Comment.Math.Univ.Carolin. 35,2 (1994)311–324

How subadditive are subadditive capacities?

George L. O'Brien, Wim Vervaat[†]*

Abstract. Subadditivity of capacities is defined initially on the compact sets and need not extend to all sets. This paper explores to what extent subadditivity holds. It presents some incidental results that are valid for all subadditive capacities. The main result states that for all hull-additive capacities (a class that contains the strongly subadditive capacities) there is countable subadditivity on a class at least as large as the universally measurable sets (so larger than the analytic sets).

Keywords: capacities, subadditive capacities, sup measures, hull-additive capacities, vague and narrow topologies, lattice of capacities

Classification: Primary 28A12; Secondary 28C15

1. Capacities and their subclasses

Let E be a Hausdorff space, often assumed to have additional properties. By $\mathcal{P}(E)$, $\mathcal{F}(E)$, $\mathcal{G}(E)$, $\mathcal{K}(E)$ and $\mathcal{B}(E)$ we denote the families of all subsets of E, the closed sets, the open sets, the compact sets and the Borel sets. We omit the argument when there is no risk of confusion.

Capacities as considered here have been introduced and studied at length in O'Brien & Vervaat (1991, 1993), O'Brien (1993a), and with emphasis on sup measures in Vervaat (1988). So we confine ourselves here to a review of the necessary definitions and properties.

A *capacity* on E is a function $c\colon \mathcal{P}(E) \to [0,\infty]$ such that

(1.1 a) $\quad c(\emptyset) = 0,$

(1.1 b) $\quad c(A) = \sup_{K\in\mathcal{K}: K\subset A} c(K) \quad \text{for } A \in \mathcal{P} \quad \text{(inner regularity)},$

(1.1 c) $\quad c(K) = \inf_{G\in\mathcal{G}: G\supset K} c(G) \quad \text{for } K \in \mathcal{K} \quad \text{(outer regularity)}.$

These conditions imply

(1.1 d) $\quad c(A) \le c(B) \quad \text{if } A \subset B \text{ in } \mathcal{P},$

(1.1 e) $\quad c(K_n) \downarrow c(K) \quad \text{if } K_n \downarrow K \text{ in } \mathcal{K}.$

[†] I am sad to report that my good collaborator and friend Wim Vervaat died suddenly on February 1, 1994. The probability community has lost a fine colleague. – George O'Brien

* This research was supported by the Natural Sciences and Engineering Research Council of Canada and by a NATO grant for collaborative research. The second author acknowledges the hospitality of York University.

The family of all capacities is denoted by $\mathcal{C}(E)$ or $\mathcal{C}$.

By SA(E) or SA we denote the class of the *subadditive* capacities c, namely those satisfying

$$(1.2_{\mathrm{SA}}) \qquad c(K_1 \cup K_2) \le c(K_1) + c(K_2) \qquad \text{for } K_1, K_2 \in \mathcal{K}.$$

A capacity c is said to be *additive* (or 'modular') if

$$(1.2_{\mathrm{AD}}) \qquad c(K_1 \cup K_2) + c(K_1 \cap K_2) = c(K_1) + c(K_2) \qquad \text{for } K_1, K_2 \in \mathcal{K}$$

and to be *'maxitive'* or a *sup measure* if

$$(1.2_{\mathrm{SM}}) \qquad c(K_1 \cup K_2) = c(K_1) \vee c(K_2) \qquad \text{for } K_1, K_2 \in \mathcal{K}.$$

In previous papers, additivity was defined by the conjunction of subadditivity and (1.2_{AD}) restricted to disjoint K_1 and K_2. For Hausdorff E the two definitions are equivalent. The classes of all additive capacities and of all sup measures are denoted by AD and SM. Obviously, AD $\cup$ SM $\subset$ SA.

The restriction of an additive capacity to $\mathcal{B}$ turns out to be a countably additive measure (cf. Theorem 8.2). A sup measure turns out to be arbitrarily maxitive on all of $\mathcal{P}$, i.e. $c(\bigcup_\alpha A_\alpha) = \sup_\alpha c(A_\alpha)$ for arbitrary collections (A_α) in $\mathcal{P}$. Unfortunately, subadditivity does not extend so easily. Subadditive capacities need not be subadditive even on $\mathcal{B}$, as will be exhibited in Section 4. It is the major object of this paper to explore how far subadditivity goes. Sections 4 and 9 contain the results that motivated the authors to write this paper.

To characterize additive capacities, first note that Radon measures are nothing but restrictions to $\mathcal{B}$ of additive capacities that are finite on $\mathcal{K}$. Conversely, such capacities are extensions by (1.1b) of Radon measures. To handle the general case, we first define for any capacity c its *infinity support* $\operatorname{supp}_\infty c$ by

$$(1.3) \qquad \operatorname{supp}_\infty c := \{x \in E : c\{x\} = \infty\},$$

a closed set. Now all additive capacities are characterized by the property that their restrictions to $\mathcal{B}(E \setminus \operatorname{supp}_\infty c)$ are Radon measures. For a further discussion, see Section 8.

Sup measures are characterized by the identity $c(A) = \sup_{x \in A} c\{x\}$ for $A \in \mathcal{P}$. For general capacities the mapping $x \mapsto c\{x\}$ is an upper semicontinuous (usc) function from E into $[0, \infty]$, and for each usc $f : E \to [0, \infty]$ the set function $f^\vee$ defined by $f^\vee(A) := \sup_{x \in A} f(x)$ is a sup measure. So sup measures correspond to usc functions in a canonical way.

Two other subspaces of SA will be of importance for us. First, we say that a capacity c is *strongly subadditive* (or 'submodular') if

$$(1.2_{\mathrm{SSA}}) \qquad c(K_1 \cup K_2) + c(K_1 \cap K_2) \le c(K_1) + c(K_2) \qquad \text{for } K_1, K_2 \in \mathcal{K}.$$

The class of all such capacities is denoted by SSA. Obviously, $\mathrm{AD} \subset \mathrm{SSA} \subset \mathrm{SA}$. Strongly subadditive capacities, or rather their outer capacities (cf. Section 3), are also capacities in the sense of Choquet (1969) and Dellacherie & Meyer (1978, Sections 27–45), which is proved in Section 4 of Norberg & Vervaat (1989). Choquet capacities are defined by regularity conditions on all of $\mathcal{P}$. With this approach capacities behave more nicely, but their existence becomes more problematic. In practical terms, SSA constitutes the intersection of the capacities in the two approaches. We do not study SSA in this paper, but compare *our* results to what is known for SSA.

Second, we say that a capacity c is *hull-additive* if $c = \sup\{c' \in \mathrm{AD} : c' \leq c\}$ in a sense to be made precise in Section 7. The class of hull-additive capacities is indicated by HA. It will be studied thoroughly in this paper, and constitutes the class for which we will obtain the best general results about subadditivity. The class has been studied before by Anger & Lembcke (1985). From that paper and previous literature it follows that

$$\mathrm{AD} \subset \mathrm{SSA} \subset \mathrm{HA} \subset \mathrm{SA},$$

the inclusion $\mathrm{SSA} \subset \mathrm{HA}$ however under regularity conditions on E and for slightly different types of capacities. Under similar conditions Anger & Lembcke (1985) obtain another characterization for HA similar to those of the other classes: $c \in \mathrm{HA}$ iff

$$(1.2_{\mathrm{HA}}) \qquad n\, c(K) \leq \sum_{i=1}^{m} c(K_i) \qquad \text{for all instances of } n 1_K \leq \textstyle\sum_{i=1}^{m} 1_{K_i},$$

where $m, n \in \mathsf{N}$ and $K, K_1, K_2, \ldots \in \mathcal{K}$. In this paper we consider HA only as the $\vee$ semilattice generated by AD, from which it is obvious that $\mathrm{AD} \subset \mathrm{HA} \subset \mathrm{SA}$.

2. An example

In order to get an impression of the different subspaces of capacities we study the case $E = \{1, 2, 3\}$, the simplest for which all these subspaces are different. We restrict our considerations to capacities c such that $c(E) = 1$. They can be represented by the matrix

$$\begin{pmatrix} c\{2,3\} & c\{3,1\} & c\{1,2\} \\ c\{1\} & c\{2\} & c\{3\} \end{pmatrix},$$

as $c\{1, 2, 3\} = 1$ and $c(\emptyset) = 0$. In this representation, AD_1, SSA_1, HA_1 and SA_1 (the index refers to the restriction $c(E) = 1$) turn out to be simplices in $[0, 1]^6$. Here are their extreme points (with thanks to Bart Gerritse who obtained them for us by computer algebra).

Common to AD_1, SSA_1, HA_1 and SA_1:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} + 2 \text{ column permutations.}$$

This is all for AD_1.
Common to SSA_1, HA_1 and SA_1:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} + 2 \text{ column permutations}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}.$$

This is all for SSA_1.
Common to HA_1 and SA_1:

$$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix} + 2 \text{ column permutations.}$$

For HA_1 only (so these points are not extreme points of $\mathrm{SA}_1 \supset \mathrm{HA}_1$):

$$\begin{pmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{pmatrix}, \quad \begin{pmatrix} 2/3 & 2/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix} + 2 \text{ column permutations.}$$

For SA_1 only:

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}.$$

3. Outer capacities

We say that an increasing set function c is *subadditive* on a family $\mathcal{D} \subset \mathcal{P}(E)$ if $c(D_1 \cup D_2) \le c(D_1) + c(D_2)$ for $D_1, D_2 \in \mathcal{D}$. The extension to countably many sets is called countable subadditivity.

Even additive capacities need not be subadditive on all of $\mathcal{P}$. To see this, consider $E = [0,1]$ with c the inner Lebesgue measure on $\mathcal{P}[0,1]$. Let A be a subset of $[0,1]$ with inner Lebesgue measure 0 and outer Lebesgue measure 1. Then $c(E) = c(A \cup A^c) = 1 > 0 = c(A) + c(A^c)$. On the other hand, sup measures are arbitrarily subadditive on all of $\mathcal{P}$, since they are arbitrarily maxitive there.

Outer capacities behave more nicely with respect to subadditivity, so we introduce them here. If c is a capacity, then the associated *outer capacity* is the set function c^* defined by

$$c^*(A) := \inf_{G \in \mathcal{G} : G \supset A} c(G) \qquad \text{for } A \in \mathcal{P}.$$

It follows that $c \le c^*$, with equality on $\mathcal{G} \cup \mathcal{K}$.

3.1 Lemma. *If $c \in \mathrm{SA}$, then c^* (and hence also c) is arbitrarily subadditive on $\mathcal{G}$ and c^* is countably subadditive on $\mathcal{P}$.*

Proof: For the statement about $\mathcal{G}$, see the discussion around Lemma 1.3 in O'Brien & Vervaat (1991). For the other we must show
$c^*\left(\bigcup_{n=1}^\infty A_n\right) \le \sum_{n=1}^\infty c^*(A_n)$ for $A_n \in \mathcal{P}$. Only the case of a finite upper bound

needs a proof. In this case, take $\varepsilon > 0$ and then open $G_n \supset A_n$ such that $c(G_n) \le c^*(A_n) + \varepsilon 2^{-n}$. Then

$$\begin{aligned} c^*\big(\textstyle\bigcup_{n=1}^\infty A_n\big) \le c^*\big(\textstyle\bigcup_{n=1}^\infty G_n\big) &\le \sum_{n=1}^\infty c^*(G_n) \\ &\le \sum_{n=1}^\infty \big(c^*(A_n) + \varepsilon 2^{-n}\big) = \sum_{n=1}^\infty c^*(A_n) + \varepsilon, \end{aligned}$$

the second inequality by the first result of this lemma. □

From Lemma 3.1 it follows that c is countably subadditive on the class

$$\mathcal{I}_c := \{A \in \mathcal{P} \colon c(A) = c^*(A)\}. \tag{3.1}$$

Elements of $\mathcal{I}_c$ are called *capacitable* by c. This class has been studied thoroughly for strongly subadditive capacities in the literature on Choquet capacities (recall that strongly subadditive capacities are Choquet capacities). For such c all $\mathcal{K}$ analytic sets are capacitable, so certainly all sets of $\mathcal{B}$ in case E is locally compact with countable base (cf. Dellacherie & Meyer (1978, Section III.13)). However, $\mathcal{I}_c$ can be much smaller for $c \in \mathrm{SA} \setminus \mathrm{SSA}$. See Example 7.4 for an instance of a K_σ set in a compact E which is not capacitable by a hull-additive c.

In this paper we do not explore $\mathcal{I}_c$ any further, except for the following small extension beyond $\mathcal{G} \cup \mathcal{K}$.

3.2 Lemma. *Let $c \in \mathrm{SA}$ and $A_1, A_2, \ldots \in \mathcal{I}_c$ be such that $c(A_n) = 0$ for $n \in \mathsf{N}$. Then $c^*(\bigcup_{n=1}^\infty A_n) = 0$ so $\bigcup_{n=1}^\infty A_n \in \mathcal{I}_c$.*

Proof: We have $c(A_n) = c^*(A_n) = 0$. Apply Lemma 3.1. □

3.3 Corollary. *If $B \in \mathcal{K}_\sigma$ and $c(B) = 0$, then $c^*(B) = 0$.*

A drawback of outer capacities is that they behave less well under restriction to subspaces. Let $E_0 \subset E$ and $c \in \mathcal{C}(E)$. Then

$$c|_{\mathcal{P}(E_0)} \in \mathcal{C}(E_0) \tag{3.2a}$$

by (1.1b), the fact that $\mathcal{K}(E) \cap \mathcal{P}(E_0) = \mathcal{K}(E_0)$ and an easy check of (1.1c) with $\mathcal{G}$ replaced by its trace in E_0. In contrast,

$$\big(c|_{\mathcal{P}(E_0)}\big)^* \le c^*|_{\mathcal{P}(E_0)} \tag{3.2b}$$

because only $\mathcal{G}(E) \cap \mathcal{P}(E_0) \subset \mathcal{G}(E_0)$. Of course, there is equality in (3.2b) if E_0 is open.

4. Subadditivity results for all of SA

Let $c \in \mathrm{SA}$. Since $c = c^*$ on $\mathcal{G} \cup \mathcal{K}$, c is countably subadditive on $\mathcal{G} \cup \mathcal{K}$. By the following result we extend the countable subadditivity a bit further (but not the identity $c = c^*$).

4.1 Lemma. *Let $c \in$ SA and $A_1, A_2, \ldots \in \mathcal{P}$ such that for each n we have $c(A_n) = c^*(A_n)$ or $c(A_n \cap K) = c^*(A_n \cap K)$ for all $K \in \mathcal{K}$. Then $c\big(\bigcup_{n=1}^\infty A_n\big) \le \sum_{n=1}^\infty c(A_n)$.*

PROOF: Let N be the set of n for which $c(A_n) = c^*(A_n)$. By Lemma 3.1 it follows for all compact $K \subset \bigcup_{n=1}^\infty A_n$ that

$$\begin{aligned} c(K) = c\big(\bigcup\nolimits_{n=1}^\infty (A_n \cap K)\big) &\le c\big(\bigcup\nolimits_{n\in N} A_n \cup \bigcup\nolimits_{n\notin N}(A_n \cap K)\big) \\ &\le \sum_{n\in N} c(A_n) + \sum_{n\notin N} c(A_n \cap K) \le \sum_{n=1}^\infty c(A_n). \end{aligned}$$

Apply (1.1b). □

Closed sets A_n satisfy the hypotheses of Lemma 4.1, so subadditive capacities are countably subadditive on $\mathcal{I}_c \cup \mathcal{F}$. By relativization to subspaces we can also improve on this.

4.2 Lemma. *If $c \in$ SA, $A_1, A_2, \cdots \in \mathcal{P}$ and each A_n is closed or open in $\bigcup_{n=1}^\infty A_n$, then $c\big(\bigcup_{n=1}^\infty A_n\big) \le \sum_{n=1}^\infty c(A_n)$.*

By (1.1d) and trade-offs, more general cases can be reduced to that of Lemma 4.2. We demonstrate this for the union of two sets.

4.3 Lemma. *Let $c \in$ SA and $A, B \in \mathcal{P}$. If the closure of $A\backslash B$ in $A \cup B$ is contained in A, then $c(A \cup B) \le c(A) + c(B)$.*

PROOF: Let C be the closure indicated above. Then $(A \cup B)\backslash C = B\backslash C$ is open in $A \cup B$. By Lemma 4.2 we obtain

$$c(A \cup B) = c\big(C \cup (B\backslash C)\big) \le c(C) + c(B\backslash C) \le c(A) + c(B).$$

□

The following example shows that the previous results are close to the limit of what can be obtained for all of SA. Subadditivity need not even extend to G_δ or K_σ sets, nor to sets of the form $G \cup K$ or $G \cap K$. The example is due to Henk Holwerda. Stephen Watson independently found a more complicated example.

4.4 Example. Let E be the unit circle R/Z identified in notation with $[0,1)$. For $K \in \mathcal{K}$, let $c(K)$ be the least number of open intervals in E of length $\frac{1}{2}$ required to cover K. It is obvious that c satisfies (1.2_{SA}) and, after extension by (1.1b), also (1.1c). Now $E = [0, \frac{1}{2}) \cup [\frac{1}{2}, 1)$, $c(E) = 3$ and $c[0, \frac{1}{2}) = c[\frac{1}{2}, 1) = 1$.

5. Tight capacities

For several applications the regularity conditions for capacities fall short of our needs. Capacities as defined in Section 1 generalize Radon measures. If one thinks of finite measures instead, in particular probability measures, one would like to replace (1.1c,e) by

$$c(F) = \inf_{G \in \mathcal{G} : G \supset F} c(G) \qquad \text{for } F \in \mathcal{F} \tag{5.1c}$$

and

$$c(F_n) \downarrow c(F) \qquad \text{if } F_n \downarrow F \text{ in } \mathcal{F}. \tag{5.1e}$$

To attain this, we introduce the notion of tightness for capacities. A capacity c is said to be *tight* if for all $\varepsilon > 0$ there is a $K \in \mathcal{K}$ such that $\arctan c(A \cup K^c) \leq \arctan c(A) + \varepsilon$ for all $A \in \mathcal{P}$. In combination with $c(A) \leq c(A \cup K^c)$ this is equivalent to

$$\lim_{K\uparrow} c(A \cup K^c) = c(A) \quad \text{in } [0,\infty] \text{ uniformly over } A \in \mathcal{P}. \tag{5.2a}$$

Equivalent conditions are:

$$\lim_{K\uparrow} c(A \cap K) = c(A) \quad \text{in } [0,\infty] \text{ uniformly over } A \in \mathcal{P}, \tag{5.2b}$$

and in case $c \in \mathrm{SA}$:

$$\lim_{K\uparrow} c(K^c) = 0. \tag{5.2c}$$

5.1 Theorem. *If $c \in \mathcal{C}$ is tight, then c satisfies* (5.1c,e). *If E is Polish and $c \in \mathrm{SA}$ satisfies* (5.1c,e), *then c is tight.*

PROOF: See Lemma 3.5 and Theorem 3.11 in O'Brien & Vervaat (1991). The first statement is proved there for regular E, but (5.1c,e) can be derived directly for general (Hausdorff) E. $\square$

Note that (5.1e) is an additional condition here, while (1.1e) is a consequence of (1.1b,c).

We write $\mathcal{C}_t$ for the space of all tight capacities, and SA_t, AD_t, SSA_t, HA_t and SM_t for the intersections of the subspaces with $\mathcal{C}_t$. For $c \in \mathcal{C}_t$ we have that $\mathrm{supp}_\infty\, c$ is compact. Furthermore, $c \in \mathrm{AD}_t$ and $\mathrm{supp}_\infty\, c = \emptyset$ iff c restricted to $\mathcal{B}$ is a finite tight measure. And $c \in \mathrm{SM}_t$ iff $c \in \mathrm{SM}$ and $x \mapsto c\{x\}$ is *upper compact,* i.e. $\{x : c\{x\} \geq y\} \in \mathcal{K}$ for all $y > 0$ (but usually not for $y = 0$).

Obviously, $\mathcal{I}_c \supset \mathcal{G} \cup \mathcal{F}$ for tight c, and Corollary 3.3 improves to $B \in \mathcal{I}_c$ if $B \in \mathcal{F}_\sigma$ and $c(B) = 0$. For an example of $c \in \mathrm{SA} \setminus \mathrm{SA}_t$ and $F \in \mathcal{F} \setminus \mathcal{I}_c$, see Choquet (1953–54, Section 31).

Remark. In the transition from (1.1) to (5.1) we replaced $\mathcal{K}$ by $\mathcal{F}$ only where it made the hypotheses more restrictive, but not in (1.1b), which would lead to the *less* restrictive $c(A) = \sup_{F \in \mathcal{F} : F \subset A} c(F)$ for $A \in \mathcal{P}$.

6. The vague and narrow topologies

Consider the evaluations $\mathcal{C} \ni c \mapsto c(A) \in [0,\infty]$ for fixed $A \in \mathcal{P}$. The *vague* and *narrow* topologies are the coarsest such that these evaluations are lower semicontinuous (lsc) for $A \in \mathcal{G}$ and usc for $A \in \mathcal{K}$, $A \in \mathcal{F}$ respectively.

The space $\mathcal{C}(E)$ is vaguely compact for all E (even non-Hausdorff), but vaguely Hausdorff iff E is locally compact ('only if' in case E itself is Hausdorff). In the latter case, the spaces SA, AD, SSA, HA and SM are vaguely closed, hence vaguely compact. See O'Brien & Vervaat (1991) for SA, AD and SM (SSA can be handled similarly) and Holwerda & Vervaat (1993) for HA. If E is locally compact with countable base, then $\mathcal{C}$ has a countable base for its vague topology, so then all spaces mentioned are compact and metrizable. If E is second countable or metrizable, then $\mathcal{C}(E)$ is vaguely sequentially Hausdorff, i.e. sequences can converge to at most one limit. See O'Brien (1993a).

If E is regular, then $\mathcal{C}$ is narrowly Hausdorff, and SA, AD, SSA and SM are narrowly closed in $\mathcal{C}$. See the same references as in the previous paragraph; the proofs in O'Brien & Vervaat (1991) are formulated for the subcollections of tight capacities, but are also valid without this restriction. The space HA is narrowly closed if E is normal (Anger & Lembcke (1985)). We do not know the situation for regular E. If E is regular with a countable base, then SA_t with the narrow topology is separable and metrizable, moreover Polish if E is Polish (cf. O'Brien & Vervaat (1993)).

Relative narrow compactness is characterized by the following generalization of Prohorov's theorem for bounded measures (cf. Theorem 3.11 in O'Brien & Vervaat (1991) complemented by Theorem 3.5 in O'Brien (1993a)). We say that a subset $\Pi \subset \mathcal{C}$ is *equitight* if the uniformity in (5.2a) (equivalently (5.2b), and (5.2c) in case $\Pi \subset \mathrm{SA}$) is extended to $c \in \Pi$.

6.1 Theorem. (a) *Let E be regular and $\Pi \subset \mathrm{SA}$. If Π is equitight, then Π is narrowly relatively compact.*

(b) *Let E be Polish and $\Pi \subset \mathrm{SA}_t$. If Π is narrowly relatively compact, then Π is equitight.*

Anger & Lembcke (1985) proved that HA can be characterized by (1.2_{HA}) in case E is locally compact, and HA_t by (1.2_{HA}) for closed sets in case E is normal (not regular). Then closedness of HA and HA_t can be proved in the same way as for the other subspaces.

7. Capacities as a lattice

We introduce an order on $\mathcal{C}$ by defining $c_1 \le c_2$ iff $c_1(A) \le c_2(A)$ for all $A \in \mathcal{P}$, equivalently, for all $A \in \mathcal{K}$ or all $A \in \mathcal{G}$. We write $\downarrow c := \{d \in \mathcal{C} : d \le c\}$. Capacities provided with this order are studied at length in Holwerda & Vervaat (1993) (cf. also O'Brien (1993b)). Here are some basic results from this paper.

7.1 Theorem. *The partially ordered space $\mathcal{C}$ is a complete lattice, and*

$$\Big(\bigvee_\alpha c_\alpha\Big)(G) = \bigvee_\alpha \big(c_\alpha(G)\big) \qquad \text{for } G \in \mathcal{G},$$
$$\Big(\bigwedge_\alpha c_\alpha\Big)(K) = \bigwedge_\alpha \big(c_\alpha(K)\big) \qquad \text{for } K \in \mathcal{K}.$$

We now are able to define the collection HA of *hull-additive* capacities formally by

$$\mathrm{HA} := \{\sup \Pi \colon \Pi \subset \mathrm{AD}\}. \tag{7.1}$$

The formulae in Theorem 7.1 need not extend to all of $\mathcal{P}$. The following lemma characterizes when this happens. It is obvious by (1.1c), (1.1b) and Theorem 7.1.

7.2 Lemma. *Let $\Pi \subset \mathcal{C}$.*

(a) $\sup_{d\in\Pi} d(A) \le (\sup \Pi)(A)$ *for $A \in \mathcal{P}$.*

(b) *If c is a capacity such that $c(K) = \sup_{d\in\Pi} d(K)$, for $K \in \mathcal{K}$, then the same formula holds with K replaced by $A \in \mathcal{P}$, so $c = \sup \Pi$.*

Here is another improvement on Theorem 7.1. Similar results (for the nonsequential part) were discovered independently by Henk Holwerda, and appeared in a more abstract setting in Holwerda & Vervaat (1993).

7.3 Theorem. *Let $\Pi \subset \mathcal{C}$. If E is locally compact and Π is vaguely compact, or E is regular and Π is narrowly compact, or E is second countable or metrizable and Π is vaguely sequentially compact, then*

$$(\sup \Pi)(A) = \sup_{c\in\Pi} c(A) \qquad \text{for } A \in \mathcal{P}, \tag{7.2a}$$

$$(\sup \Pi)(K) = \max_{c\in\Pi} c(K) \qquad \text{for } K \in \mathcal{K}. \tag{7.2b}$$

Proof: By the compactness properties of Π and Lemma 7.2 we need only show that

$$\sup_{c\in\Pi} c(K) \ge (\sup \Pi)(K) \qquad \text{for all } K \in \mathcal{K}. \tag{7.3}$$

First consider the regular case. So suppose $\sup_{c\in\Pi} c(K) < x$. For each $c \in \Pi$ there exists $G_c \in \mathcal{G}$ such that $K \subset G_c$ and $c(G_c) < x$ by (1.1c), and then by regularity there exist $F_c \in \mathcal{F}$ and $H_c \in \mathcal{G}$ with $K \subset H_c \subset F_c \subset G_c$. The collection $\big\{\{d \in \mathcal{C} \colon d(F_c) < x\} \colon c \in \Pi\big\}$ is a narrowly open cover for Π. Letting H be the intersection of the H_c's corresponding to a finite subcover, we see that

$$(\sup \Pi)(K) \le (\sup \Pi)(H) = \sup_{c\in\Pi} c(H) \le x,$$

which proves the result. In the locally compact case we may take each F_c in $\mathcal{K}$, which makes the cover vaguely open.

In the last case there exists a decreasing sequence (G_m) in $\mathcal{G}$ such that $\bigcap_{m=1}^{\infty} G_m = K$ and every open G containing K contains some G_m. Suppose $(\sup \Pi)(K) > x$. Then $\sup_{c\in\Pi} c(G_m) = (\sup \Pi)(G_m) > x$ for all m, so there is a sequence (c_n) in Π with $c_n(G_m) > x$ for $n = m$ and hence for all $n \geq m$. By Lemma 2.1 in O'Brien (1993a) there is for each m a compact $K_m \subset G_m$ with $\liminf_{n\to\infty} c_n(K_m) \geq x$. By sequential compactness there is a $c \in \Pi$ with $c(K_m) \geq x$ for all m, so that $c(G_m) \geq x$ for all m, so that $c(K) \geq x$. This proves (7.3). $\square$

We now are able to present the example, announced in Section 3, of a K_σ set in a compact E which is not capacitable by a hull-additive c. It is essentially due to Fuglede (1971) via Anger & Lembcke (1985).

7.4 Example. Let $E = [0,1]^2$ and $c = \sup_{x\in[0,1]}(\iota_x \otimes \mathrm{Leb})$, where ι_x denotes the Dirac measure at x and 'Leb' Lebesgue measure on $[0,1]$. The collection behind the supremum is compact because the mapping $[0,1] \ni x \mapsto \iota_x \otimes \mathrm{Leb} \in \mathrm{AD}$ is continuous. So Theorem 7.3 applies. Let $B = \big([0,\frac{1}{2}] \times [\frac{1}{2},1]\big) \cup \big((\frac{1}{2},1] \times [0,\frac{1}{2}]\big)$. Then $c(B) = \frac{1}{2}$ and $c^*(B) = 1$.

The following lemma will be needed to apply Theorem 7.3.

7.5 Lemma. (a) *The set $\downarrow c$ $(c \in \mathcal{C})$ is vaguely closed and compact. If E is locally compact, then also $\downarrow c \cap \mathrm{SA}$ and $\downarrow c \cap \mathrm{AD}$ are vaguely closed and compact.*

(b) *If E is regular and $c \in \mathrm{SA}_t$, then $\downarrow c \cap \mathrm{SA}$ and $\downarrow c \cap \mathrm{AD}$ are narrowly closed and compact.*

(c) *If E is second countable, then $\downarrow c \cap \mathrm{SA}$ and $\downarrow c \cap \mathrm{AD}$ are vaguely sequentially closed* (*all vague limits of sequences in the sets are themselves in the sets*) *and vaguely sequentially compact.*

PROOF: (a) $\downarrow c = \bigcap_{G\in\mathcal{G}} \{d : d(G) \leq c(G)\}$ is vaguely closed, and so are SA and AD in case E is locally compact. These sets are vaguely compact as well because the whole space $\mathcal{C}$ is vaguely compact.

(b) If E is regular, then SA and AD are narrowly closed. Furthermore, $\downarrow c$ is vaguely closed, hence narrowly closed, so $\downarrow c \cap \mathrm{SA}$ is narrowly closed. If $c \in \mathrm{SA}_t$, then $\downarrow c \cap \mathrm{SA}$ is equitight as (5.2c) holds uniformly, so $\downarrow c \cap \mathrm{SA}$ is narrowly relatively compact by Theorem 5.1.

(c) By (a), $\downarrow c$ is (vaguely) closed and hence sequentially closed. The assertions now follow from the results in O'Brien (1993a) that $\mathcal{C}$ is sequentially compact and that SA and AD are sequentially closed in $\mathcal{C}$. $\square$

The argument to prove Lemma 7.5 (c) cannot be used in the case where E is metrizable but not second countable, since $\mathcal{C}$ need not be sequentially compact in that case. We can however use a similar property to prove (7.2a,b) directly for metrizable E and $\Pi = \downarrow c \cap \mathrm{AD}$.

7.6 Lemma. *Let E be metrizable and let $\Pi := \downarrow c \cap \mathrm{AD}$, where $c \in \mathrm{SA}$. Then* (7.2a) *and* (7.2b) *hold.*

Proof: Let $K \in \mathcal{K}$. If $c(K) = \infty$, then $c(\{x_0\}) = \infty$ for some $x_0 \in K$, so we may define $d \in \Pi$ by $d(A) = \infty$ if $x_0 \in A$, 0 otherwise. Then $(\sup \Pi)(K) = \infty = d(K)$.

Now suppose $c(K) < \infty$. Choose an open $G \supset K$ with $c(G) < \infty$ and let $\Pi_G := \{d \in \downarrow c \cap \mathrm{AD} : d(G^c) = 0\}$. Note that $(\sup \Pi_G)(A) = (\sup \Pi)(A)$ for any open $A \subset G$ and hence for $A = K$. Now every $d \in \Pi_G$ is a finite measure and, by (1.1b), must be tight. By results of O'Brien (1993a), every sequence in Π_G has a vaguely convergent subsequence, whose limit must be in AD. By Lemma 7.5 (a), the limit must also be in $\downarrow c$, so Π_G is vaguely sequentially compact. By Theorem 7.3, $(\sup \Pi)(K) = (\sup \Pi_G)(K) = d(K)$ for some $d \in \Pi_G \subset \Pi$. By Lemma 7.2 (a), we obtain (7.2). □

8. Additive capacities and measurability

The relation between additive capacities and Radon measures was already sketched in the paragraph around (1.3). In general, additive capacities need not be σ-finite, even when restricted to the complement of their infinity supports. Sufficient conditions for the latter are that E is locally compact with countable base or that the capacity is tight.

Because of the lack of σ-finiteness we can define measurability with respect to an additive capacity only by a Carathéodory-type criterion locally in compact sets.

8.1 Definition. *Let $c \in \mathrm{AD}$. A set $A \in \mathcal{P}$ is c* ***measurable*** *if $c(K) = c(K \cap A) + c(K \cap A^c)$ for all $K \in \mathcal{K}$ such that $c(K) < \infty$. The collection of all c measurable sets is denoted by $\mathcal{A}$.*

8.2 Theorem. *Let $c \in \mathrm{AD}$. Then $\mathcal{A}$ is a σ-field that contains $\mathcal{B}$ and $\mathcal{P}(\mathrm{supp}_\infty c)$, and c restricted to $\mathcal{A}$ is a countably additive measure.*

Proof: It is obvious that $\mathcal{P}(\mathrm{supp}_\infty c) \subset \mathcal{A}$ and that the restriction of c to it is countably additive, as c assigns the value ∞ to each nonempty subset of $\mathrm{supp}_\infty c$. It remains to consider c restricted to the subsets of $(\mathrm{supp}_\infty c)^c$, where it is finite-valued on all compact sets. In this context the theorem has been proved by Berg, Christensen & Ressel (1984, Th.2.1.4), even for a slightly wider class of Radon premeasures. □

8.3 Remarks. For an alternative proof that $c \in \mathrm{AD}$ restricted to $\mathcal{B}$ is a Radon measure, see Norberg & Vervaat (1989, Th.3.7). In fact, it is the purpose of that paper to establish this result more generally for non-Hausdorff E.

In addition to the results of Section 4 for $c \in \mathrm{SA}$ we have for $c \in \mathrm{AD}$ that $c(B) = c^*(B)$ in case $B \in \mathcal{B}$ and $c^*(B) < \infty$. However, c^* restricted to $\mathcal{B}$ is another extension of c on $\mathcal{K}$ to a countably additive measure on $\mathcal{B}$, which need not be Radon. See Berg, Christensen & Ressel (1984, pp. 61–64) for a discussion of these and related results.

In the particular case that c is σ-finite on the complement of its infinity support we can characterize $\mathcal{A}$ alternatively as consisting of those sets that differ from a Borel set by the union of a c^* null set and a subset of $\operatorname{supp}_\infty c$.

Now let Π be any subset of AD. We say that a set $A \in \mathcal{P}$ is Π *measurable* if it is measurable with respect to each $\mu \in \Pi$, and *universally measurable* if it is AD measurable. It is known that the universally measurable sets contain the analytic sets, at least in case E is locally compact with countable base or E is regular with countable base.

9. Subadditivity of hull-additive capacities

The following is a basic ingredient.

9.1 Lemma. *If* $\Pi \subset \mathrm{AD}$ *and* c *is a capacity such that* $c(K) = \sup_{\mu\in\Pi} \mu(K)$ *for all* $K \in \mathcal{K}$*, then* c *is countably subadditive on the* Π *measurable sets.*

PROOF: By Lemma 7.2 (b) the formula for $c(K)$ extends to all of $\mathcal{P}$. So we have for Π measurable $A_1, A_2, \ldots$:

$$c\Big(\bigcup\nolimits_{n=1}^{\infty} A_n\Big) = \sup_{\mu\in\Pi} \mu\Big(\bigcup\nolimits_{n=1}^{\infty} A_n\Big) \le \sup_{\mu\in\Pi} \sum_{n=1}^{\infty} \mu(A_n) \le \sum_{n=1}^{\infty} c(A_n).$$

□

We now present our main result.

9.2 Theorem. *Let* $c \in \mathrm{HA}$*. Suppose that* E *is locally compact, second countable or metrizable, or that* c *is tight and* E *is regular.*

(a) *There exists* $\Pi \subset {\downarrow}c \cap \mathrm{AD}$ *such that for each* $K \in \mathcal{K}$ *there exists* $\mu \in \Pi$ *with* $\mu(K) = c(K)$ *and* $\mu(K^c) = 0$.

(b) *For any such* Π, c *is countably subadditive on the* Π *measurable sets.*

PROOF: By Theorem 7.3 and Lemma 7.5 or Lemma 7.6 there is for each $K \in \mathcal{K}$ a ν in ${\downarrow}c \cap \mathrm{AD}$ with $\nu(K) = c(K)$, so that also $\mu(K) = c(K)$, where $\mu := 1_K \nu \in {\downarrow}c \cap \mathrm{AD}$. This gives (a). By Lemma 9.1 we get (b). □

A very wasteful but general conclusion is that under the conditions of Theorem 9.2 any hull additive capacity is subadditive on the universally (= AD) measurable sets, which is already more than the analytic sets. However, the theorem admits much sharper conclusions in specific cases. As a test we present two examples for which we know the conclusion beforehand by other means.

9.3 Examples. Assume that the conditions of Theorem 9.2 hold.

(a) Let $c \in \mathrm{SM}$. Then Π consisting of all measures $c\{x\}\iota_x$ for $x \in E$ ($\iota_x :=$ Dirac measure at x) satisfies the conditions of Theorem 9.2 (a). Now all of $\mathcal{P}$ is Π measurable, so c is countably subadditive on $\mathcal{P}$.

(b) Let $c \in \mathrm{AD}$. Then Π consisting of all measures $c1_K$ for $K \in \mathcal{K}$ satisfies the conditions of Theorem 9.2 (a). The Π measurable sets coincide with the c measurable sets, and on them c is indeed countably subadditive.

Without proof we state the following result, whose conclusion may well be sharper than that of Theorem 9.2 (b), but is harder to apply (try it on Example 9.3).

Capacities can be multiplied by a nonnegative scalar and added, so that $\mathcal{C}$ is a *cone.* The subset $\downarrow c \cap \mathrm{AD}$ is convex, and we define $\mathcal{E}_c$ to be the set of its extreme points.

9.4 Theorem. *Let $c \in \mathrm{HA}$. If E is locally compact or if E is regular and c is tight, then c is countably subadditive on the $\mathcal{E}_c$ measurable sets.*

For the more special case $c \in \mathrm{SSA}$, $\mathcal{E}_c$ has been studied by El Kaabouchi (1991).

10. Remarks

10.1 Remark. Let $\mathrm{HA}_{(1.2)}$ be the set of capacities that satisfy (1.2_{HA}), and let $\mathrm{HA}_{(7.1)}$ be defined by (7.1). Here is a guide for proving equality of the two, under restrictive regularity conditions.

First, let $c \in \mathrm{HA}_{(7.1)}$. Then, by Lemma 7.5 and Theorem 7.3, and because of $\mathrm{AD} \subset \mathrm{HA}_{(1.2)}$ we have for m, n, K, K_1, $K_2, \ldots$ as indicated below (1.2_{HA}):

$$n\, c(K) = \sup_{\mu \in \downarrow c \cap \mathrm{AD}} n\, \mu(K) \leq \sup_{\mu \in \downarrow c \cap \mathrm{AD}} \sum_{i=1}^{m} \mu(K_i) \leq \sum_{i=1}^{m} c(K_i),$$

which proves $\mathrm{HA}_{(7.1)} \subset \mathrm{HA}_{(1.2)}$, without further restrictions.

Second, Anger & Lembcke (1985) prove that the condition of Lemma 7.2 (b) holds with $\Pi = \downarrow c \cap \mathrm{AD}$ in case $c \in \mathrm{HA}_{(1.2)}$ and E is locally compact. So $\mathrm{HA}_{(1.2)} \subset \mathrm{HA}_{(7.1)}$ for such E.

10.2 Remark. In this remark we write ['94] for the present paper and ['91] for O'Brien & Vervaat (1991).

In ['91] subadditivity was occasionally extended a bit too freely beyond $\mathcal{G} \cup \mathcal{F}$ (cf. ['91], Section 4). The results of ['94], Section 4 serve to fill in all gaps of this type, except for one in Theorem 4.5 (c,d) of ['91], which is, however, correct under the additional assumption that $E_0 \in \mathcal{K}_\sigma$, by Corollary 3.3 ['94]. If in addition the capacities are required to be tight, then ${E_0}^c \in \mathcal{F}_\sigma$ suffices, by the third paragraph after Theorem 5.1 ['94].

Theorem 4.5 ['91] was used only at one later part of ['91], namely in the proof of Lemma 5.4. Fortunately, the amended version of Theorem 4.5 (c,d) ['91] is still adequate for this purpose.

Acknowledgments. The paper benefitted from remarks and contributions by Bart Gerritse, Henk Holwerda and Stephen Watson.

References

[1] Anger B., Lembcke J. (1985), *Infinitely subadditive capacities as upper envelopes of measures*, Z. Wahrsch. theor. verw. Gebiete **68**, 403–414.

[2] Berg C., Christensen J.P.R., Ressel P. (1984), *Harmonic Analysis on Semigroups*, Springer.

[3] Choquet G. (1953–54), *Theory of capacities*, Ann. Inst. Fourier **5**, 131–295.

[4] ——— (1969), *Lectures on Analysis*, Benjamin.

[5] Dellacherie C., Meyer P.A. (1978), *Probabilities and Potential*, Vol. 1, North-Holland.

[6] El Kaabouchi A. (1991), *Points extrémaux du convexe des mesures majorées par une capacité*, C.R. Acad. Sci. Paris **313**, 37–40.

[7] Fuglede B. (1971), *Capacity as a sublinear functional generalizing an integral*, Danske Vid. Selsk. Mat.-Fys. Medd. **38** 7.

[8] Holwerda H., Vervaat W. (1993), *Order and topology in spaces of capacities*, preprint. In: Holwerda H., *Topology and Order*, Ph.D. Thesis, Cath. Univ. Nijmegen, 1993.

[9] Norberg T., Vervaat W. (1989), *Capacities on non-Hausdorff spaces*, Report **1989-11**, Dept. Math., Chalmers Univ. Techn. and Univ. Göteborg. To appear in [15].

[10] O'Brien G.L. (1993a), *Sequences of capacities, with connections to large deviation theory*, Report Dept. Math. Stat. York Univ., to appear.

[11] ——— (1993b), *One-sided limits of capacities*, Report Dept. Math. Stat. York Univ., to appear.

[12] O'Brien G.L., Vervaat W. (1991), *Capacities, large deviations and loglog laws*, in Stable Processes (eds. S. Cambanis, G. Samorodnitsky & M.S. Taqqu), pp. 43–83, Birkhäuser, Boston.

[13] ——— (1993), *Capacities and large deviations: an improved toolkit*, in preparation.

[14] Vervaat W. (1988), *Random upper semicontinuous functions and extremal processes*, Report **MS-8801**, Center for Mathematics and Computer Science, Amsterdam. To appear in [15].

[15] Vervaat W. (editor) (1994), *Probability and Lattices*, CWI Tracts, Center for Math and Comp. Sci., Amsterdam, to appear.

Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada M3J 1P3

E-mail: obrien@clid.yorku.ca

(*Received* November 22, 1993)