# The nil radical of an Archimedean partially ordered ring with positive squares

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*Abstract.* Let $R$ be an Archimedean partially ordered ring in which the square of every element is positive, and $N(R)$ the set of all nilpotent elements of $R$. It is shown that $N(R)$ is the unique nil radical of $R$, and that $N(R)$ is locally nilpotent and even nilpotent with exponent at most 3 when $R$ is 2-torsion-free. $R$ is without non-zero nilpotents if and only if it is 2-torsion-free and has zero annihilator. The results are applied on partially ordered rings in which every element $a$ is expressed as $a = a_1 - a_2$ with positive $a_1$, $a_2$ satisfying $a_1a_2 = a_2a_1 = 0$.

*Keywords:* partially ordered ring, Archimedean, nil radical, nilpotent

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## 0. Introduction

In this paper we give some descriptions of the set $N(R)$ of all nilpotent elements of an Archimedean partially ordered ring $R$ in which the square of every element is positive. It is shown among other things that $N(R)$ is the unique nil radical of $R$, and that $N(R)$ is locally nilpotent and even nilpotent with $N(R)^3 = \{0\}$ when $R$ is 2-torsion-free. Furthermore, we extend on $R$ some results of Diem [3] and of Bernau and Huijsmans [1], who have investigated the properties of $N(R)$ for a lattice-ordered $R$. In particular, we prove that the index of nilpotency of each $a \in N(R)$ does not exceed 4 (3 when $R$ is 2-torsion-free), and that $R$ is without non-zero nilpotents if and only if it is 2-torsion-free and has zero annihilator $ann(R)$. Also an application on partially ordered rings in which every element $a$ is expressed as $a = a_1 - a_2$ with positive $a_1$, $a_2$ satisfying $a_1a_2 = a_2a_1 = 0$ is given at the end of paper.

For the theory of rings and nil radicals we refer the reader to [4], [7] and [9]. We only recall that an ideal $I$ of a ring $R$ is called a *nil radical*, if every element of $I$ is nilpotent and if $A/I$ does not contain non-zero nilpotent ideals.

For the theory of partially ordered rings we refer the reader to [5]. We briefly review some standard terminology. A ring $R$ is said to be a *partially ordered ring* if there is a partial ordering $\leq$ on $R$ which is compatible with the algebraic structure of $R$. The positive cone $R^+ = \{a \in R : 0 \leq a\}$ of a partially ordered ring $R$ is closed for the addition and the multiplication, and determines on $R$ the ordering $\leq$ by $a \leq b$ if and only if $b - a \in R^+$.x A partially ordered ring $R$ is said to be *Archimedean* if $a = 0$ whenever $0 \leq b + na$ for some $b \in R$ and for all integers $n$. A partially ordered ring $R$ in which the square of every element is positive will be called a *ps-ring*.

## 1. The nil radical

Throughout this section $R$ denotes an associative Archimedean $ps$-ring, $N(R)$ the set of all nilpotent elements of $R$, and $N_k(R) = \{a \in R : a^k = 0\}$ the set of all elements $a \in N(R)$ with index of nilpotency not greater than $k$. An important role will play also the set $T_2(R) = \{a \in R : 2a = 0\}$ of all elements $a \in R$ with torsion 2.

The following description of $N_2(R)$ will be useful in our further work.

**Lemma 1.1.** *Let $R$ be an Archimedean $ps$-ring. Then*

(1) $N_2(R) = \{a \in R : ab + ba = 0$ *for all* $b \in R\}$.
(2) $abc = bac = bca \in T_2(R)$ *for all* $a \in N_2(R)$, $b, c \in R$.

PROOF: If $a \in N_2(R)$, $b \in R$, then

$$0 \leq (b + na)^2 = b^2 + n(ab + ba)$$

holds for all $n \in \mathbb{Z}$, hence $ab + ba = 0$, since $R$ is Archimedean. Conversely, if $a \in R$ satisfies $ab + ba = 0$ for all $b \in R$, then $2a^2 = 0$. Since $R$ has positive squares, this implies $a \in N_2(R)$, and (1) follows.

To prove (2) let $a \in N_2(R)$ and $b, c \in R$. Then use three times successively (1) to get

$$a(bc) = -(bc)a = -b(ca) = b(ac) = (ba)c = -(ab)c,$$

which implies (2), so the proof is complete. □

**Corollary 1.2.** *Let $m$ be an arbitrary positive integer. Then the left annihilator $ann_l(R^m)$ and the right annihilator $ann_r(R^m)$ of $R^m$ coincide, so*

$$ann_l(R^m) = ann_r(R^m) = ann(R^m).$$

PROOF: If $a \in N_2(R)$, $b \in R$, then by Lemma 1.1 $ab = 0$ if and only if $ba = 0$. Since $ann_l(R)$ and $ann_r(R)$ are contained in $N_2(R)$, this implies that $ann_l(R) = ann_r(R)$. We now proceed by induction. Suppose that $ann_l(R^k) = ann_r(R^k)$ for each $k \leq m$. Then $a \in ann_l(R^{m+1})$ is equivalent to $aR \subset ann_l(R^m) = ann_r(R^m)$, which holds if and only if $R^m a \subset ann_l(R) = ann_r(R)$. Since the latter is equivalent to $a \in ann_r(R^{m+1})$, the proof is complete. □

Diem has proved ([3, Theorem 3.9. (ii)]) that the index of a positive nilpotent element of an Archimedean lattice-ordered ring with positive squares does not exceed 3. Our next result shows that this is true for every Archimedean $ps$-ring.

**Proposition 1.3.** *Let $R$ be an Archimedean ps-ring. Then*

(1) $N(R) = N_4(R) = \{a \in R : 2a^3 = 0\}$.
(2) $N(R) \cap R^+ \subset N_3(R)$.

Proof: If $a \in R$ satisfies $a^{2m} = 0$ for some natural $m > 2$, then $a^{4m-6} = 0$, hence

$$0 \le (a + na^{2m-3})^2 = a^2 + n(2a^{2m-2})$$

holds for all $n \in \mathbb{Z}$. Since $R$ is Archimedean and has positive squares, this implies that $a^{2m-2} = 0$. It follows that $N(R) = N_4(R)$.

Let now $a \in N_4(R)$. Then

$$0 \le (a + na^2)^2 = a^2 + n(2a^3)$$

holds for all $n \in \mathbb{Z}$, hence $2a^3 = 0$. Conversely, $2a^3 = 0$ implies $2a^4 = 0$, and consequently $a^4 = 0$. The proof of (1) is complete, while (2) evidently follows from (1). □

**Lemma 1.4.** *Let $R$ be an Archimedean ps-ring and let $a \in N(R)$. Then*

(1) $(ab)^2 = (ba)^2 = 0$ *for all* $b \in R$.
(2) $abc = -cab = bca$ *for all* $b, c \in R$.
(3) $abcd = bacd = bcad = bcda \in T_2(R)$ *for all* $b, c, d \in R$.

Proof: Since by Proposition 1.3 we have $a^2 \in N_2(R)$, Lemma 1.1. (2) implies $2aba^2 = 0$ for all $b \in R$. Therefore $2(aba)^2 = 0$ and consequently $(aba)^2 = 0$, since $R$ has positive squares. It follows that

$$0 \le (b + naba)^2 = b^2 + n((ab)^2 + (ba)^2)$$

holds for all $n \in \mathbb{Z}$, which implies (1).

To prove (2), combine (1) and Lemma 1.1. (1), while to obtain (3), note that by (1) $ab, ba, ad, da \in N_2(R)$ for all $b, d \in R$, and then use Lemma 1.1. (2). □

We are prepared to prove a generalization of a result of Diem [3] and Bernau, Huijsmans [1], Propositions 3.1 and 3.2.

**Theorem 1.5.** *Let $R$ be an Archimedean ps-ring. Then $N(R)$ is an order-convex ideal of $R$, satisfying*

$$\begin{aligned} N(R) = N_4(R) &= \{a \in R : abc + cab = 0 \ \textit{ for all } \ b, c \in R\} \\ &= \{a \in R : bca + cab = 0 \ \textit{ for all } \ b, c \in R\} \\ &= \{a \in R : 2a^3 = 0\} \\ &= \{a \in R : 2a \in ann(R^3)\}. \end{aligned}$$

Proof: Combine Proposition 1.3 and Lemma 1.4 to obtain the required equalities for $N(R)$. The last one implies that $N(R)$ is an ideal, and hence it is convex. □

It can be shown similarly that $N_2(R)$ is an order-convex ideal of $R$.

**Corollary 1.6.** *Let $R$ be an Archimedean ps-ring and $Z(R)$ its center. Then*

(1) $N(R)^2R = N(R)RN(R) = RN(R)^2 \subset T_2(R)$;
(2) $N_2(R) \subset Z(R)$ *implies* $N(R)R^2 = RN(R)R = R^2N(R) \subset T_2(R)$;
(3) $R = R^2$ *implies* $N(R) \subset Z(R)$, $N(R) = N_2(R)$, $N(R)R = RN(R) \subset T_2(R)$.

*If $R$ is also 2-torsion-free, then*

(4) $N(R) = N_3(R) = ann(R^3)$;
(5) $N_2(R) \subset Z(R)$ *implies* $N(R) = ann(R^2)$;
(6) $R = R^2$ *implies* $N(R) = N_2(R) = ann(R)$.

Proof: (1) Use Lemma 1.4. (2) to see that $a, b \in N(R)$, $c \in R$ implies $abc = -cab = bca = -abc$, and the result follows.

(2) If $a \in N(R)$, $b, c \in R$, then $ab \in N_2(R) \subset Z(R)$, and therefore $abc = cab$. By Lemma 1.4. (2) we have $2abc = abc + cab = 0$, so (2) follows easily.

(3) Lemma 1.4. (2) implies that $N(R) \subset Z(R)$, thus by (2) $N(R)R = RN(R) \subset T_2(R)$. It follows that each $a \in N(R)$ satisfies $2a^2 = 0$, hence $a \in N_2(R)$, and the proof is complete.

(4), (5), (6) Use Theorem 1.5 and (1), (2), (3). □

**Remark 1.7.** An Archimedean $ps$-ring $R$ with generating cone $R^+$ is torsion-free. Indeed, if $a = a_1 - a_2$ with $a_1, a_2 \in R^+$ satisfies $ma = 0$ for some $m \in \mathbb{N}$, then it follows that

$$0 \le m(a_1 + a_2) + na \quad \text{for all} \quad n \in \mathbb{Z},$$

which implies that $a = 0$.

It can be proved, that the set $T(R)$ of all torsion elements of $R$ is an order-convex ideal of $R$, and that the quotient ring $R/T(R)$ is an Archimedean torsion-free $ps$-ring.

**Theorem 1.8.** *Let $R$ be an Archimedean ps-ring. Then $N(R)$ is the unique nil radical of $R$. It is locally nilpotent and satisfies $N(R)^3 \subset T_2(R)$. If $R$ is 2-torsion-free, then $N(R)$ is nilpotent with $N(R)^3 = \{0\}$.*

Proof: A nil radical of bounded index is contained in the lower nil radical ([7, p. 232]), hence by Theorem 1.5 $N(R)$ is the unique nil radical of $R$. Thus $N(R)$ equals the Levitzki nil radical $L(R)$ of $R$, which is locally nilpotent ([9, Proposition 21.2]). The required inclusion follows by Corollary 1.6 and implies the remaining part of the theorem. □

It is proved in [3, Theorem 3.9. (v)] that an Archimedean lattice-ordered ring with positive squares and with zero left or right annihilator has no nonzero positive nilpotents. The following generalization of this result is a simple consequence of the last equation of Theorem 1.5.

**Proposition 1.9.** *If $R$ is an Archimedean ps-ring such that $ann(R) = \{0\}$, then $N(R) = T_2(R)$.* □

**Corollary 1.10.** *Let $R$ be an Archimedean ps-ring. Then the following statements are equivalent.*

(i) *$R$ is semiprime.*
(ii) *$R$ is reduced, i.e. without non-zero nilpotents.*
(iii) *$R$ is 2-torsion-free and satisfies $ann(R) = \{0\}$.*

□

Note that by Remark 1.7 a lattice-ordered $R$ is torsion-free, hence the above characterization of reduced rings $R$ improves [3, Theorem 3.9. (v)].

**Corollary 1.11.** *A unital Archimedean ps-ring is reduced (or semiprime) if and only if it is 2-torsion-free.* □

## 2. Examples

By Theorem 1.5 an Archimedean *ps*-ring satisfies $N(R) = N_4(R)$. We show that in general $N(R) \neq N_3(R)$.

**Example 2.1.** Let $R = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}_2c$ be the ring with multiplication defined by

$$a^2 = b,\ ab = ba = c,\ b^2 = c^2 = bc = cb = ac = ca = 0,$$

and ordered by the positive cone $R^+ = \mathbb{Z}^+ b$. It is easy to see that $R$ is an Archimedean *ps*-ring with

$$N_3(R) = \mathbb{Z}(2a) \oplus \mathbb{Z}b \oplus \mathbb{Z}_2c$$

and $N_4(R) = R$, thus $N_3(R) \neq N(R)$.

By Theorem 1.8 every nil Archimedean *ps*-ring $R$ is locally nilpotent, and even nilpotent with $R^3 = \{0\}$ when $R$ is 2-torsion-free. In the following example we present a nil Archimedean *ps*-ring $R$ which is not nilpotent.

**Example 2.2.** If $m \in \mathbb{Z}^+$, let $\beta_0(m), \beta_1(m), \cdots$ be the digits in the binary representation

$$m = \sum_{i=0}^{\infty} \beta_i(m) 2^i,\ \beta_i(m) \in \{0, 1\}$$

of $m$. Define the function $\varphi : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ by

$$\varphi(p,q) = \begin{cases} 0, & \text{if } \ \beta_i(p) = \beta_i(q) = 1 \ \text{ for some } \ i \\ p+q, & \text{otherwise} \end{cases}$$

and consider the ring $R = (\mathbb{Z}_2)^\infty$ with coordinatewise addition and with the multiplication defined by

$$e_p e_q = e_{\varphi(p,q)},\ \ p, q \in \mathbb{N},$$

where $e_n = (\underbrace{0, \cdots, 1}_{n}, 0, \cdots)$ and $e_0 = 0 = (0, 0, \cdots)$.

It is easy to see that $R$ is an associative ring which satisfies $R = T_2(R) = N_2(R)$. It follows that the positive cone $R^+ = \{0\}$ turn $R$ into a nil Archimedean $ps$-ring. Note that for each $n \in \mathbb{N}$

$$e_{2^n-1} = e_1 e_2 e_4 \cdots e_{2^{n-1}} \in R^n,$$

thus $R$ is not nilpotent.

If $R$ is 2-torsion-free, then by Corollary 1.6 the chain $ann(R) \subset ann(R^2) \subset \cdots$ is finite and satisfies $ann(R^k) = N(R)$ for all $k \geq 3$. Example 2.2 show that this is not the case in a general situation. Moreover, there exists an Archimedean $ps$-ring $S$ with strictly increasing chain $ann(S) \subset ann(S^2) \subset \cdots$.

**Example 2.3.** Let $R$ be the ring from Example 2.2. For each $n \in \mathbb{N}$ denote by $R_n$ the additive subgroup of $R$ generated by elements $e_k$ with $1 \leq k \leq 2^n - 1$, and observe that $R_n$ is a subring of $R$, since

$$0 \leq i, j \leq 2^n - 1 \quad \text{implies} \quad \varphi(i,j) \leq 2^n - 1.$$

Let $S$ be the direct product of all $R_n$ with componentwise defined operations. Then $S$ is an Archimedean $ps$-ring with the chain of annihilators

$$ann(S) \subset ann(S^2) \subset ann(S^3) \subset \cdots$$

strictly increasing and contained in $N_2(S) = S$.

It may be asked (see Corollary 1.11) if a unital Archimedean $ps$-ring is automatically 2-torsion-free, and therefore reduced. The following example shows that this is not the case.

**Example 2.4.** Denote by $J$ the principal ideal of $\mathbb{Z}_4$ generated by 2, and consider its unitization ring $R = J \oplus \mathbb{Z}$ ordered by the positive cone $R^+ = \{0\} \oplus \mathbb{Z}^+$.

It is easy to see that $R$ is a unital Archimedean $ps$-ring satisfying

$$T_2(R) = N(R) = J \oplus \{0\},$$

thus $R$ is not 2-torsion-free.

## 3. An application

We shall need a result on quotient ring $R/N(R)$.

**Lemma 3.1.** *Let $R$ be an Archimedean $ps$-ring. Then $R/N(R)$ is a reduced Archimedean $ps$-ring.*

Proof: Since by Theorem 1.5 $N(R)$ is an order-convex ideal of $R$, the quotient ring $R/N(R)$ is partially ordered by the positive cone $\pi(R^+)$, where $\pi$ denotes the canonical projection of $R$ onto $R/N(R)$. A simple verification shows that $R/N(R)$ is a reduced $ps$-ring.

The proof will be complete by proving that $R/N(R)$ is Archimedean. To this end suppose that elements $\pi(a), \pi(b) \in R/N(R)$ satisfy $0 \leq \pi(b) + n\pi(a)$ for all $n \in \mathbb{Z}$. Then there exist elements $a_n \in N(R)$, $n \in \mathbb{Z}$ such that

$$0 \leq b + na + a_n.$$

Multiplying this inequality by $2a^4$ and observing that $2a^4 a_n = 0$ we get

$$0 \leq 2a^4 b + n(2a^5)$$

for all $n \in \mathbb{Z}$. Since $R$ is Archimedean this implies $2a^5 = 0$. It follows that $a \in N(R)$, hence $\pi(a) = 0$, as required. □

Recall that a partially ordered ring $R$ is said to be an *$f$-ring* if it is lattice-ordered and if $a, b \in R$ with $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in R^+$. It is well known that an Archimedean $f$-ring is commutative. Let us apply our results on partially ordered rings which are closely related to $f$-rings.

We shall say that a partially ordered ring $R$ is *$f$-decomposable*, if every element $a \in R$ is expressed as $a = a_1 - a_2$ with $a_1, a_2 \in R^+$ and $a_1 a_2 = a_2 a_1 = 0$.

Observe that an Archimedean $f$-decomposable ring is a $ps$-ring with generating cone $R^+$, and therefore torsion-free by Remark 1.7.

**Theorem 3.2.** *Let $R$ be an Archimedean $f$-decomposable partially ordered ring. Then*

(1) *$R/N(R)$ is an Archimedean f-ring;*
(2) *all triples of elements of $R$ commute, that is*

$$a_1 a_2 a_3 = a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} \quad \text{for all } a_1, a_2, a_3 \in R,\ \sigma \in S_3.$$

Proof: The quotient ring $R/N(R)$ is reduced and $f$-decomposable, hence (1) follows by [6] and Lemma 3.1.

To prove (2) note that $R$ satisfies

$$N_2(R) = R^+ \cap N_2(R) - R^+ \cap N_2(R),$$

which by Lemma 1.1. (1) implies that $N_2(R) = ann(R)$. It follows by Corollary 1.6 that $N(R) = ann(R^2)$, hence

$$N(R) \cap R^2 \subset N_2(R) = ann(R).$$

The commutativity of the Archimedean $f$-ring $R/N(R)$ implies that $ab - ba \in N(R) \cap R^2 \subset ann(R)$, therefore (2) follows. □

Applying Proposition 1.9 we get

**Corollary 3.3.** *Let $R$ be an Archimedean $f$-decomposable partially ordered ring.*

(1) *If $ann(R) = \{0\}$, then $R$ is an Archimedean semiprime $f$-ring.*

(2) *If $R = R^2$, then $R$ is commutative.* □

Let $R$ be an Archimedean $f$-decomposable ring. It can be seen (using for example [1, Proposition 1.3]) that if $R$ is lattice-ordered then it is an almost $f$-ring, thus Corollary 3.3. (1) generalizes [1, Theorem 1.11 (ii)]. Moreover, in this case $R$ is commutative by [1, Theorem 2.15], and it might be interesting to know whether $R$ is commutative also in general case.

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