Semi-symmetric $\mathfrak{P}$-spaces

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Abstract. We determine explicitly the local structure of a semi-symmetric $\mathfrak{P}$-space.

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1. Introduction

Symmetric spaces play a central role in differential geometry and have been studied intensively because of their interesting characterizations and their remarkable geometric properties. These spaces are well-understood now. In the course of research, various generalizations were defined, sharing some of the special features of symmetric spaces. We mention here two of them.

The first generalization is given by the so-called *semi-symmetric spaces*. These are Riemannian manifolds (M, g) of which the curvature tensor R satisfies the algebraic condition $R_{XY} \cdot R = 0$ for all vector fields X and Y on M, where R_{XY} acts as a derivation on R. This means that, at each point p of M, the curvature tensor R_p is the same as the curvature tensor of some symmetric space. This symmetric space can change with the point p, in general. Basic theorems about semi-symmetric spaces have been proved in the fundamental papers by Z.I. Szabó ([Sz1], [Sz2]). For explicit classifications and for more up-to-date information, we refer to the recent papers [K], [BKV], [B2].

The second generalization has been given very recently by J. Berndt and L. Vanhecke in [BeV1]. Their starting point is the following characterization of symmetric spaces by means of the *Jacobi operators* R_γ along geodesics γ in the space (see Section 3 for the definitions): a Riemannian manifold (M, g) is locally symmetric if and only if, along every geodesic γ, the Jacobi operator R_γ has constant eigenvalues and parallel eigenspaces. The authors then define $\mathfrak{C}$-*spaces* (resp. $\mathfrak{P}$-*spaces*) as Riemannian manifolds for which the Jacobi operator R_γ along every geodesic γ has constant eigenvalues (resp. R_γ can be diagonalized by a parallel frame). They give non-trivial examples and prove nice properties of these spaces. See also [BeV2], [BeV3], [BePV] for further properties.

In view of these two generalizations, the following natural question arises: "What are the relations between the class of semi-symmetric spaces and the class of $\mathfrak{C}$-, resp. $\mathfrak{P}$-spaces?" This question is all the more interesting as this study

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could provide us with new examples of $\mathfrak{C}$- and $\mathfrak{P}$-spaces in the class of semi-symmetric spaces. Especially for $\mathfrak{P}$-spaces this is worthwhile as few examples of $\mathfrak{P}$-spaces are known, whereas a wealth of examples of $\mathfrak{C}$-spaces have been found. It was proved by J.T. Cho ([C]) and, independently and by a different method, by the present author ([B1]), that every semi-symmetric $\mathfrak{C}$-space is locally symmetric. Hence, within the class of semi-symmetric spaces no new examples of $\mathfrak{C}$-spaces exist. The case of semi-symmetric $\mathfrak{P}$-spaces was also considered by J.T. Cho in [C], but not in its full generality. Indeed, he studies only the case of *complete* semi-symmetric $\mathfrak{P}$-spaces and the case of semi-symmetric $\mathfrak{P}$-spaces of *cone type* ([Sz1], [Sz2]). The only new examples of $\mathfrak{P}$-spaces he obtains are the cones. The general, non-complete case is not treated in his work. In this short paper, we fill this gap. Using results of our earlier research ([B2]), we show the link between the class of semi-symmetric $\mathfrak{P}$-spaces and the class of *planar* semi-symmetric spaces as defined in [B2]. This allows us to give *explicitly* the local form for the metrics of such spaces (Theorem 3.7).

The paper is organized as follows: in Section 2, a short review about semi-symmetric spaces is given insofar as it is needed for the purpose of this article. In Section 3, we introduce $\mathfrak{P}$-spaces and determine the structure of a semi-symmetric $\mathfrak{P}$-space.

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2. Semi-symmetric spaces

In this section we review briefly some facts about semi-symmetric spaces, in particular about semi-symmetric spaces foliated by Euclidean spaces of codimension two. For a more detailed treatment, the reader should consult the papers by Z.I. Szabó ([Sz1], [Sz2]) and [B2]. We shall restrict ourselves here to the minimum necessary for the next section.

We start with the following local structure theorem of Szabó ([Sz1, Theorem 4.5]).

Proposition 2.1. *Around the points of an everywhere dense open subset, a semi-symmetric space is locally a de Rham product of symmetric spaces, two-dimensional surfaces, six types of cones and semi-symmetric Riemannian manifolds foliated by Euclidean leaves of codimension two.*

Symmetric spaces and two-dimensional surfaces are well-understood and the cones were explicitly constructed in [Sz1] and [Sz2]. The foliated semi-symmetric manifolds appearing in the above proposition have not been explored much until recently. For the recent developments, see [K], [BKV], [B2]. In the following we give the necessary information about this class.

In [B2, Theorem 3.1], it is shown that the metric g of a foliated semi-symmetric space can locally always be given in the form $g = \sum_{i=1}^{n+2} \omega^i \otimes \omega^i$, where

$$\begin{aligned} \omega^1 &= f(w,x,y^1,\dots,y^n)\,dw, \\ \omega^2 &= A(w,x,y^1,\dots,y^n)\,dx + C(w,x,y^1,\dots,y^n)\,dw, \\ \omega^{\alpha+2} &= dy^\alpha + H^\alpha(w,x,y^1,\dots,y^n)\,dw, \qquad \alpha = 1,\dots,n, \end{aligned} \tag{2.1}$$

in the coordinates $(w,x,y^1,\dots,y^n)$, and the following partial differential equations are satisfied:

(A1)	$(AB)'_\alpha + (B_\alpha)'_x = 0,$	(A2)	$(B_\alpha)'_w - R'_\alpha = 0,$
(B1)	$(AB_\alpha)'_\beta + A'_\alpha B_\beta = 0,$	(B2)	$(S_\alpha)'_\beta + T_\alpha B_\beta = 0,$
(C1)	$A''_{\alpha\beta} - AB_\alpha B_\beta = 0,$	(C2)	$(T_\alpha)'_\beta - S_\alpha B_\beta = 0,$
(D1)	$(H^\alpha)''_{\beta\gamma} = 0,$	(D2)	$(H^\alpha)''_{\beta x} + (AT_\alpha)'_\beta - (AT_\beta)'_\alpha = 0.$

Here B_α, B, R, S_α and T_α are given by

$$\begin{aligned} B_\alpha &= (2Af)^{-1}\big((H^\alpha)'_x + (AC'_\alpha - CA'_\alpha)\big), \\ B &= (Af)^{-1}(A'_w - C'_x - \sum_\alpha A'_\alpha H^\alpha), \\ R &= A^{-1} f'_x - CB + \sum_\alpha B_\alpha H^\alpha, \\ S_\alpha &= f'_\alpha + CB_\alpha, \\ T_\alpha &= C'_\alpha - fB_\alpha. \end{aligned}$$

In this notation, the Euclidean leaves are given by $\omega^1 = 0$, $\omega^2 = 0$ and the curvature tensor R has the following form

$$R = 4k\,\omega^1 \wedge \omega^2 \otimes \omega^1 \wedge \omega^2. \tag{2.2}$$

Here, $k(w,x,y^1,\dots,y^n)$ is the sectional curvature of the plane section determined by $\omega^{\alpha+2} = 0$ for $\alpha = 1,\dots,n$. It is given by the formula

$$k = -(Af)^{-1}\big((AB)'_w + R'_x + \sum_\alpha (AS_\alpha)'_\alpha\big).$$

This function never vanishes by definition ([Sz1], [B2]), because the index of nullity of the curvature tensor R is supposed to be constant.

In the next section we will need the covariant derivative of R. For that reason we also give the connection forms for the metric (2.1). These are given by

$$\begin{aligned} \omega_2^1 &= (Af)^{-1} f'_x\,\omega^1 - B\,\omega^2 + \sum_\alpha b^\alpha\,\omega^{\alpha+2}, \\ \omega_{\alpha+2}^1 &= a^\alpha\,\omega^1 + b^\alpha\,\omega^2, \\ \omega_{\alpha+2}^2 &= c^\alpha\,\omega^1 + e^\alpha\,\omega^2, \\ \omega_{\beta+2}^{\alpha+2} &= f^{-1}(H^\alpha)'_\beta\,\omega^1, \end{aligned} \tag{2.3}$$

where we put, for the sake of simplicity,

$$a^\alpha = f^{-1} f'_\alpha, \qquad\qquad b^\alpha = B_\alpha,$$
$$c^\alpha = (Af)^{-1}(AC'_\alpha - CA'_\alpha) - B_\alpha, \qquad e^\alpha = A^{-1}A'_\alpha.$$

In the explicit classification of foliated semi-symmetric spaces in dimension three ([K]) and also in higher dimensions ([B2]) the existence of a special distribution on the manifold, the *asymptotic distribution*, plays a major role.

Definition 2.2. An $(n+1)$-dimensional distribution E on a foliated semi-symmetric space (M^{n+2}, g) is an *asymptotic distribution*, if it is integrable, contains the nullity vector space of R at each point and is parallel along each (n-dimensional) Euclidean leaf.

This means the following: let $(E_1, \ldots, E_{n+2})$ be the local orthonormal frame dual to the coframe $(\omega^1, \ldots, \omega^{n+2})$. Then $E_{\alpha+2}$, $\alpha = 1, \ldots, n$, span the tangent space of the corresponding Euclidean leaf at each point, which, by (2.2), is the nullity vector space of R. The above definition says that an $(n+1)$-dimensional distribution E is an asymptotic distribution if the following three conditions hold for all vector fields X and Y in E:

(i) $[X, Y] \in E$,
(ii) $D_{E_{\alpha+2}} X \in E, \quad \alpha = 1, \ldots, n,$
(iii) $E_{\alpha+2} \in E, \quad \alpha = 1, \ldots, n.$

It is proved in [B2] that E must satisfy the equations

$$c^\alpha (\omega^1)^2 + (e^\alpha - a^\alpha)\, \omega^1 \omega^2 - b^\alpha (\omega^2)^2 = 0, \qquad \alpha = 1, \ldots, n. \tag{2.4}$$

Definition 2.3. A foliated semi-symmetric space which admits infinitely many asymptotic distributions is said to be *planar*.

From (2.4) it follows that in a planar semi-symmetric space we have, in a neighbourhood of each point,

$$c^\alpha = b^\alpha = 0, \qquad e^\alpha = a^\alpha, \qquad \alpha = 1, \ldots, n. \tag{2.5}$$

Conversely, if (2.5) is satisfied, the space will be planar in the corresponding neighbourhood.

Concerning the planar foliated semi-symmetric spaces, the following result is proved in [B2].

Proposition 2.4. *(M^{n+2}, g) is a planar semi-symmetric space if and only if (M^{n+2}, g) is locally isometric either to the product of a two-dimensional surface with $\mathbb{R}^n$, or to $M^3 \times \mathbb{R}^{n-1}$ where the metric of M^3 is locally determined by the orthonormal coframe*

$$\begin{aligned} \omega^1 &= f(w,x)\, y\, dw, \\ \omega^2 &= y\, dx, \\ \omega^3 &= dy \end{aligned} \tag{2.6}$$

with f an arbitrary positive function of the variables w and x, and $y \in \mathbb{R}_0^+$.

Remark. The metric given by (2.6) clearly belongs to a warped product $M_1 \times_h M_2$ of a one-dimensional manifold M_1 with a two-dimensional manifold M_2, where the warping function h is a *linear* function of the coordinate y on M_1. The class of warped products $M_1 \times_h M_2$ occurs in the classification theorem for three-dimensional $\mathfrak{P}$-spaces given in [BeV1]. However, the result there guarantees that such a warped product is a $\mathfrak{P}$-space only in case the manifolds M_1 and M_2 and the function h are *analytic*. For the metric (2.6) above, we will show in the next section that analyticity is not needed.

3. Semi-symmetric $\mathfrak{P}$-spaces

Let (M, g) be a Riemannian manifold with curvature tensor R and let γ be a geodesic in (M, g). The symmetric operators $R_\gamma := R_{\cdot\dot\gamma}\dot\gamma$ along γ determine the *Jacobi operator field* along γ. Using these operators, a special class of Riemannian manifolds is defined in [BeV1].

Definition 3.1. A Riemannian manifold (M, g) is a $\mathfrak{P}$*-space* if the Jacobi operator along any geodesic can be diagonalized by a parallel orthonormal coframe along the geodesic.

Moreover, the authors prove

Proposition 3.2. *If (M, g) is a $\mathfrak{P}$-space, then the curvature condition $[R'_X, R_X] = 0$ is satisfied for all $X \in TM$. (Here $R_X = R_\gamma(0)$ and $R'_X = D_{\dot\gamma} R_\gamma(0)$, where γ is the unique geodesic such that $\dot\gamma(0) = X$.) Moreover, for analytic manifolds, this condition is also sufficient.*

As both semi-symmetric manifolds and $\mathfrak{P}$-spaces are generalizations of symmetric spaces, it is interesting to know the relation between these two classes of manifolds. This problem was first studied by J.T. Cho ([C]). He considered the different factors in Szabó's local structure theorem for semi-symmetric spaces (Proposition 2.1) and checked whether these factors are $\mathfrak{P}$-spaces. This obviously is true for every two-dimensional surface (see [BeV1]). For the semi-symmetric cones and for foliated semi-symmetric manifolds, Cho obtained the following results:

Proposition 3.3. *Any semi-symmetric space of cone type is a $\mathfrak{P}$-space.*

Theorem 3.4. *Let (M, g) be a complete, semi-symmetric $\mathfrak{P}$-space. Then M is a local product space of symmetric spaces and of $LP(M^2, \mathbb{R}^k)$-spaces. In the analytic case, also the converse holds.*

Here $LP(M^2, \mathbb{R}^k)$ denotes the class of local product spaces of a two-dimensional Riemannian space and a Euclidean space. Note that the last theorem only deals with *complete* semi-symmetric manifolds, and that the spaces of cone type are never complete. In what follows, we will determine *all* semi-symmetric $\mathfrak{P}$-spaces, i.e. we drop the completeness condition.

As the only factor in Szabó's structure theorem that still requires treatment, concerns foliated semi-symmetric spaces, we suppose that we have a metric g of the form (2.1) with curvature tensor R given by (2.2). We now calculate $D_X R$ using the well-known formulas $D_X\omega^i = \sum_k \omega^i_k(X)\,\omega^k$ and (2.3). We obtain

$$\begin{aligned} D_X R =& 4dk(X)\,(\omega^1\wedge\omega^2\otimes\omega^1\wedge\omega^2)\\ &- 4k\big[\omega^1(X)\big(\sum a^\alpha\,\omega^{\alpha+2}\wedge\omega^2 + \sum c^\alpha\;\omega^1\wedge\omega^{\alpha+2}\big)\\ &\quad + \omega^2(X)\big(\sum b^\alpha\,\omega^{\alpha+2}\wedge\omega^2 + \sum e^\alpha\;\omega^1\wedge\omega^{\alpha+2}\big)\big]\otimes\omega^1\wedge\omega^2\\ &- 4k\,\omega^1\wedge\omega^2\otimes\big[\omega^1(X)\big(\sum a^\alpha\,\omega^{\alpha+2}\wedge\omega^2 + \sum c^\alpha\;\omega^1\wedge\omega^{\alpha+2}\big)\\ &\quad + \omega^2(X)\big(\sum b^\alpha\,\omega^{\alpha+2}\wedge\omega^2 + \sum e^\alpha\;\omega^1\wedge\omega^{\alpha+2}\big)\big]. \end{aligned} \tag{3.1}$$

This and (2.2) yield

$$\begin{aligned} R_{E_1}R'_{E_1}E_{\beta+2} &= k^2 c^\beta\,E_2,\\ R'_{E_1}R_{E_1}E_{\beta+2} &= 0. \end{aligned}$$

Recall that the function k is never zero. Hence, if the space is a $\mathfrak{P}$-space, we must have

$$c^\beta = 0, \qquad \text{for all } \beta. \tag{3.2}$$

Similarly, from

$$\begin{aligned} R_{E_2}R'_{E_2}E_{\beta+2} &= k^2 b^\beta\,E_1,\\ R'_{E_2}R_{E_2}E_{\beta+2} &= 0, \end{aligned}$$

it follows that

$$b^\beta = 0, \qquad \text{for all } \beta. \tag{3.3}$$

Finally, from

$$\begin{aligned} R_{E_1+E_2}R'_{E_1+E_2}E_{\beta+2} &= 2k^2(a^\beta - e^\beta)\,(E_1 - E_2),\\ R'_{E_1+E_2}R_{E_1+E_2}E_{\beta+2} &= 0, \end{aligned}$$

we find the condition

$$a^\beta = e^\beta, \qquad \text{for all } \beta. \tag{3.4}$$

Conversely, if (3.2)–(3.4) are satisfied, then we have

$$\begin{aligned} R_X &= k\,\big(\omega^1(X)\omega^2 - \omega^2(X)\omega^1\big)\otimes\big(\omega^1(X)E_2 - \omega^2(X)E_1\big),\\ R'_X &= \big[dk(X) - 2k\big(\sum a^\alpha\omega^{\alpha+2}(X)\big)\big]\\ &\qquad \big(\omega^1(X)\omega^2 - \omega^2(X)\omega^1\big)\otimes\big(\omega^1(X)E_2 - \omega^2(X)E_1\big), \end{aligned}$$

from which it is clear that $[R_X, R'_X] = 0$ for all $X\in TM$.

On the other hand, the conditions (3.2)–(3.4) determine exactly the class of the *planar* semi-symmetric spaces (see the previous section). So, we have proved

Theorem 3.5. *Every foliated semi-symmetric $\mathfrak{P}$-space is planar.*

Concerning the converse of this theorem, we have

Theorem 3.6.

(i) *The space (M^3, g) given by the metric (2.6) is a $\mathfrak{P}$-space.*
(ii) *Every planar foliated semi-symmetric space is a $\mathfrak{P}$-space.*

Proof: (i) For the metric (2.6) we calculate explicitly the eigenspaces of the Jacobi operator field R_γ along an arbitrary geodesic γ. The form (2.3) for the connection forms specializes for the metric (2.6) to

$$\begin{aligned} \omega_2^1 &= (f'_x/fy)\,\omega^1, \\ \omega_3^1 &= (1/y)\,\omega^1, \\ \omega_3^2 &= (1/y)\,\omega^2, \end{aligned} \tag{3.5}$$

and the formula (2.2) for the curvature tensor R to

$$R = -(4/fy^2)(f''_{xx} + f)\,\omega^1 \wedge \omega^2 \otimes \omega^1 \wedge \omega^2. \tag{3.6}$$

Further, if (E_1, E_2, E_3) is the dual orthonormal frame of $(\omega^1, \omega^2, \omega^3)$ given by (2.6), then by the standard formulas $D_X E_i = -\sum_j \omega_i^j(X)\,E_j$ and by (3.5), we have

$$\begin{aligned} D_X E_1 &= (f'_x/fy)\,\omega^1(X)\,E_2 + (1/y)\,\omega^1(X)\,E_3, \\ D_X E_2 &= -(f'_x/fy)\,\omega^1(X)\,E_1 + (1/y)\,\omega^2(X)\,E_3, \\ D_X E_3 &= -(1/y)\,\omega^1(X)\,E_1 - (1/y)\,\omega^2(X)\,E_2. \end{aligned} \tag{3.7}$$

Now, let $\gamma(t) = (w(t), x(t), y(t))$ be an arbitrary unit-speed geodesic. We decompose its velocity vector field $\dot\gamma$ with respect to the frame (E_1, E_2, E_3):

$$\dot\gamma(t) = a_1(t)\,E_1(\gamma(t)) + a_2(t)\,E_2(\gamma(t)) + a_3(t)\,E_3(\gamma(t)). \tag{3.8}$$

The functions $a_1(t)$, $a_2(t)$ and $a_3(t)$ are given by

$$\begin{aligned} a_1(t) &= f(w(t), x(t))y(t)\dot w(t), \\ a_2(t) &= y(t)\dot x(t), \\ a_3(t) &= \dot y(t). \end{aligned} \tag{3.9}$$

Using (3.7), we find that $D_{\dot\gamma}\dot\gamma = 0$ is equivalent to the system of ordinary differential equations (in the unknown functions $w(t)$, $x(t)$ and $y(t)$)

$$\begin{aligned} \dot a_1 - (f'_x/fy)\,a_1a_2 - (1/y)\,a_1a_3 &= 0, \\ \dot a_2 + (f'_x/fy)\,{a_1}^2 - (1/y)\,a_2a_3 &= 0, \\ \dot a_3 + (1/y)\,({a_1}^2 + {a_2}^2) &= 0. \end{aligned} \tag{3.10}$$

By (3.6) and (3.8), the Jacobi operator field R_γ has the following matrix form with respect to the frame (E_1, E_2, E_3)

$$((f''_{xx} + f)/fy^2)\begin{pmatrix} {a_2}^2 & -a_1a_2 & 0 \\ -a_1a_2 & {a_1}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

If $a_1(t_0) = a_2(t_0) = 0$ for some t_0, then it follows readily from (3.9) that $\dot{w}(t_0) = \dot{x}(t_0) = 0$. By the uniqueness of geodesics we then have $\gamma(t) = (w_0, x_0, y_0 + t)$ and we see that $a_1(t) = a_2(t) = 0$ for all t. In that case R_γ is identically zero along γ and hence, obviously diagonalizable by a parallel orthonormal frame along γ.

So, we suppose that ${a_1}^2 + {a_2}^2$ is nowhere zero. Then the eigenvalues of (3.11) are given by ${a_1}^2 + {a_2}^2$ with multiplicity one, and 0 with multiplicity two. The eigenspace belonging to the first eigenvalue is spanned by $a_2\,E_1 - a_1\,E_2$, the eigenspace belonging to the eigenvalue 0 by $a_1\,E_1 + a_2\,E_2$ and E_3. Using (3.7) and (3.10), it is easy to show that both are parallel along γ. This proves that the metric (2.6) determines a $\mathfrak{P}$-space.

(ii) We remark that the product of $\mathfrak{P}$-spaces is again a $\mathfrak{P}$-space if all the manifolds involved are *analytic* ([BeV1]). We show now that analyticity is not necessary when one of the spaces is a Euclidean space $\mathbb{R}^n$. In view of Proposition 2.4 and part (i) above, this is sufficient to prove Theorem 3.6 (ii). So, let M^m be an m-dimensional $\mathfrak{P}$-space and $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ a geodesic in $M^m \times \mathbb{R}^n$. Clearly γ_1 (resp. γ_2) is a geodesic in M^m (resp. in $\mathbb{R}^n$). Moreover, $R_{XY}Z = 0$ whenever X, Y or Z is tangent to $\mathbb{R}^n$. Taking these considerations into account, it is clear that $R_\gamma Y = R_{\gamma_1} Y_1$, where Y_1 is the component of Y tangent to M^m. As M^m is a $\mathfrak{P}$-space, we can find a parallel orthonormal frame $(F_1(\gamma_1(t)), \ldots, F_m(\gamma_1(t)))$ along γ_1 in M^m. We can suppose that $F_m = \dot{\gamma}_1$. Using the natural embedding of $T_{\gamma_1(t)}(M^m)$ in $T_{\gamma(t)}(M^m \times \mathbb{R}^n)$, we can consider the orthonormal vectors $F_1(\gamma(t)), \ldots, F_{m-1}(\gamma(t))$ along γ in $M^m \times \mathbb{R}^n$. These are clearly parallel along γ, and eigenvectors of $R_\gamma(t)$. Moreover, their orthogonal complement consists of eigenvalues of R_γ with eigenvalue zero and is also parallel along γ. Hence, $M^m \times \mathbb{R}^n$ is also a $\mathfrak{P}$-space. □

The above theorems, together with Proposition 3.3 and the classification theorem by Szabó then yield:

Theorem 3.7. *Around the points of an everywhere dense open subset, a semi-symmetric $\mathfrak{P}$-space is locally a product of symmetric spaces, semi-symmetric cones, two-dimensional surfaces and three-dimensional spaces with the metric given by* (2.6)*. Moreover, in the analytic case, the converse also holds.*

The last statement in this theorem follows from general results about the products of $\mathfrak{P}$-spaces proved in [BeV1].

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