# Boundary value problems and periodic solutions for semilinear evolution inclusions

Nikolaos S. Papageorgiou

*Abstract.* We consider boundary value problems for semilinear evolution inclusions. We establish the existence of extremal solutions. Using that result, we show that the evolution inclusion has periodic extremal trajectories. These results are then applied to closed loop control systems. Finally, an example of a semilinear parabolic distributed parameter control system is worked out in detail.

*Keywords:* evolution operator, multifunction, Hausdorff metric, extremal solution, periodic solution, Fredholm alternative, control system, parabolic system

*Classification:* 34G20

## 1. Introduction

In this paper we consider the following boundary value problem for semilinear evolution inclusions, defined on $T = [0, b]$ with values in a separable Banach space $X$:

$$\left\{ \begin{array}{c} \dot{x}(t) \in A(t)x(t) + \text{ext } F(t, x(t)) \\ L(x) = w \end{array} \right\} . \tag{1}$$

Here $\{A(t)\}_{t\in T}$ is a family of generally unbounded, densely defined linear operators which generates an evolution operator $K(t, s) \in \mathcal{L}(X) = \{$space of bounded linear operators from $X$ into itself$\}$, $F(t, x)$ is an orientor field (i.e. a set-valued vector field) and ext $F(t, x)$ denotes the set of extreme points of $F(t, x)$ and finally $L : C(T, X) \to X$ is a continuous, linear operator. By a solution of (1), we understand a mild solution; i.e. a function $x(\cdot) \in C(T, X)$ such that $x(t) = K(t, 0)x(0) + \int_0^t K(t, s)f(s)\, ds$ for all $t \in T$ and with $f \in L^1(T, X) = L^1(X)$, $f(t) \in \text{ext } F(t, x(t))$ a.e., while $L(x) = w$.

From an abstract viewpoint, our study here is a direct attempt to extend the work done in [11]. We remark that the theory developed in [11] (in particular Theorem 2, the "nonconvex" existence result) can no longer be applied in (1) since the multifunction $(t, x) \to \text{ext } F(t, x)$ is not necessarily closed valued and, in general, we cannot say anything about its continuity properties. At the same time, we generalize the earlier works of Anichini [1], Kartsatos [5] and Zecca-Zezza [17]. From the above works, Anichini [1] considered single-valued boundary value problems in $\mathbb{R}^n$ over a compact interval. Kartsatos [5] considered semilinear single-valued boundary value problems in $\mathbb{R}^n$ over an unbounded interval, while Zecca-Zezza [17] extended the work of Kartsatos to differential inclusions defined in a separable Banach space. However, their formulation does not allow for the presence of unbounded operators and therefore precludes the possibility of applying their work to partial differential inclusions (distributed parameter systems).

The plan of the paper is as follows. In Section 2, we present some background material on multifunctions that will be needed in the sequel. Section 3 is concerned with the existence of mild solutions for problem (1). In Section 4, we use the existence result of Section 3, to establish the existence of periodic extremal (bang-bang) trajectories. In Section 5, we show how our work can be used on control systems and finally in Section 6, we work out in detail an example of a parabolic partial differential inclusion.

## 2. Preliminaries

Throughout this paper, $T$ denotes the interval $[0, b]$ (endowed with the Lebesgue measure $\lambda(\cdot)$) and $X$ stands for a separable Banach space with norm $\|\cdot\|$ and whose dual $X^*$ has the Radon-Nikodym Property (RNP). The following notation will also be used: $P_{f(c)}(X) = \{B \subseteq X : B \text{ is nonempty, closed (and convex)}\}$, $P_{(w)k(c)} = \{B \subseteq X : B \text{ is nonempty, (weakly-) compact (and convex)}\}$. A multifunction (set-valued function) $F : T \to P_f(X)$ is said to be measurable, if for every $x \in X$, the $\mathbb{R}_+$-valued function $t \to d(x, F(t)) = \inf\{\|x - z\| : z \in F(t)\}$ is measurable. By $S_F^p$ $(1 \leq p \leq \infty)$ we will denote the set of selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^p(T, X) = L^p(X)$, namely $S_F^p = \{f \in L^p(X) : f(t) \in F(t) \text{ a.e.}\}$. This set may be empty. It is easy to check that for a measurable $F(\cdot)$, $S_F^p$ is nonempty if and only if $t \to \inf\{\|x\| : x \in F(t)\} \in L_+^p$ with $L_+^p$ being the positive cone in the Lebesgue space $L^p(T, \mathbb{R})$. Furthermore $S_F^p$ is a decomposable subset of $L^p(X)$; i.e. if $f_1, f_2 \in S_F^p$ and $B$ is a Borel subset of $T$, then $\chi_B f_1 + \chi_{B^c} f_2 \in S_F^p$, where $\chi_B(\cdot)$ stands for the characteristic function of $B$.

On $P_f(X)$, we can define a generalized metric, known in the literature as the Hausdorff metric, by

$$h(B, C) = \max\left[\sup_{b \in B} d(b, C),\ \sup_{c \in C} d(c, B)\right]$$

for all $B, C \in P_f(X)$. The space $(P_f(X), h)$ is complete. A multifunction $G : X \to P_f(X)$ is said to be Hausdorff continuous ($h$-continuous), if it is continuous from $X$ into the metric space $(P_f(X), h)$.

Let $Y$ be a complete, separable metric space (i.e. a Polish space). A multifunction $F : T \times Y \to P_f(X)$ is said to satisfy the Scorza-Dragoni Property (S-D Property), if for every $\varepsilon > 0$, there exists $T_\varepsilon \subseteq T$ closed, with $\lambda(T \setminus T_\varepsilon) < \varepsilon$ such that $F(\cdot, \cdot)$ restricted on $T_\varepsilon \times Y$ is $h$-continuous. Conversely, if $F : T \times Y \to P_{kc}(X)$ is measurable in $t \in T$ and $h$-continuous in $y \in Y$ (i.e. $F(\cdot, \cdot)$ is a Carathéodory-type multifunction), then $F(\cdot, \cdot)$ satisfies the Scorza-Dragoni Property. This follows immediately from the classical Scorza-Dragoni theorem and the fact that $(P_{kc}(X), h)$

is a Polish space (i.e. is complete and separable). Recall that the classical Scorza-Dragoni theorem says that if $V$ is a separable metric space, $W$ a Polish space and $u : T \times V \to W$ is a Carathéodory function (i.e. measurable in $t \in T$, and continuous in $v \in V$ for almost all $t \in T$), then given $\varepsilon > 0$ we can find $T_\varepsilon \subseteq T$ closed with $\lambda(T \setminus T_\varepsilon) < \varepsilon$ such that $u(\cdot, \cdot)$ restricted on $T_\varepsilon \times V$ is continuous.

Let $\Delta = \{(t, s) \in T \times T : 0 \leq s \leq t \leq b\}$. Then $K : \Delta \to \mathcal{L}(X)$ is an evolution operator if

(i) $K(t, t) = I$ for all $t \in [0, b]$,
(ii) $K(t, r)K(r, s) = K(t, s)$ if $0 \leq s \leq r \leq t \leq b$

and

(iii) $K(\cdot, \cdot)$ is a strongly continuous function.

Note that (iii) implies that there exists $M > 0$ such that $\sup_{(t,s)\in\Delta} \|K(t, s)\|_{\mathcal{L}} \leq M$. Clearly if $S(t)$ is a strongly continues semigroup, then $S(t-s)$ is an evolution operator. For further details, we refer to Tanabe [14, Chapter 4].

## 3. Existence theorem

In this section we present an existence result for problem (1). To this end, we will need the following hypotheses on the data:

$\underline{H(A)}$: $\{A(t) : t \in T\}$ is a family of generally unbounded, densely defined linear operators that generates an evolution operator $K(t, s)$ which is compact for $t - s > 0$.

$\underline{H(F)}$: $F : T \times X \to P_{wkc}(X)$ is a multifunction s.t.

(1) $(t, x) \to F(t, x)$ satisfies the S-D Property,
(2) $\sup_{\|x\|\leq k} |F(t, x)| = \sup_{y\in F(t,x)} \|y\| \leq \varphi_k(t)$ a.e. with $\varphi_k(\cdot) \in L^p_+$ $1 < p < \infty$ and $\underline{\lim}_{k\to\infty} \frac{1}{k} \int_0^b \varphi_k(t)\, dt = \beta < \infty$.

$\underline{H(L)}$: $L : C(T, X) \to X$ is a linear, continuous operator.

Let $\widehat{L} \in \mathcal{L}(X)$ be defined by $\widehat{L}(v) = L(K(\cdot)v)$. We will assume that following about $\widehat{L}$:

$\underline{H_0}$: $\widehat{L}$ is a bijection (hence by Banach's theorem $\widehat{L}^{-1} \in \mathcal{L}(X)$).

$\underline{H_1}$: $(M\|\widehat{L}^{-1}\|_{\mathcal{L}}\|L\|_{\mathcal{L}} + 1)M\beta < 1$.

We will also need a simple lemma. Let $L^1_w(X)$ denote the space of equivalence classes of Bochner integrable functions $x : T \to X$, with the ("weak") norm $\|x\|_w = \sup\{\| \int_{t_1}^{t_2} x(s)\, ds\| : 0 \leq t_1 \leq t_2 \leq b\}$. The notation $\xrightarrow{\|\cdot\|_w}$ stands for convergence in $L^1_w(X)$.

**Lemma.** *$\underline{\text{If}}$ $\{f_n\}_{n\geq1} \subseteq L^p(X)$ $1 < p < \infty$, $\sup_{n\geq1} \|f_n\|_p < \infty$ and $f_n \xrightarrow{\|\cdot\|_w} 0$ as $n \to \infty$, $\underline{\text{then}}$ $f_n \to 0$ weakly in $L^p(X)$.*

Proof: Since by hypothesis $X^*$ has the RNP (see Section 2), from Theorem 1, p. 98 of Diestel-Uhl [3], $L^p(X)^* = L^q(X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let $((\cdot, \cdot))$ denote the duality pairing between $L^p(X)$ and $L^q(X^*)$. Since $\{f_n\}_{n\geq1}$ is bounded in $L^p(X)$ and the space of $X^*$-valued step functions on $T$ is dense in $L^q(X^*)$, we only need to show that $((f_n, s)) \to 0$ as $n \to \infty$ for each $s : T \to X^*$ of the form $s(t) = \sum_{k=1}^N \chi_{(t_{k-1},t_k)}(t)v_k^*$, with $v_k^* \in X^*$. We have $|((f_n, s))| = |\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (f_n(s), v_k^*)\, ds| \leq \sum_{k=1}^N \| \int_{t_{k-1}}^{t_k} f_n(s)\, ds\| \cdot \|v_k^*\|_*$, where $\|\cdot\|_*$ denotes the norm of $X^*$. It follows that $|((f_n, s))| \leq \|f_n\|_w \sum_{k=1}^N \|v_k^*\|_* \to 0$ as $n \to \infty$ and so the proof is complete. □

Now we are ready to state and prove our existence theorem concerning problem (1).

**Theorem 1.** *If hypotheses $H(A)$, $H(F)$, $H(L)$, $H_0$ and $H_1$ hold, then problem* (1) *admits a solution.*

Proof: Let $E, H : C(T,X) \to 2^{C(T,X)}$ be the multifunctions defined by

$$E(x) = \{y \in C(T,X) : y(t) = \\ = \int_0^t K(t,s)f(s)\, ds,\ t \in T,\ f \in L^p(X),\ f(t) \in F(t,x(t)) \text{ a.e.}\}$$

and $H(x) = \{y \in C(T,X) : y(t) = K(t,x)\widehat{L}^{-1}[w - L(z)] + z(t), z \in E(x)\}$.

Clearly both $E(\cdot)$ and $H(\cdot)$ have nonempty, closed and convex values (just note that for every $x(\cdot) \in C(T,X)$, $S^p_{F(\cdot,x(\cdot))} \in P_{fc}(L^p(X))$; cf. hypothesis $H(F)$). We claim that there exists a positive integer $m \geq 1$ s.t. $H(B_m) \subseteq B_m$, where $B_m = \{z \in C(T,X) : \|z\|_\infty \leq m\}$. Suppose not. Then for every $n \geq 1$, we can find $x_n \in B_n$ such that $|H(x_n)| = \sup\{\|z\|_\infty : z \in H(x_n)\} > n$. Hence we have:

$$1 < \frac{|H(x_n)|}{n} \leq \\ \leq \frac{M\|\widehat{L}^{-1}\|_{\mathcal{L}} \cdot \|w\|}{n} + \sup_{z \in E(x_n)} \frac{M \cdot \|\widehat{L}^{-1}\|_{\mathcal{L}}\|L\|_{\mathcal{L}}\|z\|_\infty}{n} + \sup_{z \in E(x_n)} \frac{\|z\|_\infty}{n} \tag{2}$$

From the definition of the multifunction $E(\cdot)$, we know that if $z \in E(x_n)$, then

$$z(t) = \int_0^t K(t,s)f(s)\, ds,\ t \in T$$

$$\Longrightarrow \|z(t)\| \leq \int_0^t M\|f(s)\|\, ds$$

$$\Longrightarrow \|z\|_\infty \leq M \int_0^b \varphi_n(s)\, ds$$

$$\Longrightarrow \frac{\|z\|_\infty}{n} \leq \frac{M}{n} \int_0^b \varphi_n(s)\, ds$$

$$\Longrightarrow \sup_{z \in E(x_n)} \frac{\|z\|_\infty}{n} \leq \frac{M}{n} \int_0^b \varphi_n(s)\, ds$$

$$\Longrightarrow \underline{\lim}_{n\to\infty} \sup_{z \in E(x_n)} \frac{\|z\|_\infty}{n} \leq M\beta.$$

Therefore by taking $\underline{\lim}$ of both sides in (2), we get

$$1 \leq (M \cdot \|\widehat{L}^{-1}\|_{\mathcal{L}} \|L\|_{\mathcal{L}} + 1) M\beta < 1 \quad \text{(cf. hypothesis } H_1)$$

a contradiction. So indeed there exists $m \geq 1$ s.t. $H(B_m) \subseteq B_m$.

Let $\Gamma \subseteq C(T, X)$ be defined by

$$\Gamma = \{y \in C(T, X) : y(t) = \int_0^t K(t,s) g(s)\, ds,\ t \in T,\ \|g(t)\| \leq \varphi_m(t) \text{ a.e.}\}$$

Clearly this is a nonempty, closed and convex subset of $C(T, X)$. We claim that it is also compact. Since by hypothesis $H(A)$, $K(t,s)$ is compact for $t - s > 0$, from Proposition 2.1 of [10], we know that given $\varepsilon > 0$, we can find $\delta_1 > 0$ such that if $t < t'$, $t' - t < \delta_1$, then

$$\|K(t', s) - K(t,s)\|_{\mathcal{L}} < \frac{\varepsilon}{3\|\varphi_m\|_1}$$

for all $s \in [0, t - \delta]$, $0 < \delta < t$. Also, we can find $\delta_2 > 0$ such that if $\lambda(C) < \delta_2$, then

$$2M \int_C \varphi_m(s)\, ds < \frac{\varepsilon}{3}$$

(here, as before, $\lambda(\cdot)$ stands for the Lebesgue measure on $T$). Let $\delta_3 = \min[\delta_1, \delta_2]$. Let $0 < \delta < \delta_3$, let $y \in \Gamma$ and let $0 < t < t'$, $t' - t < \delta_3$. We have:

$$\begin{aligned} &\|y(t') - y(t)\| \\ &\leq \|\int_t^{t'} K(t', s) g(s)\, ds\| + \|\int_{t-\delta}^{t} (K(t', s) - K(t,s)) g(s)\, ds\| \\ &\quad + \|\int_0^{t-\delta} (K(t', s) - K(t,s)) g(s)\, ds\| \\ &\leq M \int_t^{t'} \varphi_m(s)\, ds + 2M \int_{t-\delta}^{t} \varphi_m(s)\, ds + \int_0^{t-\delta} \frac{\varepsilon}{3\|\varphi_m\|_1} \varphi_m(s)\, ds \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So $\Gamma$ is equicontinuous on $[0, b]$. For $t = 0$, using the strong continuity of the evolution operator $K(\cdot, 0)$ and the absolute continuity of the indefinite Lebesgue integral, we can easily see that $\Gamma$ is equicontinuous at $t = 0$ too. Therefore $\Gamma$ is an equicontinuous subset of $C(T, X)$.

Also let $t \in T$. Then

$$\Gamma(t) = \{y(t) : y \in \Gamma\} \subseteq \int_0^t K(t,s) V(s)\, ds$$

with $V(s) = \{x \in X : \|x\| \leq \varphi_m(s)\}$. Since the evolution operator $K(t,s)$ is compact for $t - s > 0$, $K(t,s)V(s) \in P_{kc}(X)$ for almost all $s \in [0,t]$. Furthermore, since $\varphi_m(\cdot)$ is measurable $\Rightarrow V(\cdot)$ is measurable and so from the Radström embedding theorem (see Klein-Thompson [6]), we get that $\int_0^t K(t,s)V(s)\,ds \in P_{kc}(X)$ for all $t \in T$. Thus we have that $\overline{\Gamma(t)} \in P_{kc}(X)$ for all $t \in T$ and so from the Arzela-Ascoli theorem we conclude that $\Gamma \subseteq C(T,X)$ is compact.

Let $\Gamma_1 = \{y \in C(T,X) : y(t) = K(t,0)\widehat{L}^{-1}[w - L(z)] + z(t),\ z \in \Gamma\}$. Clearly then $\Gamma_1$ is a nonempty, compact and convex subset of $C(T,X)$. Let $R : \Gamma_1 \to P_{fc}(L^1(X))$ be defined by

$$R(x) = S^1_{F(\cdot,x(\cdot))}\,.$$

Using Theorem 4.5 of [9], we get that $R(\cdot)$ is $h$-continuous and also from Proposition 3.1 of [8], we know that it has values in $P_{wkc}(L^1(X))$. Apply Theorem 1.1 of Tolstonogov [15], to get $r : \Gamma_1 \to L^1_w(X)$ a continuous map s.t. $r(x) \in \text{ext } R(x)$ for all $x \in \Gamma_1$. From Benamara [2], we know that $\text{ext } R(x) = \text{ext } S^1_{F(\cdot,x(\cdot))} = S^1_{\text{ext } F(\cdot,x(\cdot))}$ for all $x \in \Gamma_1$. So $r(x) \in S^1_{\text{ext } F(\cdot,x(\cdot))}$ for all $x \in \Gamma_1$.

Let $\theta : \Gamma_1 \to \Gamma_1$ be defined by

$$\theta(x)(t) = K(t,0)\widehat{L}^{-1}[w - L(\eta(x))] + \eta(x)(t),\ t \in T$$

with $\eta(x)(t) = \int_0^t K(t,s)r(x)(s)\,ds$. Using the continuity of the selector $r(\cdot)$ and the lemma (recall that for every $k \geq 1$, $\varphi_k \in L^p_+$, $1 < p < \infty$), we can easily see that $\theta(\cdot)$ is continuous. Apply Schauder's fixed point theorem to get $x = \theta(x)$. Clearly then, $x(\cdot) \in C(T,X)$ is the desired of (1). □

## 4. Periodic extremal solutions

In this section we use Theorem 1 to establish the existence of extremal (bang-bang) solutions for the following periodic boundary value problem:

$$\left\{ \begin{matrix} \dot{x}(t) \in A(t)x(t) + \text{ext } F(t,x(t)) \\ x(0) = x(b) \end{matrix} \right\}. \tag{3}$$

We will need the following two hypotheses.

$\underline{H(F)_1}$: $F : T \times X \to P_{wkc}(X)$ is a multifunction s.t.

(1) $(t,x) \to F(t,x)$ has the S-D Property,
(2) $\sup_{\|x\| \leq k} |F(t,x)| = \sup_{\|y \in F(t,x)\|} \|y\| \leq \varphi_k(t)$ a.e., with $\varphi_k \in L^p_+$ $1 < p < \infty$, $\underline{\lim}_{k\to\infty} \frac{1}{k} \int_0^b \varphi_k(s)\,ds = 0$.

$\underline{H_2}$: $K(b,0)v$ if and only if $v = 0$.

**Theorem 2.** *$\underline{\text{If}}$ hypotheses $H(A)$, $H(F)_1$ and $H_2$ hold, $\underline{\text{then}}$ problem (3) admits a solution.*

Proof: In this case $L : C(T,X) \to X$ is defined by $L(x) = x(b) - x(0)$. Then $\widehat{L}(v) = K(b,0)\ v - v \Rightarrow \widehat{L} = K(b,0) - I$. From hypothesis $H(A)$, we know that $K(b,0)$ is compact. So from hypothesis $H_2$ and the Fredholm alternative theorem, we get that $\widehat{L}^{-1} \in \mathcal{L}(X)$. Thus we have satisfied hypothesis $H_0$. Apply Theorem 1 to conclude the existence of a periodic extremal trajectory. $\square$

## 5. Control systems

In this section, we consider semilinear feedback (closed loop) control systems and establish the existence of periodic trajectories generated by bang-bang controls (extremal or bang-bang trajectories). The control space is modelled by $Y$ a separable Banach space. The control system under consideration is the following:

$$\left\{ \begin{array}{c} \hat{x}(t) = A(t)x(t) + B(t,x(t))u(t) \\ x(0) = x(b) \\ u(t) \in U(t,x(t)) \text{ a.e.} \end{array} \right\}. \tag{4}$$

A trajectory $x(\cdot) \in C(T,X)$ is called extremal if it is generated by a control $u(t) \in \operatorname{ext} U(t,x(t))$ a.e. ("bang-bang" control).

We will need the following hypotheses:

$\underline{H(B)}$: $\quad B : T \times X \to \mathcal{L}(Y,X)$ is a map s.t.

(1) for every $\varepsilon > 0$, there exists $T_\varepsilon \subseteq T$ closed with $\lambda(T \setminus T_\varepsilon) < \varepsilon$ s.t. $B$ restricted on $T_\varepsilon \times X$ is continuous into $\mathcal{L}(Y,X)$ equipped with operator norm topology,

(2) $\sup_{\|x\| \le k} \|B(t,x)\|_{\mathcal{L}} \le \varphi_k(t)$ a.e. with $\underline{\lim}_{k\to 0} \frac{1}{k} \int_0^b \varphi_k(s)\,ds = 0$, with $\varphi_k(\cdot) \in L^p_+$, $1 < p < \infty$.

$\underline{H(U)}$: $\quad U : T \times X \to P_{wkc}(Y)$ is a multifunction s.t.

(1) $(t,x) \to U(t,x)$ has the S-D Property,

(2) $|U(t,x)| = \sup\{\|u\| : u \in U(t,x)\} \le \widehat{M}$ for all $(t,x) \in T \times X$.

**Theorem 3.** *$\underline{\text{If}}$ hypotheses $H(A)$, $H(B)$, $H(U)$ and $H_2$ hold, $\underline{\text{then}}$ control system* (4) *admits a periodic extremal trajectory.*

Proof: Let $F : T \times X \to P_{wkc}(X)$ be defined by

$$F(t,x) = B(t,x)U(t,x).$$

Given $\varepsilon > 0$, because of hypothesis $H(U)$ (1), we can find $T^1_\varepsilon \subseteq T$ closed with $\lambda(T \setminus T^1_\varepsilon) < \frac{\varepsilon}{2}$ such that $U \mid_{T^1_\varepsilon \times X}$ is $h$-continuous. Similarly because of hypothesis $H(B)$ (1), we can find $T^2_\varepsilon \subseteq T^1_\varepsilon$ closed with $\lambda(T^1_\varepsilon \setminus T^2_\varepsilon) < \frac{\varepsilon}{2}$ such that $B \mid_{T^2_\varepsilon \times X}$ is continuous. Then $\lambda(T \setminus T^2_\varepsilon) < \varepsilon$ and for $(t,x)$, $(t',x') \in T^2_\varepsilon \times X$ we have

$$\begin{aligned} h(F(t,x),F(t',x')) &= h(B(t,x)U(t,x), B(t',x')U(t',x')) \\ &\le h(B(t,x)U(t,x), B(t,x)U(t',x')) + h(B(t,x)U(t',x'), B(t',x')U(t',x')) \\ &\le \|B(t,x)\|_{\mathcal{L}} h(U(t,x),U(t',x')) + \widehat{M}\|B(t,x) - B(t',x')\|_{\mathcal{L}}\,. \end{aligned}$$

So we deduce that $F\mid_{T^2_\varepsilon\times X}$ is $h$-continuous, i.e. $F(\cdot,\cdot)$ satisfies the S-D Property.

Finally note that

$$\sup_{\|x\|\le k}|F(t,x)| = \sup_{\|x\|\le k}\sup_{y\in F(t,x)}\|y\| \le \widehat{M}\sup_{\|x\|\le k}\|B(t,x)\|_{\mathcal{L}} \le \widehat{M}\varphi_k(t),\ t\in T.$$

A straightforward application of Aumann's selection theorem (see Wagner [16, Theorem 5.10]), shows that control system (4) is actually equivalent to the following evolution inclusion (deparametrized system):

$$\left\{\begin{matrix}\dot{x}(t)\in A(t)x(t)+F(t,x(t))\\ x(0)=x(b)\end{matrix}\right\}.$$

By virtue of Theorem 2, we know that there exists $x(\cdot)\in C(T,X)$ a mild solution of

$$\left\{\begin{matrix}\dot{x}(t)\in A(t)x(t)+\mathrm{ext}\ F(t,x(t))\\ x(0)=x(b)\end{matrix}\right\}.$$

Note that $\mathrm{ext}\ F(t,x(t)) = \mathrm{ext}\ B(t,x(t))U(t,x(t)) \subseteq B(t,x(t))\,\mathrm{ext}\ U(t,x(t))$. Let $f\in l^p(X)$ be such that $x(t)=K(t,0)x(0)+\int_0^t K(t,s)f(s)\,ds$, $t\in T$. Then let

$$\Theta(t)=\{u\in \mathrm{ext}\ U(t,x(t)) : f(t)=B(t,x(t))u\}\neq\emptyset.$$

From Theorem 9.3 of Himmelberg [4], we know that $Gr\,\mathrm{ext}\ U(\cdot,x(\cdot))=\{(t,u)\in T\times Y : u\in \mathrm{ext}\ U(t,x(t))\}\in\mathcal{L}(T)\times B(Y)$ with $\mathcal{L}(T)$ being the Lebesgue $\sigma$-field of $T$ and $B(Y)$ the Borel $\sigma$-field of $Y$. Apply Aumann's selection theorem to get $u : T\to Y$ measurable s.t. $u(t)\in\Theta(t)$ for all $t\in T$. Then $x(t)=K(t,0)x(0)+\int_0^t K(t,s)B(s,x(s))u(s)\,ds$, with $y(t)\in \mathrm{ext}\ U(t,x(t))$ a.e. and $x(0)=x(b)$. So $x(\cdot)$ is the desired periodic extremal trajectory of (4). $\square$

We can drop the separability hypothesis on the control space $Y$ and instead assume the following:

$\underline{H_3}$: $Y=Z^*$ with $Z$ being a separable Banach space.

**Remark.** This alternative formulation incorporates in our abstract framework, systems whose control space is $L^\infty$.

**Theorem 4.** *<u>If</u> hypotheses $H(A)$, $H(B)$, $H(U)$, $H_2$ and $H_3$ hold, <u>then</u> control system* (4) *admits a periodic extremal trajectory.*

Proof: Note that $F(t,x)=B(t,x)U(t,x)\in P_{wkc}(X)$. Indeed let $\{y_n\}_{n\ge1}\subseteq F(t,x)$. Then $y_n=B(t,x)u_n$, $u_n\in U(t,x)$. Note that since $Z$ is separable, $U(t,x)$ equipped with the $w^*$-topology is compact-metrizable. So by passing to a subsequence if necessary, we may assume that $u_n\xrightarrow{w^*}u$. Then for all $x^*\in X^*$, we have:

$$(x^*,B(t,x)u_n)_{X,X^*}=(B(t,x)^*x^*,u_n)_{Z,Z^*}\to 0 \text{ as } n\to\infty$$
$$\Rightarrow y_n\to y \text{ weakly in } X \text{ and clearly } y=B(t,x)u,\ u\in U(t,x)$$
$$\Rightarrow F(t,x)\in P_{wkc}(X).$$

The rest of the proof is the same as that of Theorem 3. $\square$

## 6. An example

In this section we present an example of a semilinear parabolic control system, illustrating the applicability of our abstract results.

So let $Z$ be a bounded domain in $\mathbb{R}^n$, with boundary $\Gamma = \partial Z$. For $z \in \mathbb{R}^n$, we have $z = (z_1, \dots, z_N)$ and $D_k = \frac{\partial}{\partial z_k}$ is the elementary differential operator, $k = 1, \dots, N$. We consider the following periodic control system:

$$(5)\qquad \left\{ \begin{array}{c} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{i,j=1}^{N} D_i(a_{ij}(t,z)D_j x) = b(t,z,x(t,z))u(t,z) \text{ on } T \times Z \\ x\mid_{T\times\Gamma} = 0,\ \|u(t,\cdot)\|_{L^\infty(Z)} \le r(t, x(t,\cdot)) \text{ a.e.} \\ x(0,z) = x(b,z) \text{ a.e.} \end{array} \right\}.$$

We will need the following hypotheses:

$\underline{H(a)}$: $a_{ij} \in L^\infty(T \times Z)$, $a_{ij} = a_{ji}$, $c_0\|\eta\|^2 \le \sum_{i,j=1}^N a_{ij}(t,z)\,\eta_i\eta_j \le c_1\|\eta\|^2$ with $0 < c_0 < c_1$ and for all $\eta \in \mathbb{R}^n$, and $|a_{ij}(t',z) - a_{ij}(t,z)| \le \xi(z)|t'-t|$ a.e. with $\xi(\cdot) \in L^\infty_+(Z)$.

$\underline{H(b)}$: $b : T \times Z \times \mathbb{R} \to \mathbb{R}$ is a function s.t.

(1) $(t,z) \to b(t,z,x)$ is measurable,
(2) $x \to b(t,z,x)$ is continuous,
(3) $\sup_{\|x\|_{L^2(Z)}\le k} \|b(t,\cdot,x(\cdot))\|_{L^2(Z)} \le \varphi_k(t)$ a.e. with $\varphi_k \in L^2_+$ and $\underline{\lim}_{k\to\infty} \frac{1}{k}\int_0^b \varphi_k(t)\,dt = 0$.

$\underline{H(r)}$: $r : T \times L^2(Z) \to \mathbb{R}_+$ is a function s.t.

(1) $t \to r(t,x)$ is measurable,
(2) $x \to r(t,x)$ is continuous,
(3) $r(t,x) \le \widehat{M}$ for all $(t,x) \in T \times L^2(Z)$.

We have the following result concerning (5).

**Theorem 5.** *If hypotheses $H(a)$, $H(b)$ and $H(r)$ hold, then control system* (5) *admits a periodic extremal trajectory $x(t,z)$ s.t.*

Proof: Let $a : T \times H^1_0(Z) \times H^1_0(Z) \to \mathbb{R}$ be defined by

$$a(t,x,y) = \int_Z \sum_{i,j=1}^N a_{ij}(t,z) D_i x(z) D_j y(z)\,dz.$$

Using the Cauchy-Schwartz inequality and the fact that $(\sum_{i=1}^N \|D_i x\|^2_{L^2(Z)})^{1/2}$ is an equivalent norm for the Sobolev space $H^1_0(Z)$, we get that for some $\theta > 0$

$$|a(t,x,y)| \le \theta\|x\|_{H^1_0(Z)} \cdot \|y\|_{H^1_0(Z)},\ x,y \in H^1_0(Z).$$

Also using hypothesis $H(a)$ we can easily check that for some $\theta_1 > 0$

$$\theta_1 \|x\|^2_{H^1_0(Z)} \leq a(t,x,x), \ \ x \in H^1_0(Z).$$

Furthermore because of the Lipschitz property of the coefficients $a_{ij}(\cdot, z)$ we get that for some $\theta_2 > 0$

$$|a(t',x,y) - a(t,x,y)| \leq \theta_2 |t'-t| \ \|x\|_{H^1_0(Z)} \cdot \|y\|_{H^1_0(Z)}, \ t,t' \in T, \ x,y \in H^1_0(Z).$$

Let $A : T \times H^1_0(Z) \to H^{-1}(Z) = H^1_0(Z)^*$ be defined by

$$\langle A(t)x, y\rangle = a(t,x,y), (t,x,y) \in T \times H^1_0(Z) \times H^1_0(Z),$$

where by $\langle \cdot,\cdot\rangle$ we denote the duality pairing of $(H^1_0(Z), H^{-1}(Z))$. From Theorem 5.4.1 of Tanabe [14], we know that $\{A(t) : t \in T\}$ generates an evolution operator $K(t,s) \in \mathcal{L}(H)$, with $H = L^2(Z)$. Let $A_H(t) = A(t) \mid_{L^2(Z)\times L^2(Z)}$ be the operator defined by

$$D(A_H(t)) = \{x \in H^1_0(Z) : \sum_{i,j=1}^N D_i(a_{ij}(t,z)D_j x(z)) \in L^2(Z)\}$$

$$\text{and } A_H(t)x = -\sum_{i,j=1}^N D_i(a_{ij}(t,z)D_j x(z)).$$

We know that this is a linear, densely defined and self-adjoint operator (recall by hypothesis $H(a)$, $a_{ij} = a_{ji}$), with $A(t)$ being its energetic extension. Let $\varphi : T \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi(t,x) = \begin{cases} \frac{1}{2}\int_Z \sum_{i,j=1}^N a_{ij}(t,z) D_i x(z) D_j y(z)\, dz & x \in H^1_0(Z) \\ +\infty & \text{otherwise} \end{cases}.$$

We know (see for example [12]), that $\varphi(t,\cdot) \in \Gamma_0(H)$ (i.e. $\varphi(t,\cdot)$ is proper, l.s.c, convex function on $H$), $\partial\varphi(t,x) = A_H(t)x$ and furthermore exploiting the fact that $H^1_0(Z)$ embeds compactly in $L^2(Z)$, we can see that for every $\lambda \in \mathbb{R}$, the level set $\{x \in L^2(Z) : \|x\|^2_{L^2(Z)} + \varphi(t,x) \leq \lambda\}$ is compact (indeed it is closed, convex and bounded in $H^1_0(Z)$ and so by the compactness of the embedding of $H^1_0(Z)$ in $L^2(Z)$, is compact in $L^2(Z)$). So for every $t \in T$, $\varphi(t,\cdot)$ is of compact type (see [12]) and so the resolvent $(I + \lambda\partial\varphi(t,\cdot))^{-1}$ is compact for every $\lambda > 0$. Also from the Konishi-Brezis theorem [7], we know that for every $t \in T$, $\partial\varphi(t,\cdot)$ generates a compact semigroup. So using inequality (4.12), p. 39 of Pavel [13], we deduce that $K(t,s)$ is equicontinuous on bounded subsets of $H = L^2(Z)$ (i.e. given $\varepsilon > 0$ and $C \subseteq H$ a bounded subset, there is $\delta > 0$ such that $\|K(t',s)v - K(t,s)v\|_{\mathcal{L}} < \varepsilon$

for all $|t' - t| < \delta$ and all $v \in C$). Thus finally, by invoking Theorem 5.1, p. 46 of Pavel [13], we get that $K(t,s) \in \mathcal{L}(H)$ is compact for $t - s > 0$.

Next, let $Y = L^\infty(Z)$ be the control space and $U(t,x) = \{u \in L^\infty(Z) : \|u\|_{L^\infty(Z)} \leq r(t,x)\}$ for all $(t,x) \in T \times L^2(Z)$. Then by applying the classical Scorza-Dragoni theorem on $r(\cdot,\cdot)$ we can get that $U(\cdot,\cdot)$ satisfies the S-D Property.

Finally because of hypothesis $H(r)\,(3)$, we have $|U(t,x)| \leq \widehat{M}$. Also let $B : T \times H \to \mathcal{L}(Y,H)$ be defined by $B(t,x)(\cdot) = b(t,\cdot,x(\cdot))$. Clearly because of $H(b)$ and the classical Scorza-Dragoni theorem, this satisfies hypotheses $H(B)$ and $H_3$. Then rewrite (5) in the following equivalent abstract form:

$$\left\{\begin{array}{c} \dot{x}(t) + A(t)x(t) = B(t,x(t))u(t) \\ x(0) = x(b) \\ u(t) \in U(t,x(t)) \text{ a.e.} \end{array}\right\}.$$

Apply Theorem 4 to get a periodic extremal trajectory for (5). From Tanabe [14] we know that $x \in C(T, L^2(Z)) \cap L^2(T, H_0^1(Z))$ and $\frac{\partial x}{\partial t} \in L^2(T, H^{-1}(Z))$. □

**Remark.** Note that the bang-bang control $u(t,z)$ generating the periodic extremal trajectory $x(t,z)$, is easily realized, since $\lambda\{t \in T : \|u(t,\cdot)\|_\infty \neq r(t,x(t,\cdot))\} = 0$. Here $\lambda$ is the Lebesgue measure on $T$. So at almost all times $t \in T$, $\operatorname*{ess\,sup}_{z \in Z} |u(t,z)| = r(t,x(t,\cdot))$.

**Acknowledgement.** The author wishes to thank the referee for his (her) corrections and remarks.

## References


[1] Anichini G., *Nonlinear problems for systems of differential equations*, Nonlinear Anal.-TMA **1** (1977), 691–699.

[2] Benamara M., *Points Extremaux, Multi-applications et Fonctionelles Intégrales*, Thèse du 3ème cycle, Université de Grenoble, 1975.

[3] Diestel J., Uhl J., *Vector Measures*, Math. Surveys, Vol. 15, AMS, Providence, RI, 1977.

[4] Himmelberg C., *Measurable relations*, Fund. Math. **87** (1975), 57–91.

[5] Kartsatos A., *Locally invertible operators and existence problems in differential systems*, Tohoku Math. Jour. **28** (1976), 167–176.

[6] Klein E., Thompson A., *Theory of Correspondences*, Wiley, New York, 1984.

[7] Konishi Y., *Compacité des résolvantes des opérateurs maximaux cycliquement monotones*, Proc. Japan Acad. **49** (1973), 303–305.

[8] Papageorgiou N.S., *On the theory of Banach space valued multifunctions. Part 1: Integration and conditional expectation*, J. Multiv. Anal. **17** (1985), 185–206.

[9] ———, *Convergence theorems for Banach space valued integrable multifunctions*, Intern. J. Math. and Math. Sci. **10** (1987), 433–442.

[10] ———, *On multivalued evolution equations and differential inclusions in Banach spaces*, Comm. Math. Univ. S.P. **36** (1987), 21–39.

[11] ———, *Boundary value problems for evolution inclusions*, Comment. Math. Univ. Carolinae **29** (1988), 355–363.

[12] ———, *On evolution inclusion associated with time dependent convex subdifferentials*, Comment. Math. Univ. Carolinae **31** (1990), 517–527.

[13] Pavel N., *Nonlinear Evolution Operators and Semigroups*, Lecture Notes in Math. **1260**, Springer, Berlin, 1987.
[14] Tanabe H., *Equations in Evolution*, Pitman, London, 1979.
[15] Tolstonogov A., *Extreme continuous selectors of multivalued maps and the "bang-bang" principle for evolution inclusions*, Soviet Math. Doklady **317** (1991), 481–485.
[16] Wagner D., *Survey of measurable selection theorems*, SIAM J. Control. Optim. **15** (1977), 859–903.
[17] Zecca P., Zezza P., *Nonlinear boundary value problems in Banach spaces for multivalued differential equations on a non-compact interval*, Nonlinear Anal.-TMA **3** (1979), 347–352.

National Technical University, Department of Mathematics, Zografou Campus, Athens 15773, Greece
mailing address:
Florida Institute of Technology, Department of Applied Mathematics, 150 West University Blvd., Melbourne, Florida 32901–6988, USA

(*Received* December 29, 1992)