On maximum principle for weak subsolutions of degenerate parabolic linear equations

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Abstract. Sufficient conditions are obtained so that a weak subsolution of (0.1), bounded from above on the parabolic boundary of the cylinder Q, turns out to be bounded from above in Q.

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1. Introduction

In a recent Note [1] this author has indicated sufficient conditions allowing a weak subsolution [(1)] of the parabolic differential equation

$$(0.1)\qquad -\sum_1^m{}_i \frac{\partial}{\partial x_i}\left(\sum_1^m{}_j a_{ij}\frac{\partial u}{\partial x_j}+d_i u\right)+\left(\sum_1^m{}_i b_i\frac{\partial u}{\partial x_i}+cu-f\right)+\frac{\partial u}{\partial t}=0$$

bounded from above on the parabolic boundary of the cylinder Q, to turn out to be bounded from above in Q, assuming the ellipticity condition to be of the following kind:

$$\sum_1^m{}_{ij} a_{ij}(x,t)\xi_i\xi_j \geq \mu\sum_1^m{}_i \xi_i^2$$

with $\mu=\nu(x)\psi(t)$, ν and ψ satisfactory hypotheses sufficiently general.

Such results do not generally require the subsolutions of (0.1) to have second derivatives with respect to the space variables or the derivative with respect to t.

In this Note similar results are obtained regarding a class of subsolutions less weak as compared to the ones considered in [1], working, however, on more restrictive hypotheses concerning the functions f and ψ.

Moreover, the comparison between these results and the results cited above would require either the functions ψ and ψ^{-1} to be essentially bounded, or, working on more restrictive hypotheses concerning the coefficients of (0.1), $\psi \in C^0(]0,T[)\cap L^\infty(0,T)$ and ψ^{-1} to be r-integrable with $r\geq 1$.[(2)] When μ is

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[(1)] see Definition 1, p. 134 of [1].

constant, sufficient conditions for the boundedness of weak subsolutions may be obtained from [2], [6] and [9], whilst the case where μ depends on x and t has been studied by A.V. Ivanov in [4] (see Theorem 5.3) but with a further hypothesis (Condition II) obviously limiting the kind of degeneration (with respect to the variable t) and which has been suppressed in [4] (see the remark at p. 41), assuming, however, that the subsolutions have square-integrable derivative on t.

2. Functional spaces

Let $\mathbb{R}^m$ be the Euclidean m-dimensional space having generical point $x \equiv (x_1, \dots, x_m)$, Ω an open and bounded set of $\mathbb{R}^m$, T a positive number.

The symbol meas_x(meas) will henceforth indicate the m-dimensional ($m+1$-dimensional) LEBESGUE's measure.

If $u(x,t)$ is a function defined in Q and k is a real number, we will indicate with $\Omega(t,k)$, $t \in]0,T[$, the set of points of Ω in which $u(x,t) > k$.

Hypothesis 2.1. *Let $\nu(x)$ be a positive function defined in Ω such that:*

$$\nu(x) \in L^1(\Omega), \ \nu^{-1}(x) \in L^1_{\mathrm{loc}}(\Omega).$$

$\tilde{H}^1(\nu,\Omega)$ indicates the completion of $C^1(\bar{\Omega})$ with respect to the norm

$$\|u\|_1 = \left(\int_\Omega \left(|u|^2 + \sum_1^m {}_i \nu(x) \left| \frac{\partial u}{\partial x_i} \right|^2 \right) dx \right)^{1/2} .$$

$\tilde{H}^1_0(\nu,\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $\tilde{H}^1(\nu,\Omega)$.

Hypothesis 2.2. *Let $\psi(t)$ be a positive function defined in $]0,T[$ such that:*

$$\psi(t) \in L^1(0,T).$$

$\tilde{H}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$ $(0 \le \tau_1 < \tau_2 \le T)$ stands for the completion of $C^1(\overline{Q(\tau_1,\tau_2)})$ with respect to the norm

$$\|u\|_{1,0,(\tau_1,\tau_2)} = \left(\int_{Q(\tau_1,\tau_2)} \left(|u|^2 + \sum_1^m {}_i \nu(x)\psi(t) \left| \frac{\partial u}{\partial x_i} \right|^2 \right) dx\, dt \right)^{1/2} ;$$

$$\|u\|_{1,0} = \|u\|_{1,0,(0,T)} \cdot$$

$\tilde{H}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$ is a HILBERT space with respect to the norm $\|u\|_{1,0,(\tau_1,\tau_2)}$.

$\overset{0}{\tilde{H}}{}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$ $(0 \le \tau_1 < \tau_2 \le T)$ is the closure of $C_0^\infty(Q(\tau_1,\tau_2))$ in $\tilde{H}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$.

Finally, we will denote with $\overset{*}{H}{}^{1,0}(\nu\psi, Q)$ the space of functions $u(x,t)$ belonging to $\tilde{H}^{1,0}(\nu\psi, Q)$, continuous in $[0,T]$ with respect to values in $L^2(\Omega)$.

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(2) if $\psi \in C^0(]0,T[) \cap L^1(0,T)$ and $\psi^{-1} \in L^1_{\mathrm{loc}}(0,T)$, the space $C_0^\infty(Q)$ is dense in: $W_\psi = \Big\{ w \mid w \in L^2_\psi(0,T; H^1_0(\nu,\Omega)), w_t \in L^2_{1/\psi}(0,T; L^2(\Omega)), w(x,0) = w(x,T) = 0 \text{ a.e. in } \Omega \Big\}$ endowed with the graph norm.

Hypothesis 2.3. *Let us assume that:*

$$\psi, \psi^{-1} \in L^{\infty}_{\mathrm{loc}}(0,T)\,.$$ (3)

Definition 1. We will say that a subsolution of the equation (0.1) is a function $u(x,t) \in \overset{*}{H}{}^{1,0}(\nu\psi, Q)$ such that

$$\int_Q \left\{ \sum_1^m {}_{ij} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_1^m {}_i b_i \frac{\partial u}{\partial x_i} \varphi + cu\varphi + \sum_1^m {}_i d_i u \frac{\partial \varphi}{\partial x_i} - \right. \\ \left. -u \frac{\partial \varphi}{\partial t} \right\} dx\,dt \le \int_Q f\varphi\,dx\,dt \tag{2.1}$$

for any $\varphi \in C_0^\infty(Q)$ such that $\varphi(x,t) \ge 0$ a.e. in Q.

Definition 2. Given a real number k, if $u \in \tilde{H}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$ $(0 \le \tau_1 < \tau_2 \le T)$, we will say that $u(x,t) \le k$ on $\partial\Omega \times [\tau_1,\tau_2]$ if there exists a sequence $\{u_n\}$ of functions of $C^1(\overline{Q(\tau_1,\tau_2)})$ such that

$$u_n(x,t) \le k \quad \text{on} \quad \partial\Omega \times [\tau_1,\tau_2]$$

and

$$\lim_{n\to\infty} \|u_n - u\|_{1,0,(\tau_1,\tau_2)} = 0\,. \tag{2.2}$$

If k is such that $u(x,t) \le k$ on $\partial\Omega \times [\tau_1,\tau_2]$, we will say that $u(x,t)$ is bounded from above on $\partial\Omega \times [\tau_1,\tau_2]$. In this case, the symbol $\sup\limits_{[\tau_1,\tau_2]} {}^* u$ stands for the greatest lower bound of the real numbers k such that $u(x,t) \le k$ on $\partial\Omega \times [\tau_1,\tau_2]$;

$$\sup{}^* u = \sup_{[0,T]}{}^* u\,.$$

Definition 3. We shall say that a function $u(x,t)$ belonging to $\overset{*}{H}{}^{1,0}(\nu\psi, Q)$ is bounded from above on $(\Omega \times \{t=0\}) \cup (\partial\Omega \times [0,T])$ if $u(x,0)$ is bounded from above in Ω and, also, if $u(x,t)$ is bounded from above on $\partial\Omega \times [0,T]$.

(3) hypotheses 2.2 and 2.3 do not imply ψ and ψ^{-1} to be essentially bounded in $]0,T[$; e.g. it will be sufficient to consider:

$$\psi(t) = \begin{cases} \sqrt{t} & 0 < t \le \frac{1}{2} \\ (\sqrt{1-t})^{-1} & \frac{1}{2} < t \le 1\,. \end{cases}$$

3. Hypotheses on coefficients

Let us denote with $\mathcal{A}$ the set of pairs (α^*, α), with $2 \leq \alpha^*,\ \alpha \leq +\infty$, such that there exists a positive constant β for which

$$\|u\|_{\alpha^*,\alpha} \leq \beta(\|u\|_{2,\infty} + \|u\|_{1,0}) \text{ (4)}$$

for any $u \in L^{2,\infty}(Q) \cap \tilde{H}^{1,0}(\nu\psi, Q)$. The set $\mathcal{A}$ obviously contains the pair $(2, +\infty)$. Let us indicate with $\mathcal{B}$ the subset of $\mathcal{A}$ formed by the pairs (α^*, α) with $2 < \alpha^*,\ \alpha < +\infty$.

We will need the following

Hypothesis 3.1. *The set $\mathcal{B}$ is not empty.* (5)

It is therefore reasonable to postulate the following hypotheses on the coefficients of (0.1):

Hypothesis 3.2. *The functions $a_{i,j}$, b_i, c, d_i, f $(i, j = 1, \ldots, m)$ are defined and measurable in Q;*

$$a_{i,j}(\nu\psi)^{-1} \in L^\infty(Q)\,, \quad b_i(\nu\psi)^{-1/2} \in L^{p^*,p}(Q)\,,$$

$$c \in L^{q^*,q}(Q)\,, d_i(\nu\psi)^{-1/2} \in L^{r^*,r}(Q)\,, f \in L^{g^*,g}(Q)\,,$$

where p^*, p, q^*, q, r^*, r, g^*, g are to be such that $2 < 2g^*,\ 2g < +\infty$

$$\frac{1}{p^*} + \frac{1}{\alpha_1^*} = \frac{1}{2}\,, \frac{1}{p} + \frac{1}{\alpha_1} = \frac{1}{2}\,, \frac{1}{q^*} + \frac{2}{\alpha_2^*} = 1$$

$$\frac{1}{q} + \frac{2}{\alpha_2} = 1\,, \frac{1}{r^*} + \frac{1}{\alpha_3^*} = \frac{1}{2}\,, \frac{1}{r} + \frac{1}{\alpha_3} = \frac{1}{2}$$

$$\frac{1}{g^*} + \frac{2}{\alpha_4^*} < 1\,, \frac{1}{g} + \frac{2}{\alpha_4} < 1$$

with (α_1^*, α_1), (α_2^*, α_2), (α_3^*, α_3), belonging to $\mathcal{A}$ and (α_4^*, α_4) belonging to $\mathcal{B}$.

Moreover, if $p = +\infty$ $[q = +\infty, r = +\infty]$ and $p^* < +\infty$ $[q^* < +\infty, r^* < +\infty]$, then there exists a function $\eta_1(\sigma)$ $[\eta_2(\sigma), \eta_3(\sigma)]$, defined for $\sigma \geq 0$, non decreasing, vanishing for σ approaching zero, having such a property as to give, for almost any t in the interval $]0, T[$:

$$\sum_1^m {}_i \left(\int_E \left(\frac{|b_i(x,t)|}{\sqrt{\nu(x)}} \right)^{p^*} dx \right)^{1/p} \leq \eta_1(\sigma)\sqrt{\psi(t)}$$

$$\left[\left(\int_E (|c(x,t)| - c(x,t))^{q^*}\, dx \right)^{1/q} \leq \eta_2(\sigma)\,, \right.$$

$$\left. \sum_1^m {}_i \left(\int_E \left(\frac{|d_i(x,t)|}{\sqrt{\nu(x)}} \right)^{r^*} dx \right)^{1/r} \leq \eta_3(\sigma)\sqrt{\psi(t)} \right]$$

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(4) if $1 \leq p, q \leq +\infty$, $\|\cdot\|_{p,q,(\tau_1,\tau_2)}$ stands for the norm in $L^{p,q}(Q(\tau_1,\tau_2))$ $(0 \leq \tau_1 < \tau_2 \leq T)$; $\|\cdot\|_{p,q} = \|\cdot\|_{p,q,(0,T)}$.

(5) sufficient conditions so that Hypothesis 3.1 holds may be obtained from § 2 of [7].

for all measurable subsets E of Ω such that $\operatorname{meas}_x E \le \sigma$.

Hypothesis 3.3. *The following inequality results a.e. in Q for all the real numbers $\xi_1, \xi_2, \dots, \xi_m$*

$$\sum_1^m {}_{ij} a_{ij}(x,t)\xi_i\xi_j \ge \nu(x)\psi(t) \sum_1^m {}_i \xi_i^2 \,.$$

Hypothesis 3.4. *There exists a nonnegative constant ϱ:*

$$c - \sum_1^m {}_i \frac{\partial d_i}{\partial x_i} \ge -\varrho$$

in the distributional sense over Q.

In § 5 we will prove the following

Theorem. *Let us assume Hypotheses* 2.1, 2.2, 2.3, 3.1, 3.2, 3.3, 3.4 *hold and let $u(x,t)$ be a subsolution of the equation* (0.1) *bounded from above on* $(\Omega \times \{t = 0\}) \cup (\partial\Omega \times [0,T])$. *Then u is bounded from above in Q; moreover,*

$$\operatorname*{ess\,sup}_Q u \le e^{\varrho T}\{\max(0, \operatorname*{ess\,sup}_\Omega u(x,0), \sup{}^* u) + \gamma\|f\|_{g^*,g}\}.\ ^{(6)}$$

4. Preliminary lemmas

Lemma 4.1. *Let $u \in \tilde{H}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$ $(0 \le \tau_1 < \tau_2 \le T)$ bounded from above on $\partial\Omega \times [\tau_1,\tau_2]$ and $h > \sup\limits_{[\tau_1,\tau_2]}{}^* u$, then there exists a sequence of functions $\{U_\nu\}_\nu$ such that*

$$U_\nu \in C^1(\overline{Q(\tau_1,\tau_2)}), \quad U_\nu(x,t) < h \quad \text{on} \quad \partial\Omega \times [\tau_1,\tau_2]$$

for any $\nu \in \mathbb{N}$ and

$$\lim_{\nu\to\infty} \|U_\nu - u\|_{1,0,(\tau_1,\tau_2)} = 0\,.$$

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(6) having fixed a number

$$l \ge \max(1, T, \beta, \operatorname{meas}_i \Omega, \sum_1^m {}_i \| \frac{b_i(x,t)}{\sqrt{\nu\psi}} \|_{p^*,p}, \|c\|_{q^*,q}, \sum_1^m {}_i \| \frac{d_i(x,t)}{\sqrt{\nu\psi}} \|_{r^*,r})$$

if $p < +\infty$ or $p^* = p = +\infty$, $q < +\infty$ or $q^* = q = +\infty$, $r < +\infty$ or $r^* = r = +\infty$, γ stands for a constant dependent on m, p, q, r, l. If $p = +\infty$ $[q = +\infty, r = +\infty]$ and $p^* < +\infty$ $[q^* < +\infty, r^* < +\infty]$ γ stands for a constant dependent on m, p, q, r, l, $\eta_1(\sigma)$ $[\eta_2(\sigma), \eta_3(\sigma)]$. The constants dependent on the same arguments will be denoted with the same symbol although their values are different.

It is possible to get, for any $\nu \in \mathbb{N}$, a sequence $\{u_{\nu,n}\}_n$ such that:

$$u_{\nu,n} \in C^1(\overline{Q(\tau_1,\tau_2)}), \quad u_{\nu,n} < h + \frac{1}{\nu} \text{ on } \partial\Omega \times [\tau_1,\tau_2] \text{ for any } n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \|u_{\nu,n} - u\|_{1,0,(\tau_1,\tau_2)} = 0\,.$$

Let us fix $n_\nu \in \mathbb{N}$ such that $\|u_{\nu,n_\nu} - u\|_{1,0,(\tau_1,\tau_2)} < \frac{1}{\nu}$, and assume for any $\nu \in \mathbb{N}$, $U_\nu = u_{\nu,n_\nu} - \frac{1}{\nu}$.

We get $U_\nu \in C^1(\overline{Q(\tau_1,\tau_2)})$, $U_\nu(x,t) < h$ on $\partial\Omega \times [\tau_1,\tau_2]$ for any $\nu \in \mathbb{N}$ and also:

$$\|U_\nu - u\|_{1,0,(\tau_1,\tau_2)} \leq \|u_{\nu,n} - u\|_{1,0,(\tau_1,\tau_2)} + \frac{1}{\nu}(\operatorname{meas} Q)^{1/2}\,.$$

Lemma 4.2. *Let $u \in \tilde{H}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$ $(0 \leq \tau_1 < \tau_2 \leq T)$ bounded from above on $\partial\Omega \times [\tau_1,\tau_2]$ and $k > \sup\limits_{[\tau_1,\tau_2]} {}^*u$, then $v = u - \min(u,k)$ in $Q(\tau_1,\tau_2)$ belongs to $\overset{0}{\tilde{H}}{}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$.* (7)

Let us fix $k_1 : \sup\limits_{[\tau_1,\tau_2]} {}^*u < k_1 < k$, then (Lemma 4.1) there exists a sequence of functions $\{u_n\}_n$ such that

$$u_n(x,t) \in C^1(\overline{Q(\tau_1,\tau_2)}), \quad u_n(x,t) < k_1 \text{ on } \partial\Omega \times [\tau_1,\tau_2] \text{ for any } n \in \mathbb{N}$$

and (2.2) holds.

As $u_n < k_1$ on $\partial\Omega \times [\tau_1,\tau_2]$, u_n is uniformly continuous in $\overline{Q(\tau_1,\tau_2)}$, it is possible to determine a $\delta = \delta(n)$, $\delta > 0$ such that for any (x,t) belonging to $\overline{Q(\tau_1,\tau_2)}$ with

$$d((x,t), \partial\Omega \times [\tau_1,\tau_2]) < \delta\,,$$

we get: $u_n(x,t) < k$.

Consequently, assuming $\psi_n = u_n - \min(u_n,k)$ in $\overline{Q(\tau_1,\tau_2)}$, for any $n \in \mathbb{N}$, we get:

$$\begin{gathered}\|\psi_n\|^2_{1,0,(\tau_1,\tau_2)} \leq 2\|u_n\|^2_{1,0,(\tau_1,\tau_2)} + 2k^2 \operatorname{meas} Q(\tau_1,\tau_2) \;\text{ and}\\ \operatorname{supp}\{\psi_n(x,t)\} \subset \Omega \times [\tau_1,\tau_2].^{(8)}\end{gathered} \tag{4.1}$$

We call, for $\mu \in \mathbb{N}$ great enough, $\alpha_\mu(t)$ the characteristic function of the interval $]\tau_1 + \frac{1}{\mu}, \tau_2 - \frac{1}{\mu}[$ and we assume in $\overline{Q(\tau_1,\tau_2)} : \chi_{\mu,n} = \alpha_\mu(t)\psi(x,t)$.

(7) the function v does not generally belong to $\tilde{H}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$.

(8) if $g : C \to \mathbb{R}$, we denote with the symbol $\operatorname{supp}\{g\}$ the support of g in C.

The functions $\chi_{\mu,n}$, $\frac{\partial \chi_{\mu,n}}{\partial x_i}$ $(i = 1, 2, \ldots, m)$ are bounded and have a compact support in $Q(\tau_1, \tau_2)$ for any $\mu, n \in \mathbb{N}$; moreover,

$$\lim_{n\to\infty} \|\chi_{\mu,n} - \psi_n\|_{1,0,(\tau_1,\tau_2)} = 0\,. \tag{4.2}$$

Thus, fixed μ and n, a sequence $\{d_\lambda\}$ of nonnegative equibounded functions of $C_0^\infty(Q(\tau_1, \tau_2))$ converging a.e. in $Q(\tau_1, \tau_2)$ to $\chi_{\mu,n}$ can be constructed via a well-known regularization procedure; (9) moreover, also the functions in the sequence $\{\frac{\partial d_\lambda}{\partial x_i}\}_\lambda$ are equibounded in $Q(\tau_1, \tau_2)$ and the sequence converges to $\frac{\partial \chi_{\mu,n}}{\partial x_i}$ a.e. in $Q(\tau_1, \tau_2)$ $(i = 1, 2, \ldots, m)$.

We deduce from LEBESGUE's theorem that the function $\chi_{\mu,n}$ belongs to $\overset{0}{\tilde H}{}^{1,0}(\nu\psi, Q(\tau_1, \tau_2))$ for any $\mu, n \in \mathbb{N}$.

Recalling (4.1) and (4.2), it is also proved that, for any $n \in \mathbb{N}$,

$$\psi_n \in \overset{0}{\tilde H}{}^{1,0}(\nu\psi, Q(\tau_1, \tau_2)).$$

From (4.1) we deduce that a subsequence $\{\psi_{n_k}\}_k$ weakly converging in $\overset{0}{\tilde H}{}^{1,0}(\nu\psi, Q(\tau_1, \tau_2))$ can be obtained from the sequence $\{\psi_n\}_n$.

On the other hand, ψ_n converges to v in $L^2(Q(\tau_1, \tau_2))$, so that $v \in \overset{0}{\tilde H}{}^{1,0}(\nu\psi, Q(\tau_1, \tau_2))$. (10)

Remark 4.1. In the particular case where $u \in C^1(\overline{Q(\tau_1, \tau_2)})$ and $k > u$ on $\partial\Omega \times [\tau_1, \tau_2]$, the function $v = u - \min(u, k)$ is the limit in $\overset{0}{\tilde H}{}^{1,0}(\nu\psi, Q(\tau_1, \tau_2))$ of a sequence $\{d_\lambda\}_\lambda$ of nonnegative equibounded functions of $C_0^\infty(Q(\tau_1, \tau_2))$ such that the functions of the sequence $\{\frac{\partial d_\lambda}{\partial x_i}\}_\lambda$ are equibounded $(i = 1, 2, \ldots, m)$.

Lemma 4.3. *Let us assume that Hypotheses* 2.1, 2.2, 2.3, 3.1, 3.2, *hold and let* $u(x, t)$ *be a subsolution of the equation* (0.1) *bounded from above on* $(\Omega \times \{t = 0\}) \cup (\partial\Omega \times [0, T])$*. Then, if* $0 \le \tilde\tau_1 < \tau < T$ *and* $k > \sup^* u$*, we get*

$$\begin{aligned}
&\int_{Q(\tilde\tau_1,\tau)} \left\{ \sum_1^m {}_{ij} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_1^m {}_i b_i \frac{\partial v}{\partial x_i} v + cuv + \right.\\
&\left. + \sum_1^m {}_i d_i u \frac{\partial v}{\partial x_i} \right\} dx\,dt + \frac{1}{2} \int_\Omega v^2(x, \tau)\,dx \le \\
&\le \int_{Q(\tilde\tau_1,\tau)} f v\,dx\,dt + \frac{1}{2} \int_\Omega v^2(x, \tilde\tau_1)\,dx
\end{aligned} \tag{4.3}$$

(9) see, e.g. [3, pp. 109–110].

(10) we get $|v - \psi_{n_k}| \le |u - u_{n_k}|$ a.e. in $Q(\tau_1, \tau_2)$ for any $k \in \mathbb{N}$.

where $v = u - \min(u,k)$ *in* Q. [11]

Let $\tilde{\tau}_1, \tau$ be such that $0 < \tilde{\tau}_1 < \tau < T$; setting $\tau_1 = \frac{\tau+T}{2}$, we denote with $C^\infty_\tau(Q)$ the set formed by those nonnegative functions of $C^\infty_0(Q)$ whose support is contained in $Q(0,\tau_1)$. Let $\varphi(x,t)$ be a function of $C^\infty_\tau(Q)$, we extend u, φ and the coefficients of $(0,1)$ in $\Omega\times]-\infty,+\infty[$, assuming that these functions are equal to zero in those points where they are not defined.

Set $\tau_2 = \frac{T-\tau}{2}$, we then define in $\Omega\times]-\infty,+\infty[$ and for any integer ϱ:

$$\Phi_\varrho(x,t) = \frac{\varrho}{\tau_2}\int_{t-(\tau_2/\varrho)}^{t} \varphi(x,\lambda)\,d\lambda\,, \quad U_\varrho(x,t) = \frac{\varrho}{\tau_2}\int_t^{t+(\tau_2/\varrho)} u(x,\lambda)\,d\lambda\,,$$

$$A_{i,\varrho}(x,t) = \frac{\varrho}{\tau_2}\int_t^{t+(\tau_2/\varrho)} \sum_1^m {}_j a_{ij}(x,\lambda)\frac{\partial u(x,\lambda)}{\partial x_i}\,d\lambda\,,$$

$$B_\varrho(x,t) = \frac{\varrho}{\tau_2}\int_t^{t+(\tau_2/\varrho)} \sum_1^m {}_i b_i(x,\lambda)\frac{\partial u(x,\lambda)}{\partial x_i}\,d\lambda\,,$$

$$C_\varrho(x,t) = \frac{\varrho}{\tau_2}\int_t^{t+(\tau_2/\varrho)} c(x,\lambda)u(x,\lambda)\,d\lambda\,, \quad F_\varrho(x,t) = \frac{\varrho}{\tau_2}\int_t^{t+(\tau_2/\varrho)} f(x,\lambda)\,d\lambda\,,$$

$$D_{i,\varrho}(x,t) = \frac{\varrho}{\tau_2}\int_t^{t+(\tau_2/\varrho)} d_i(x,\lambda)u(x,\lambda)\,d\lambda\,.$$ [12]

From (2.1), in correspondence with $\varphi = \Phi_\varrho(x,t)$, via an exchange in the order of the integrations with respect to t and λ, we get:

$$\begin{aligned}(4.4)\qquad &\int_Q \left\{ \sum_1^m {}_i A_{i,\varrho}\frac{\partial\varphi}{\partial x_i} + B_\varrho\varphi + C_\varrho\varphi + \sum_1^m {}_i D_{i,\varrho}\frac{\partial\varphi}{\partial x_i} + \frac{\partial U_\varrho}{\partial t}\varphi \right\} dx\,dt \le \\ &\le \int_Q F_\varrho\varphi\,dx\,dt\end{aligned}$$

for all φ belonging to the functional class $C^\infty_\tau(Q)$.

Let $h_2 : \sup^* u < h_2 < k$. Because $h_2 > \sup^* u$, there exists a sequence of functions $\{u_n\}_n$ such that $u_n \in C^1(\overline{Q})$, $u_n < h_2$ on $\partial\Omega\times[0,T]$ and satisfying (2.2). For all pairs of positive integer numbers ϱ and n, we assume:

$$U_{\varrho,n}(x,t) = \frac{\varrho}{\tau_2}\int_t^{t+(\tau_2/\varrho)} u_n(x,\lambda)\,d\lambda\,;$$

the function $U_{\varrho,n}(x,t)$ is defined in the closure of the cylinder $Q(0,\tau_1)$ and is therein of class C^1.

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(11) according to Lemma 4.2, $v \in \overset{0}{\tilde{H}}{}^{1,0}(\nu\psi, Q(\tau_1,\tau_2))$ for any $0 \le \tau_1 < \tau_2 \le T$.

(12) we will remark that, because $\nu \in L^1(\Omega)$ and $\psi \in L^1(0,T)$, $a_{ij}\frac{\partial u}{\partial x_i}$ is integrable in $]-\infty,+\infty[$.

Let us now introduce the function $V_{\varrho,n}(x,t)$ defined in Q assuming:

$$V_{\varrho,n}(x,t)=\begin{cases} U_{\varrho,n}(x,t)-\min(U_{\varrho,n}(x,t),k) & \text{in } Q(\tilde\tau_1,\tau)\\ 0 & \text{in } Q\setminus Q(\tilde\tau_1,\tau)\,.\end{cases}$$

Let $\{\chi_\lambda\}_\lambda$ be the sequence of nonnegative equibounded functions of $C_0^\infty(Q(\tilde\tau_1,\tau))$, having partial derivatives with respect to x_i equibounded ($i=1,2,\dots,m$), approaching $V_{\varrho,n}$ in $\overset{0}{\tilde H}{}^{1,0}(\nu\psi,Q(\tilde\tau_1,\tau))$ (see Remark 4.1). From (4.4), in correspondence with $\varphi=\chi_\lambda$, as λ diverges to $+\infty$, we can deduce the following relation:

$$\begin{aligned}\int_{Q(\tilde\tau_1,\tau)}\Big\{\sum_1^m {}_iA_{i,\varrho}\frac{\partial V_{\varrho,n}}{\partial x_i}+B_\varrho V_{\varrho,n}+C_\varrho V_{\varrho,n}+\sum_1^m {}_iD_{i,\varrho}\frac{\partial V_{\varrho,n}}{\partial x_i}+&\\ +\frac{\partial U_\varrho}{\partial t}V_{\varrho,n}\Big\}\,dx\,dt\le\int_{Q(\tilde\tau_1,\tau)}F_\varrho V_{\varrho,n}\,dx\,dt\,.&\end{aligned}\tag{4.5}$$

Setting, then, in Q:

$$V_\varrho(x,t)=U_\varrho(x,t)-\min(U_\varrho(x,t),k)\,,$$

we get:

$$\|V_\varrho\|^2_{1,0,(\tilde\tau_1,\tau)}\le\hat c\|u\|^2_{1,0}+k^2\,\mathrm{meas}\,Q,\ ^{(13)}\quad\text{for any }\varrho\in\mathbb N\,.$$

The sequence $\{V_{\varrho,n}\}_n$ converges to V_ϱ in both $\overset{0}{\tilde H}{}^{1,0}(\nu\psi,Q(\tilde\tau_1,\tau))$ and $L^{2,\infty}(Q(\tilde\tau_1,\tau))$; [14] accordingly, the function V_ϱ belongs to $\overset{0}{\tilde H}{}^{1,0}(\nu\psi,Q(\tilde\tau_1,\tau))\cap L^{2,\infty}(Q(\tilde\tau_1,\tau))$. From (4.5) we deduce, as n goes to $+\infty$, the following:

$$\begin{aligned}\int_{Q(\tilde\tau_1,\tau)}\Big\{\sum_1^m {}_iA_{i,\varrho}\frac{\partial V_\varrho}{\partial x_i}+B_\varrho V_\varrho+C_\varrho V_\varrho+\sum_1^m {}_iD_{i,\varrho}\frac{\partial V_\varrho}{\partial x_i}+&\\ +\frac{\partial U_\varrho}{\partial t}V_\varrho\Big\}\,dx\,dt\le\int_{Q(\tilde\tau_1,\tau)}F_\varrho V_\varrho\,dx\,dt\,.&\end{aligned}\tag{4.6}$$

Let us verify, for example, that:

$$\lim_{n\to\infty}\int_{Q(\tilde\tau_1,\tau)}A_{i,\varrho}\frac{\partial V_{\varrho,n}}{\partial x_i}\,dx\,dt=\int_{Q(\tilde\tau_1,\tau)}A_{i,\varrho}\frac{\partial V_\varrho}{\partial x_i}\,dx\,dt\,.$$

---

[13] the constant $\hat c$ depends on $\|\psi\|_{\infty,(\tilde\tau_1,\tau_1)}$, $\|\psi^{-1}\|_{\infty,(\tilde\tau_1,\tau_1)}$.

[14] we will remark that

$$\|V_{\varrho,n}-V_\varrho\|^2_{1,0,(\tilde\tau_1,\tau)}\le\hat c\|u_n-u\|^2_{1,0}+o(\frac{1}{n})\ \text{for any } n\in\mathbb N\,.$$

It will suffice to prove that:

$$\lim_{n\to\infty}\int_{Q(\tilde\tau_1,\tau)}|A_{i,\varrho}|\left|\frac{\partial V_{\varrho,n}}{\partial x_i}-\frac{\partial V_\varrho}{\partial x_i}\right|dx\,dt=0\,.$$

We get:

$$\begin{aligned}&\int_{Q(\tilde\tau_1,\tau)}|A_{i,\varrho}|\left|\frac{\partial V_{\varrho,n}}{\partial x_i}-\frac{\partial V_\varrho}{\partial x_i}\right|dx\,dt\le\\&\le\hat c\left\|\frac{A_{i,\varrho}}{\sqrt{\nu(x)}}\right\|_{2,2,(\tilde\tau_1,\tau)}\|V_{\varrho,n}-V_\varrho\|_{1,0,(\tilde\tau_1,\tau)}\le\\&\le\hat c\left\|\frac{a_{i,j}}{\nu\psi}\right\|_\infty\cdot\|u\|_{1,0}\cdot\|V_{\varrho,n}-V_\varrho\|_{1,0,(\tilde\tau_1,\tau)}\ \text{ for all }\ n\in\mathbb{N}\,.\end{aligned}$$

We can rewrite (4.6) as follows:

$$\begin{aligned}&\int_{Q(\tilde\tau_1,\tau)}\left\{\sum_1^m{}_iA_{i,\varrho}\frac{\partial V_\varrho}{\partial x_i}+B_\varrho V_\varrho+C_\varrho V_\varrho+\sum_1^m{}_iD_{i,\varrho}\frac{\partial V_\varrho}{\partial x_i}\right\}dx\,dt+\\&+\frac12\int_{\Omega_\varrho(\tau,k)}|U_\varrho(x,\tau)-k|^2\,dx\le\\&\le\int_{Q(\tilde\tau_1,\tau)}F_\varrho V_\varrho\,dx\,dt+\frac12\int_{\Omega_\varrho(\tilde\tau_1,k)}|U_\varrho(x,\tilde\tau_1)-k|^2\,dx.\ ^{(15)}\end{aligned}\tag{4.7}$$

We call $v=u-\min(u,k)$ in Q.
Let us remark now that we get:

$$\|V_\varrho-v\|^2_{1,0,(\tilde\tau_1,\tau)}\le\hat c\int_{Q(\tilde\tau_1,\tau)}|U_\varrho-u|^2+\nu\sum_1^m{}_i|\frac{\partial U_{\varrho,n}}{\partial x_i}-\frac{\partial u}{\partial x_i}|^2\,dx\,dt+o(\frac1\varrho)$$

for any $\varrho\in\mathbb{N}$; so, V_ϱ converges to v in $\tilde H^{1,0}(\nu\psi,Q(\tilde\tau_1,\tau))$. (16)

Moreover, because U_ϱ converges to u in $L^2(\Omega)$ uniformly with respect to $t\in[\tilde\tau_1,\tau]$, it is proved that V_ϱ converges to v in $L^{2,\infty}(Q(\tilde\tau_1,\tau))$.

From (4.7), the conclusion now follows via another passage to the limit.

---

(15) we will denote with $\Omega_\varrho(t,k)$ the set of those points of Ω in which $U_\varrho(x,t)>k$.

(16) we get (see [5], p. 85):

$$\lim_{\varrho\to\infty}\int_{Q(\tilde\tau_1,\tau)}|U_\varrho-u|^2+\nu\sum_1^m{}_i|\frac{\partial U_\varrho}{\partial x_i}-\frac{\partial u}{\partial x_i}|^2\,dx\,dt=0\,.$$

For example, we prove that:

$$\lim_{\varrho\to\infty}\int_{Q(\tilde\tau_1,\tau)} C_\varrho V_\varrho\,dx\,dt = \int_{Q(\tilde\tau_1,\tau)} cuv\,dx\,dt\,.$$

We get:

$$\begin{aligned}
&|\int_{Q(\tilde\tau_1,\tau)} C_\varrho V_\varrho - cuv\,dx\,dt| \le \\
&\le \|C_\varrho\|_{\frac{\alpha_2^*}{\alpha_2^*-1},\frac{\alpha_2}{\alpha_2-1},(\tilde\tau_1,\tau)}\|V_\varrho - v\|_{\alpha_2^*,\alpha_2,(\tilde\tau_1,\tau)}+ \\
&+\|C_\varrho - cu\|_{\frac{\alpha_2^*}{\alpha_2^*-1},\frac{\alpha_2}{\alpha_2-1}}\|v\|_{\alpha_2^*,\alpha_2} \le \\
&\le \beta\|c\|_{q^*,q}\|u\|_{\alpha_2^*,\alpha_2}(\|V_\varrho - v\|_{1,0,(\tilde\tau_1,\tau)} + \|V_\varrho - v\|_{2,\infty,(\tilde\tau_1,\tau)})+ \\
&\|C_\varrho - cu\|_{\frac{\alpha_2^*}{\alpha_2^*-1},\frac{\alpha_2}{\alpha_2-1}}\|v\|_{\alpha_2^*,\alpha_2},\ ^{(17)}\ \text{ for all } \varrho\in\mathbb{N}\,.
\end{aligned}$$

If $\tilde\tau_1 = 0$, assumed $\tau > 0$, it will suffice to consider $\tau_n = \frac{\tau}{n+1}$ for $n\in\mathbb{N}$. Accordingly, we get:

$$\begin{aligned}
&\int_{Q(\tau_n,\tau)}\{\sum_1^m {}_{ij}a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_1^m {}_i b_i\frac{\partial v}{\partial x_i}v + cuv + \sum_1^m {}_i d_i u\frac{\partial v}{\partial x_i}\}\,dx\,dt+ \\
&+\frac{1}{2}\int_\Omega v^2(x,\tau)\,dx \le \int_{Q(\tau_n,\tau)} fv\,dx\,dt + \frac{1}{2}\int_\Omega v^2(x,\frac{\tau}{n})\,dx \text{ for any } n\in\mathbb{N}\,.
\end{aligned}$$

The conclusion will follow via another passage to the limit for n approaching $+\infty$, recalling that the function $v(x,t)$ is continuous in $[0,T]$ to values in $L^2(\Omega)$.

Lemma 4.4. *Let us assume Hypotheses* 2.1, 2.2, 2.3, 3.1, 3.2, 3.3, 3.4 *with* $\varrho = 0$, *hold and let* $u(x,t)$ *be a subsolution of the equation* (0.1) *bounded from above on* $(\Omega\times\{t=0\})\cup(\partial\Omega\times[0,T])$.

Then, if $k > \max(0, \operatorname{ess\,sup}_\Omega u(x,0), \sup^* u)$, *we get:*

$$\|v\|_{1,0} + \|v\|_{2,\infty} \le \gamma\|f\psi_k\|_{s^*,s}$$

where $v = u - \min(u,k)$ *in* Q, ψ_k *is the characteristic function of the set of points of* Q *in which* $u(x,t) > k$ *and* s^*, s *are defined by the following formulas:*

$$\frac{1}{s^*} + \frac{1}{\alpha_4^*} = 1\,,\quad \frac{1}{s} + \frac{1}{\alpha_4} = 1\,.$$

---

(17) we will remark that the function which equals $V_\varrho - v$ in $Q(\tilde\tau_1,\tau)$ and vanishes in the remaining points belongs to $\overset{0}{\tilde H}{}^{1,0}(\nu\psi,Q)\cap L^{2,\infty}(Q)$ and that C_ϱ converges to cu in $L^{\alpha_2^*/(\alpha_2^*-1)}(Q)$.

According to our hypothesis, we deduce:

$$\int_Q c\varphi + \sum_1^m {}_i d_i \frac{\partial \varphi}{\partial x_i}\, dx\, dt \geq 0 \tag{4.8}$$

for any $\varphi \in C_0^\infty(Q)$ such that $\varphi(x,t) \geq 0$ a.e. in Q.

From (4.8), via the same procedure adopted in Lemma 4.3, we deduce

$$\int_{Q(\tilde\tau_1,\tau)} c\sigma + \sum_1^m {}_i d_i \frac{\partial \sigma}{\partial x_i}\, dx\, dt \geq 0\,, \tag{4.9}$$

for any $\tilde\tau_1, \tau : 0 \leq \tilde\tau_1 < \tau < T$.

Recalling that $k > \max(0, \operatorname{ess\,sup}_\Omega u(x,0), \sup^* u)$, from (4.3) and (4.9) we get:

$$\int_{Q(0,\tau)} \left\{ \sum_1^m {}_{ij} a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_1^m {}_i b_i \frac{\partial v}{\partial x_i} v + cv^2 + \right.$$
$$\left. + \sum_1^m {}_i d_i v \frac{\partial v}{\partial x_i} \right\} dx\, dt + \frac{1}{2}\int_\Omega v^2(x,\tau)\, dx \leq \int_{Q(0,\tau)} fv\, dx\, dt\,.$$

With slight modifications of the procedure followed in Lemma 4.1 of [7] the conclusion easily follows.

5. Proof of Theorem

Let us first examine the particular case where $\varrho = 0$.

There is no loss of generality if we assume that

$$\alpha_4^*(1 - \frac{1}{g^*}) \geq \alpha_4(1 - \frac{1}{g})\,; \tag{5.1}$$

in fact, the first term of the preceding inequality is greater than 2, so that the inequality will hold by decreasing g.

Let $\bar k$ be a number greater than $\max(0, \operatorname{ess\,sup}_\Omega u(x,0), \sup^* u)$ and h and k two numbers such that $\bar k \leq k < h$.

Assumed that $v = u - \min(u,k)$ in Q, we get:

$$\|v\|_{\alpha_4^*,\alpha_4} \geq (h-k)(\int_0^\tau [\operatorname{mis}_x \Omega(t,h)]^{\alpha_4/\alpha_4^*}\, dt)^{1/\alpha_4} = (h-k)\|\psi_h\|_{\alpha_4^*,\alpha_4}\,. \tag{5.2}$$

On the other hand (Lemma 4.4 and Hypothesis 3.1)

$$\|v\|_{\alpha_4^*,\alpha_4} \leq \gamma \|f\psi_k\|_{s^*,s}\,,$$

then from (5.2), we get:

$$\|\psi_h\|_{\alpha_4^*,\alpha_4} \le \frac{\gamma}{(h-k)}\|f\|_{g^*,g}\|\psi_k\|_{\lambda^*,\lambda} \tag{5.3}$$

where

$$\begin{aligned}\frac{1}{\lambda^*} &= \frac{1}{s^*} - \frac{1}{g^*} = 1 - \frac{1}{\alpha_4^*} - \frac{1}{g^*} > \frac{1}{\alpha_4^*}\,,\\ \frac{1}{\lambda} &= \frac{1}{s} - \frac{1}{g} = 1 - \frac{1}{\alpha_4} - \frac{1}{g} > \frac{1}{\alpha_4}\,.\end{aligned}$$

We get:

$$\frac{\lambda}{\lambda^*} = \frac{1-\frac{1}{\alpha_4^*}-\frac{1}{g^*}}{1-\frac{1}{\alpha_4}-\frac{1}{g}} \ge \frac{\frac{\alpha_4}{\alpha_4^*}(1-\frac{1}{g})-\frac{1}{\alpha_4^*}}{1-\frac{1}{\alpha_4}-\frac{1}{g}} = \frac{\alpha_4}{\alpha_4^*}\,,$$

from which, a.e. in $]0,T[$:

$$[\operatorname{meas}_x \Omega(t,k)]^{\lambda/\lambda^*} \le l^{\alpha_4}[\operatorname{meas}_x \Omega(t,k)]^{\alpha_4/\alpha_4^*}\,. \tag{5.4}$$

From (5.3) and (5.4) we deduce that

$$\|\psi_h\|_{\alpha_4^*,\alpha_4} \le \frac{\gamma}{(h-k)}\|f\|_{g^*,g}(\|\psi_k\|_{\alpha_4^*,\alpha_4})^{\vartheta}\,,$$

where $\vartheta = \frac{\alpha_4}{2}(1-\frac{1}{g}) > 1$.

If we assume for any $k \ge \bar{k}$:

$$\eta(k) = \|\psi_k\|_{\alpha_4^*,\alpha_4}$$

then we get (see [8], p. 212):

$$\eta(\bar{k}+d) = 0\,,\ \text{where } d = \gamma\|f\|_{g^*,g}\|\psi_{\bar{k}}\|^{\vartheta-1}_{\alpha_4^*,\alpha_4}2^{\vartheta/(\vartheta-1)}\,. \tag{5.5}$$

Remarking that $\|\psi_{\bar{k}}\|_{\alpha_4^*,\alpha_4} \le l$, from (5.5) we get:

$$u(x,t) \le \bar{k} + 2^{\vartheta/(\vartheta-1)}l^{\vartheta-1}\gamma\|f\|_{g^*,g}$$

a.e. in Q, from which the proof follows in the case where $\varrho = 0$.

Finally, if ϱ is a nonnegative constant, the proof follows as in Theorem of § 3 of [1].

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