. (See also [13] for the w^*-Opial property.)

Remark 3. The uniform (w^*-) Opial property of $(J, \|\cdot\|_j)$ with $j = 1, 2$ and JT implies that these spaces satisfy the weak (weak*) uniform Kadec-Klee property [16].

We conclude our paper with three corollaries.

Corollary 2. $(J, \|\cdot\|_j)$, $j = 1, 2$, *and* JT *have weak and weak* normal structure.*

Corollary 3. $(J, \|\cdot\|_j)$, $j = 1, 2$, *and* JT *have the* FPP *for weakly* compact convex subsets.*

Recall that a Banach space $(X, \|\cdot\|)$ is said to satisfy the (w^*-) demiclosedness principle [1], [4] if whenever C is a nonempty weakly (weakly*) compact convex subset of X and $T : C \to X$ is nonexpansive, then the mapping $I - T$, where I is the identity operator, is (w^*-) demiclosed, i.e. if $\{x_n\}$ is weakly (weakly*) convergent to x and $\{x_n - Tx_n\}$ converges strongly to y, then $x - Tx = y$. It is known that every Banach space with the (w^*-) Opial property satisfies the (w^*-) demiclosedness principle.

Corollary 4. *$(J, \|\cdot\|_j)$, $j = 1, 2$, and JT satisfy the (w^*-) demiclosedness principle.*

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