Classical global solutions of the initial boundary value problems for a class of nonlinear parabolic equations

Chen Guowang*

Abstract. The existence, uniqueness and regularities of the generalized global solutions and classical global solutions to the equation

$$u_t = -A(t)u_{x^4} + B(t)u_{x^2} + g(u)_{x^2} + f(u)_x + h(u_x)_x + G(u)$$

with the initial boundary value conditions

$$u(-\ell, t) = u(\ell, t) = 0, \quad u_{x^2}(-\ell, t) = u_{x^2}(\ell, t) = 0, \quad u(x, 0) = \varphi(x),$$

or with the initial boundary value conditions

$$u_x(-\ell, t) = u_x(\ell, t) = 0, \quad u_{x^3}(-\ell, t) = u_{x^3}(\ell, t) = 0, \quad u(x, 0) = \varphi(x),$$

are proved. Moreover, the asymptotic behavior of these solutions is considered under some conditions.

Keywords: nonlinear parabolic equation, initial boundary value problem, classical global solutions

Classification: 35K35, 35K60

1. Introduction

In the present paper, we are going to consider the following nonlinear parabolic equation

$$u_t = -A(t)u_{x^4} + B(t)u_{x^2} + g(u)_{x^2} + f(u)_x + h(u_x)_x + G(u) \tag{1}$$

where $u(x, t)$ is an unknown function, $A(t)$ and $B(t)$ are the given functions defined on $[0, T]$ $(T > 0)$, $g(s)$, $f(s)$, $h(s)$ and $G(s)$ are the given nonlinear functions defined in $\mathbb{R}$. Partial differential equations of this kind are often found in the study of biology, chemistry, physics and engineering technology. For example, in the study of growth and dispersal in populations, there arises the model equation [1]

$$u_t = -a_1 u_{x^4} + a_2 u_{x^2} + (au^3)_{x^2} + f(u), \tag{2}$$

*The project is supported by the National Natural Science Foundation of China.

which is a special case of (1). Here $a_1 > 0$, $a > 0$ and $a_2 \neq 0$ are constants. The existence and uniqueness of the classical global solutions of the periodic boundary value problem for the nonlinear parabolic equation

$$u_t = -a_1 u_{x^4} + a_2 u_{x^2} + (g(u))_{x^2} + f(u) \tag{$*$}$$

have been proved by the integral equation method in [6]. In [7] the initial value problem for the nonlinear system of parabolic type which is an analogous equation $(*)$ has been studied by the integral estimates.

In the following, we consider the initial boundary value problem for the equation (1)

$$\begin{gathered} u(-\ell,t) = u(\ell,t) = 0, \quad u_{x^2}(-\ell,t) = u_{x^2}(\ell,t) = 0, \quad 0 \le t \le T, \\ u(x,0) = \varphi(x), \quad x \in \overline{\Omega} = [-\ell,\ell], \end{gathered} \tag{3}$$

in which $\varphi(x)$ is the given function. Then, we consider the initial boundary value problem for the equation (1) $(f(u)_x \equiv 0)$

$$\begin{gathered} u_x(-\ell,t) = u_x(\ell,t) = 0, \quad u_{x^3}(-\ell,t) = u_{x^3}(\ell,t) = 0, \quad 0 \le t \le T, \\ u(x,0) = \varphi(x), \quad x \in \overline{\Omega}. \end{gathered} \tag{4}$$

By means of integral estimates and Galerkin method we prove the existence and regularities of the generalized global solutions and the classical global solutions to problems (1), (3) and (1), (4). We also prove the uniqueness of the solutions and asymptotic behavior of these solutions as $t \to \infty$. Let

$$\begin{gathered} (u,v) = \int_{-\ell}^{\ell} uv\,dx, \quad |u(\cdot,t)|^2_{L_2(\Omega)} = (u,u), \\ [u,v] = \int_0^t (u,v)\,dt = \int\int_{Q_t} uv\,dx\,dt, \\ \|u\|^2_{L_2(Q_t)} = [u,u], \end{gathered}$$

where $Q_t = \Omega \times [0,t]$.

In an other place the usual symbols of Sobolev spaces are used.

2. Initial boundary value problems (1), (3)

Let $\{y_n(x)\}$ be the orthonormal complete system composed of the eigenfunctions of the following boundary problem of the ordinary differential equation [2]

$$y^{(4)} = \lambda y, \tag{5}$$

$$y(-\ell) = y(\ell) = 0,\ y''(-\ell) = y''(\ell) = 0 \tag{6}$$

corresponding to eigenvalues λ_n $(n = 1, 2, \dots)$. Then the Galerkin approximate solution $u_N(x,t)$ for the problems (1), (3) can be expressed as

$$u_N(x,t) = \sum_{n=1}^{N} \alpha_{N,n}(t) y_n(x), \tag{7}$$

where $\alpha_{N,n}(t)$ $(n = 1, 2, \dots, N)$ are the undetermined coefficients and N is a natural number. According to the Galerkin method, the undetermined coefficients $\alpha_{N,s}(t)$ $(s = 1, 2, \dots, N)$ satisfy the system of ordinary differential equations

$$\begin{aligned}(u_{Nt}, y_s) + (A(t)u_{Nx^4}, y_s) - (B(t)u_{Nx^2}, y_s)& \\ = (g(u_N)_{x^2} + f(u_N)_x &+ h(u_{Nx})_x + G(u_N), y_s)\end{aligned} \tag{8}$$

with the initial condition

$$(u_N(x,0), y_s) = (\varphi(x), y_s), \tag{9}$$

where $s = 1, 2, \dots, N$.

Lemma 1. *Suppose that the following conditions are satisfied:*

(1) *There exist constants $a_0 > 0$, $b > 0$, such that $A(t) \geq a_0 > 0$, $B(t) \geq -b$ on $[0,T]$;*
(2) *$g \in C^2$; $\forall\, s \in \mathbb{R}$, $g'(s) \geq 0$ and $|g'(s)| \leq K_1 |s|^{\xi+1}$, $|g''(s)| \leq K_1 |s|^{\xi}$, where $0 < \xi < 3$, K_1 is a positive number;*
(3) *$f \in C^1$, $F(u) = \int_0^u f(s)\,ds$ and $|f(s)| \leq K_2 |s|^{\eta+1}$, $|f'(s)| \leq K_2 |s|^{\eta}$, where $0 < \eta < 6$ and $K_2 > 0$ is a constant;*
(4) *$h \in C^1$; $\forall\, s \in \mathbb{R}$, $h'(s) \geq 0$ and $|h(s)| \leq K_3 |s|^{\mu+1}$, where $0 < \mu < \frac{4}{3}$ and $K_3 > 0$ is a constant;*
(5) *$G \in C^1$; $\forall\, s \in \mathbb{R}$, $G'(s) \leq \gamma$ and $|G'(s)| \leq K_4 |s|^{\zeta}$, where $0 < \zeta < 8$; $K_4 > 0$ and γ are constants;*
(6) *$\varphi \in V_2$, and φ satisfies the boundary conditions, where V_2 is the closed linear extension of the orthonormal complete system $\{y_n(x)\}$ in $H^2(\Omega)$.*

Then for any N there exists a solution $u_N(x,t)$ of the initial value problems (8), (9) *in $[0,T]$ and there is the estimation*

$$|u_N(\cdot,t)|^2_{H^2(\Omega)} + \|u_N\|^2_{H^4(Q_t)} \leq C, \quad t \in [0,T], \tag{10}$$

where C is a constant independent of N.

Proof: Multiplying (8) by $\alpha_{N,s}(t)$, summing up the products for $s = 1, 2, \dots N$, integrating by parts and integrating with respect to t, we get

$$\begin{aligned}&|u_N(\cdot,t)|^2_{L_2(\Omega)} + 2[A(t)u_{Nx^2}, u_{Nx^2}] + 2[B(t)u_{Nx}, u_{Nx}] \\ &= 2[g(u_N)_{x^2} + f(u_N)_x + h(u_{Nx})_x + G(u_N), u_N] + |\varphi|^2_{L_2(\Omega)}.\end{aligned} \tag{11}$$

We have

$$(g(u_N)_{x^2}, u_N) = -(g'(u_N)u_{Nx}, u_{Nx}) \le 0, \tag{12}$$

$$(f(u_N)_x, u_N) = -(f(u_N), u_{Nx}) = -\int_{-\ell}^{\ell} \frac{\partial F}{\partial x}\, dx = 0, \tag{13}$$

$$\begin{aligned}(h(u_{Nx})_x, u_N) = -\int_{-\ell}^{\ell} h(u_{Nx})u_{Nx}\, dx \le \frac{1}{2}|h(0)|^2_{L_2(\Omega)} \\ + \frac{1}{2}|u_{Nx}|^2_{L_2(\Omega)},\end{aligned} \tag{14}$$

$$(G(u_N), u_N) \le (\gamma + \frac{1}{2})|u_N|^2_{L_2(\Omega)} + \frac{1}{2}|G(0)|^2_{L_2(\Omega)}\,. \tag{15}$$

Substituting formulas (12)–(14) into formula (11), we get

$$\begin{aligned}|u(\cdot,t)|^2_{L_2(\Omega)} + 2a_0\|u_{Nx^2}\|^2_{L_2(Q_t)} \le (2b+1)\|u_{Nx}\|^2_{L_2(Q_t)} \\ +(2\gamma+1)\|u_N\|^2_{L_2(Q_t)} + \|h(0)\|^2_{L_2(Q_t)} + \|G(0)\|^2_{L_2(Q_t)} + |\varphi|^2_{L_2(\Omega)}\,.\end{aligned} \tag{16}$$

By means of interpolation formula for $|u_{Nx}|^2_{L_2(\Omega)}$, from (16) it follows

$$\begin{aligned}&|u_N(\cdot,t)|^2_{L_2(\Omega)} + \|u_{Nx^2}\|^2_{L_2(Q_t)} \\ &\quad \le C_1\|u_N\|^2_{L_2(Q_t)} + C_2\left\{\|h(0)\|^2_{L_2(Q_t)} + \|G(0)\|^2_{L_2(Q_t)} + |\varphi|^2_{L_2(\Omega)}\right\}.\end{aligned}$$

Thus, by Gronwall's inequality we obtain

$$\begin{aligned}&|u_N(\cdot,t)|^2_{L_2(\Omega)} + \|u_{Nx^2}\|^2_{L_2(Q_t)} \\ &\quad \le C_3\left\{\|h(0)\|^2_{L_2(Q_t)} + \|G(0)\|^2_{L_2(Q_t)} + |\varphi|^2_{L_2(\Omega)}\right\}, \quad \forall\, t \in [0,T],\end{aligned} \tag{17}$$

where C_3 is a constant independent of N.

Multiplying (8) by $\lambda_s \alpha_{N,s}(t)$, summing up the products for $s = 1, 2, \dots, N$, integrating by parts and integrating with respect to t, we get

$$\begin{aligned}|u_{Nx^2}(\cdot,t)|^2_{L_2(\Omega)} + 2a_0\|u_{Nx^4}\|^2_{L_2(Q_t)} \le 2b|u_{Nx^3}|^2_{L_2(Q_t)} + 2[g(u_N)_{x^2} \\ +f(u_N)_x + h(u_N)_x + G(u_N), u_{Nx^4}] + |\varphi_{x^2}|^2_{L_2(\Omega)}\,.\end{aligned} \tag{18}$$

By means of interpolation formulas [3], assumptions, Hölder's inequality and

Young's inequality, we have

$$|(g(u_N)_{x^2}, u_{Nx^4})| \le |g''(u_N)|_{L_\infty(\Omega)} |u_{Nx}|^2_{L_4(\Omega)} |u_{Nx^4}|_{L_2(\Omega)}$$
$$+|g'(u_N)|_{L_\infty(\Omega)} |u_{Nx^2}|_{L_2(\Omega)} |u_{Nx^4}|_{L_2(\Omega)} \le C_4 |u_N|^{\frac{\xi}{8}}_{H^4(\Omega)} |u_N|^{\frac{5}{8}}_{H^4(\Omega)}.$$

$$(19)\quad |u_{Nx^4}|_{L_2(\Omega)} + C_5 |u_N|^{\frac{\xi+1}{8}}_{H^4(\Omega)} |u_N|^{\frac{1}{2}}_{H^4(\Omega)} |u_{Nx^4}|_{L_2(\Omega)} \le \varepsilon |u_{Nx^4}|^2_{L_2(\Omega)} + C_6;$$

$$\begin{aligned} |(f(u_N)_x, u_{Nx^4})| &\le C_7 |u_N|^{\frac{7}{8}}_{H^4(\Omega)} |u_N|^{\frac{1}{4}}_{H^4(\Omega)} |u_{Nx^4}|_{L_2(\Omega)} \\ &\le \varepsilon |u_{Nx^4}|^2_{L_2(\Omega)} + C_8; \end{aligned} \tag{20}$$

$$\begin{aligned} |(h(u_{Nx})_x, u_{Nx^4})| &\le C_9 |u_N|^{\frac{3\mu}{8}}_{H^4(\Omega)} |u_N|^{\frac{1}{2}}_{H^4(\Omega)} |u_{Nx^4}|_{L_2(\Omega)} \\ &\le \varepsilon |u_{Nx^4}|^2_{L_2(\Omega)} + C_{10}; \end{aligned} \tag{21}$$

$$\begin{aligned} |(G(u_N), u_{Nx^4})| &\le |G'(u_N)|_{L_\infty(\Omega)} |u_{Nx}|_{L_2(\Omega)} |u_{Nx^3}|_{L_2(\Omega)} \\ &\le C_{11} |u_N|^{1+\frac{\zeta}{8}}_{H^4(\Omega)} \le \varepsilon |u_{Nx^4}|^2_{L_2(\Omega)} + C_{12}; \end{aligned} \tag{22}$$

$$2b|u_{Nx^3}|^2_{L_2(\Omega)} \le C_{13} |u_N|^{\frac{3}{2}}_{H^4(\Omega)} \le \varepsilon |u_{Nx^4}|^2_{L_2(\Omega)} + C_{14}\,. \tag{23}$$

Substituting formulas (19)–(23) into formula (18), we obtain

$$|u_{Nx^2}(\cdot,t)|^2_{L_2(\Omega)} + \|u_{Nx^4}\|^2_{L_2(Q_t)} \le C_{15}(1+|\varphi_{x^2}|^2_{L_2(\Omega)}) \le c_{16}, \quad \forall t \in [0,T], \tag{24}$$

where C_{16} is a constant independent of N. From (17) and (24) it follows (10). The existence of the solution $\alpha_{N,s}(t)$ $(s = 1, 2, \dots, N)$ is global for $0 \le t \le T$, can be proved by the fixed-point technique and the a priori bounded estimation for $\alpha_{N,s}(t)$ $(s = 1, 2, \dots, N)$, which follows immediately from the uniform boundedness of the approximate solution $u_N(x,t)$ given in (10) and the expressions $\alpha_{N,s}(t) = (u_N, y_s)$ for $(s = 1, 2, \dots, N)$. Lemma 1 has been proved. □

Lemma 2 ([4]). *Let $G(z_1, z_2, \dots, z_h)$ be the function of the variables $z_1, z_2, \dots, z_h$ and suppose that G is continuously differentiable for k-times $(k \ge 1)$ with respect to every variable. Let $z_i(x,t) \in L_\infty([0,T];\ H^k(\Omega))$ $(i = 1, 2, \dots, h)$, then the estimation*

$$\int_{-\ell}^{\ell} |D_x^k G(z_1(x,t), \dots, z_h(x,t))|^2\, dx \le C(M,k,h) \sum_{i=1}^{h} |z_i|^2_{H^k(\Omega)}$$

holds, where

$$M = \max_{i=1,\dots,h} \max_{\substack{0 \le t \le T \\ -\ell \le x \le \ell}} |z_i(x,t)|, \ D_x = \frac{\partial}{\partial x}\,.$$

Lemma 3. *Suppose that the following conditions are satisfied:*

(1) *The conditions of Lemma 1 are satisfied;*
(2) $g \in C^{2k}$, $f \in C^{2k-1}$, $h \in C^{2k-1}$, $G \in C^{2k-1}$ *and* $\varphi \in V_{2k}$ ($k \geq 1$ *is a natural number*);
(3)

$$\frac{\partial^\beta}{\partial x^\beta}[g(u)_{x^2}]|_{x=-\ell} = \frac{\partial^\beta}{\partial x^\beta}[g(u)_{x^2}]|_{x=\ell} = 0,\ \beta = 0, 2, \ldots, 2(k-1), \tag{25}$$

$$\frac{\partial^\beta}{\partial x^\beta}[f(u)_x]|_{x=-\ell} = \frac{\partial^\beta}{\partial x^\beta}[f(u)_x]|_{x=\ell} = 0,\ \beta = 0, 2, \ldots, 2(k-1), \tag{26}$$

$$\frac{\partial^\beta}{\partial x^\beta}[h(u_x)_x]|_{x=-\ell} = \frac{\partial^\beta}{\partial x^\beta}[h(u_x)_x]|_{x=\ell} = 0,\ \beta = 0, 2, \ldots, 2(k-1), \tag{27}$$

$$\frac{\partial^\beta}{\partial x^\beta}G(u)|_{x=-\ell} = \frac{\partial^\beta}{\partial x^\beta}G(u)|_{x=\ell} = 0,\ \beta = 0, 2, \ldots, 2(k-1). \tag{28}$$

Then there is the estimate for the approximate solution $u_N(x,t)$ *as*

$$|u_N(\cdot,t)|^2_{H^k(\Omega)} + \|u_N\|^2_{H^{2(k+1)}(Q_t)} \leq C_{17}, \quad \forall\, t \in [0,T], \tag{29}$$

where C_{17} *is a constant independent of* N.

Proof: In order to get further estimates of $u_N(x,t)$, the following properties of the orthonormal complete system $\{y_n(x)\}$ on the boundary points of Ω are used:

$$y_s^{(L)}(-\ell) = y_s^{(L)}(\ell) = 0, \quad L = 2\nu,\ \nu = 0, 1, \ldots, \tag{30}$$

where (L) denotes the order of the derivatives of the function $y_s(x)$.

By means of the method of induction we shall prove the estimation (29). It is known from Lemma 1 that the estimation (29) holds when $k = 1$. Suppose that when $k = p$ estimation (29) holds. Multiplying (8) by $\lambda_s^{p+1}\alpha_{N,s}(t)$, summing up the products for $s = 1, 2, \ldots, N$, taking notice of (25)–(28) and (30) and integrating by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}|u_{Nx^{2(p+1)}}(\cdot,t)|^2_{L_2(\Omega)} + 2a_0|u_{Nx^{2(p+2)}}|^2_{L_2(\Omega)} \leq 2b|u_{Nx^{2(p+2)-1}}|^2_{L_2(\Omega)}\\ &+ C_{18}\big\{|g(u_N)_{x^{2(p+1)}}|_{L_2(\Omega)} + |f(u_N)_{x^{2p+1}}|_{L_2(\Omega)} + |h(u_{Nx})_{x^{2p+1}}|_{L_2(\Omega)}\\ &+ |G(u_N)_{x^{2p}}|_{L_2(\Omega)}\big\} \cdot |u_{Nx^{2(p+2)}}|_{L_2(\Omega)}.\end{aligned} \tag{31}$$

From Lemma 2, assumptions of the method of induction and interpolation formulas, it follows

$$|g(u_N)_{x^{2(p+1)}}|_{L_2(\Omega)} \leq C_{19}|u_N|_{H^{2(p+1)}(\Omega)} \leq C_{20} + C_{21}|u_{Nx^{2(p+2)}}|^{\frac{1}{2}}_{L_2(\Omega)}. \tag{32}$$

In a similar manner we have

$$|f(u_N)_{x^{2p+1}}|_{L_2(\Omega)} \le C_{22} + C_{23}|u_{Nx^{2(p+2)}}|^{\frac{1}{4}}_{L_2(\Omega)}\,; \tag{33}$$

$$|h(u_{Nx})_{x^{2p+1}}|_{L_2(\Omega)} \le C_{24} + C_{25}|u_{Nx^{2(p+2)}}|^{\frac{1}{2}}_{L_2(\Omega)}\,; \tag{34}$$

$$u_{Nx^{2(p+2)-1}}|^2_{L_2(\Omega)} \le C_{26} + C_{27}|u_{Nx^{2(p+2)}}|^{\frac{3}{2}}_{L_2(\Omega)}\,. \tag{35}$$

Substituting formulas (32)–(35) into (31) and using Young's inequality, we obtain

$$\frac{d}{dt}|u_{Nx^{2(p+1)}}(\cdot,t)|^2_{L_2(\Omega)} + |u_{Nx^{2(p+2)}}|^2_{L_2(\Omega)} \le C_{28}\,.$$

Hence

$$|u_N(\cdot,t)|^2_{H^{2(p+1)}(\Omega)} + \|u_{Nx^{2(p+2)}}\|^2_{L_2(Q_t)} \le C_{29}, \quad \forall\, t \in [0,T], \tag{36}$$

where C_{29} is a constant independent of N. Lemma 3 has been proved. □

Lemma 4. *Suppose that the conditions of Lemma* 3 *are held and* $A'(t)$ *and* $B'(t)$ *are bounded in* $[0,T]$. *If* $k \ge 2$, $k = 2+p_0$, $p_0 \ge 0$, *then there exists the estimation*

$$|u_{Nt}(\cdot,t)|^2_{H^{2p_0}(\Omega)} + \|u_{Nt}\|^2_{H^{2(p_0+1)}(Q_t)} \le C_{30}, \quad \forall\, t \in [0,T], \tag{37}$$

where C_{30} *is a constant independent of* N.

Proof: We apply the method of induction. Differentiating (8) with respect to t, multiplying it by $\alpha'_{N,s}(t)$, summing up the products for $s = 1, 2, \ldots, N$ and integrating by parts, we get

$$\begin{aligned} &\frac{d}{dt}|u_{Nt}(\cdot,t)|^2_{L_2(\Omega)} + 2a_0|u_{Nx^2t}|^2_{L_2(\Omega)} \\ &\le 2b|u_{Nxt}|^2_{L_2(\Omega)} - 2(A'(t)u_{Nx^2}, u_{Nx^2t}) - 2(B'(t)u_{Nx}, u_{Nxt}) \\ &+ 2(g(u_N)_{x^2t} + f(u_N)_{xt} + h(u_{Nx})_{xt} + G(u_N)_t, u_{Nt}). \end{aligned} \tag{38}$$

It is easy to prove that

$$\begin{aligned} (g(u_N)_{x^2t}, u_{Nt}) &= -(g'(u_N)u_{Nxt} + g''(u_N)u_{Nt}u_{Nx}, u_{Nxt}) \\ &\le -(g''(u_N)u_{Nx}u_{Nt}, u_{Nxt}); \end{aligned} \tag{39}$$

$$(f(u_N)_{xt}, u_{Nt}) = -(f'(u_N)u_{Nt}, u_{Nxt}); \tag{40}$$

$$(h(u_{Nx})_{xt}, u_{Nt}) = -(h'(u_{Nx})u_{Nxt}, u_{Nxt}) \le 0; \tag{41}$$

$$(G(u_N)_t, u_{Nt}) \le \gamma|u_{Nt}|^2_{L_2(\Omega)}\,. \tag{42}$$

Substituting formulas (39)–(42) into (38), by means of Cauchy's inequality and interpolation formula, we have

$$\frac{d}{dt}|u_{Nt}(\cdot,t)|^2_{L_2(\Omega)} + |u_{Nx^2t}|^2_{L_2(\Omega)} \le C_{31}|u_{Nt}|^2_{L_2(\Omega)} + C_{32}\,. \tag{43}$$

Let us now prove that $|u_{Nt}(\cdot,0)|^2_{L_2(\Omega)}$ is uniformly bounded with respect to N. Multiplying (8) by $\alpha'_{N,s}(t)$, summing up the products for $s = 1,2,\dots,N$ and putting $t = 0$, we obtain

$$\begin{aligned}|u_{Nt}(\cdot,t)|^2_{L_2(\Omega)} &\le C_{33}\Big\{|u_{Nx^4}(\cdot,0)|^2_{L_2(\Omega)} + |u_{Nx^2}(\cdot,0)|^2_{L_2(\Omega)}\\ &+ |g(u_N(\cdot,0))_{x^2}|^2_{L_2(\Omega)} + |f(u_N(\cdot,0))_x|^2_{L_2(\Omega)} + |h(u_{Nx}(\cdot,0))_x|^2_{L_2(\Omega)}\\ &+ |G(u_N(\cdot,0))_{x^2}|^2_{L_2(\Omega)}\Big\}.\end{aligned}$$

By virtue of the assumptions of φ, g, f, h and G, the right side of the above inequality is uniformly bounded, then $|u_{Nt}(\cdot,0)|^2_{L_2(\Omega)}$ is uniformly bounded with respect to N. From (43) and using Gronwall's inequality, we have

$$|u_{Nt}(\cdot,t)|^2_{L_2(\Omega)} + \|u_{Nx^2t}\|^2_{L_2(Q_t)} \le C_{34}, \quad \forall\, t\in[0,T],$$

where C_{34} is a constant independent of N.

Now suppose that when $0 \le p_0 \le n$, the estimation (37) holds. We can prove that when $p_0 = n+1$, the estimation (37) holds, too. Differentiating (8) with respect to t, multiplying it by $\lambda_s^{n+1}\alpha'_{N,s}(t)$, summing up the products for $s = 1,2,\dots,N$, taking notice of (25)–(28) and (30) and integrating by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}|u_{Nx^{2(n+1)}t}|^2_{L_2(\Omega)} + 2a_0|u_{Nx^{2(n+2)}t}|^2_{L_2(\Omega)} \le 2b|u_{Nx^{2(n+2)-1}t}|^2_{L_2(\Omega)}\\ &+ 2|(A'(t)u_{Nx^{2(n+2)}}, u_{Nx^{2(n+2)}t})| + 2|(B'(t)u_{Nx^{2(n+1)}}, u_{Nx^{2(n+2)}t})|\\ &+ 2\big\{|g(u_N)_{x^{2(n+1)}t}|_{L_2(\Omega)} + |f(u_N)_{x^{2n+1}t}|_{L_2(\Omega)}\\ &+ |h(u_{Nx})_{x^{2n+1}t}|_{L_2(\Omega)} + |G(u_N)_{x^{2n}t}|_{L_2(\Omega)}\big\}|u_{Nx^{2(n+2)}t}|_{L_2(\Omega)}\,.\end{aligned} \tag{44}$$

Let $g(u_N)_{x^{2(n+1)}t} = w(u_N, u_{Nt})_{x^{2(n+1)}}$. From Lemma 2 and the interpolation formula it follows

$$\begin{aligned}&|g(u_N)_{x^{2(n+1)}t}|_{L_2(\Omega)} = |w(u_N,u_{Nt})_{x^{2(n+1)}}|_{L_2(\Omega)}\\ &\le C_{35}\big(|u_N|_{H^{2(n+1)}(\Omega)} + |u_{Nt}|_{H^{2(n+1)}(\Omega)}\big) \le C_{36} + C_{37}|u_{Nt}|^{\frac{n+1}{n+2}}_{H^{2(n+2)}(\Omega)}\,.\end{aligned} \tag{45}$$

Similarly, we get

$$|f(u_N)_{x^{2n+1}t}|_{L_2(\Omega)} \le C_{38} + C_{39}|u_{Nt}|^{\frac{2n+1}{2(n+2)}}_{H^{2(n+2)}(\Omega)}\,; \tag{46}$$

$$|h(u_{Nx})_{x^{2n+1}t}|_{L_2(\Omega)} \le C_{40} + C_{41}|u_{Nt}|^{\frac{n+1}{n+2}}_{H^{2(n+2)}(\Omega)}\,; \tag{47}$$

$$|G(u_N)_{x^{2n}t}|_{L_2(\Omega)} \le C_{42} + C_{43}|u_{Nt}|^{\frac{n}{n+2}}_{H^{2(n+2)}(\Omega)}\,. \tag{48}$$

We also have

$$|u_{Nx^{2n+3}t}|_{L_2(\Omega)} \le C_{44}|u_{Nt}|^{\frac{2n+3}{2(n+2)}}_{H^{2(n+2)}(\Omega)}\,. \tag{49}$$

Substituting formulas (45)–(49) into formula (44), by means of Young's inequality, we obtain

$$\frac{d}{dt}|u_{Nx^{2(n+1)}t}|^2_{L_2(\Omega)} + |u_{Nx^{2(n+2)}t}|^2_{L_2(\Omega)} \le C_{45}\,. \tag{50}$$

Now, since $k = 3+n$, it is easy to see that $|u_{Nx^{2(n+1)}t}(\cdot,0)|^2_{L_2(\Omega)}$ is uniformly bounded with respect to N. Hence, from (50) and using Gronwall's inequality, it follows that there is

$$|u_{Nx^{2(n+1)}t}(\cdot,t)|^2_{L_2(\Omega)} + \|u_{Nx^{2(n+2)}t}\|^2_{L_2(Q_t)} \le C_{46}, \quad \forall\, t\in[0,T],$$

where C_{46} is a constant independent of N. Then, the estimation (37) holds. This completes the proof of Lemma 4. $\square$

Lemma 5. *Suppose that the conditions of Lemma 4 are satisfied. Let* $k = 2r+p_{r-1}$, $r\ge 1$, $p_{r-1}\ge 0$. *If* $k\ge 2r$ $(r=1,2,\dots)$ *and* $A(t)$, $B(t)$ *are for* r*-times continuously differentiable in* $[0,T]$, *then there exists an estimation*

$$|u_{Nt^r}|^2_{H^{2p_{r-1}}(\Omega)} + \|u_{Nt^r}\|^2_{H^{2+2p_{r-1}}(Q_t)} \le C_{47}, \quad \forall\, t\in[0,T],\ (r=2,3,\dots), \tag{51}$$

where C_{47} *is a constant independent of* N.

Proof: We first prove that the estimation (51) holds when $r=2$. If $r=2$, then $k=4+p_1$. Differentiating (8) with respect to t for 2-times, multiplying it by $\lambda_s^{p_1}\alpha''_{N,s}(t)$, summing up the products for $s=1,2,\dots,N$, and integrating by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}|u_{Nx^{2p_1}t^2}|^2_{L_2(\Omega)} + 2a_0|u_{Nx^{2+2p_1}t^2}|^2_{L_2(\Omega)} \le 2b|u_{Nx^{1+2p_1}t^2}|^2_{L_2(\Omega)}\\ &- 2(2A'(t)u_{Nx^{2+2p_1}t}, u_{Nx^{2+2p_1}t^2}) - 2(A''(t)u_{Nx^{2+2p_1}}, u_{Nx^{2+2p_1}t^2})\\ &- 2(2B'(t)u_{Nx^{1+2p_1}t}, u_{Nx^{2+2p_1}t^2}) - 2(B''(t)u_{Nx^{1+2p_1}}, u_{Nx^{2+2p_1}t^2})\\ &+ 2\big(g(u_N)_{x^{2+2p_1}} + f(u_N)_{x^{1+2p_1}t^2} + h(u_{Nx})_{x^{1+2p_1}t^2}\\ &+ G(u_N)_{x^{2p_1}t^2}, u_{Nx^{2p_1}t^2}\big).\end{aligned} \tag{52}$$

Putting $p_1=0$, using Cauchy's inequality and from (52) and the results obtained, we have

$$\begin{aligned}&\frac{d}{dt}|u_{Nt^2}|^2_{L_2(\Omega)} + 2a_0|u_{Nx^2t^2}|^2_{L_2(\Omega)} \le 2b|u_{Nxt^2}|^2_{L_2(\Omega)} + C_{48}|u_{Nt^2}|^2_{L_2(\Omega)}\\ &+ C_{49} + \varepsilon\{|u_{Nx^2t^2}|^2_{L_2(\Omega)} + |g(u_N)_{x^2t^2}|^2_{L_2(\Omega)} + |f(u_N)_{xt^2}|^2_{L_2(\Omega)}\\ &+ |h(u_{Nx})_{xt^2}|^2_{L_2(\Omega)} + |G(u_N)_{t^2}|^2_{L_2(\Omega)}\},\end{aligned} \tag{53}$$

where $\varepsilon > 0$. Using Lemma 2 we get

$$|g(u_N)_{x^2t^2}|^2_{L_2(\Omega)} \leq C_{50} \sum_{i=0}^{2} |u_{Nt^i}|^2_{H^2(\Omega)}$$

$$\leq C_{51}\{1 + |u_{Nt^2}|^2_{L_2(\Omega)} + |u_{Nx^2t^2}|^2_{L_2(\Omega)}\}; \tag{54}$$

$$|f(u_N)_{xt^2}|^2_{L_2(\Omega)} \leq C_{52}\{1 + |u_{Nt^2}|^2_{L_2(\Omega)} + |u_{Nxt^2}|^2_{L_2(\Omega)}\}; \tag{55}$$

$$|h(u_{Nx})_{xt^2}|^2_{L_2(\Omega)} \leq C_{53}\{1 + |u_{Nt^2}|^2_{L_2(\Omega)} + |u_{Nx^2t^2}|^2_{L_2(\Omega)}\}; \tag{56}$$

$$|G(u_N)_{t^2}|^2_{L_2(\Omega)} \leq C_{54}\{1 + |u_{Nt^2}|^2_{L_2(\Omega)}\}. \tag{57}$$

From the interpolation formula, we have

$$\begin{aligned} |u_{Nxt^2}|^2_{L_2(\Omega)} \leq C_{55}|u_{Nt^2}|_{L_2(\Omega)}|u_{Nt^2}|_{H^2(\Omega)} \leq \varepsilon|u_{Nx^2t^2}|^2_{L_2(\Omega)} \\ + C_{56}|u_{Nt^2}|^2_{L_2(\Omega)}. \end{aligned} \tag{58}$$

Substituting formulas (54)–(58) into (53), taking that ε is sufficiently small, we obtain

$$\frac{d}{dt}|u_{Nt^2}|^2_{L_2(\Omega)} + |u_{Nx^2t^2}|^2_{L_2(\Omega)} \leq C_{57}|u_{Nt^2}|^2_{L_2(\Omega)} + C_{58}. \tag{59}$$

It is easy to prove that $|u_{Nt^2}(\cdot, 0)|^2_{L_2(\Omega)}$ is uniformly bounded with respect to N. From (59) and by Gronwall's inequality it follows that

$$|u_{Nt^2}(\cdot, t)|^2_{L_2(\Omega)} + \|u_{Nx^2t^2}\|^2_{L_2(Q_t)} \leq C_{59}, \quad \forall\, t \in [0, T]. \tag{60}$$

Similarly, we can prove that when $p_1 \geq 1$, there is the following estimation

$$|u_{Nx^{2p_1}t^2}|^2_{L_2(\Omega)} + \|u_{Nx^{2+2p_1}t^2}\|^2_{L_2(Q_t)} \leq C_{60}, \quad \forall\, t \in [0, T],$$

where C_{60} is a constant independent of N. Similarly, we can prove that the estimation (51) holds for $r = 3, 4, \dots$. The lemma is proved. □

Theorem 1. *Under the conditions of Lemma 5, if $k \geq 3$, then there exists a unique generalized global solution $u(x,t)$ of the initial boundary value problems (1), (3) and the solution has continuous derivatives u_{x^s} $(0 \leq s \leq 2k-5)$ and the generalized derivatives $u_{x^st^r}$ $(0 \leq s+4r \leq 2k,\ r = 0,1)$. If $k \geq 5$, then there exists a unique classical global solution $u(x,t)$ of the initial boundary value problems (1), (3) and the solution $u(x,t)$ has continuous derivatives $u_{x^st^r}$ $(0 \leq s+4r \leq 2k-5,\ r = 0,1,2,\dots)$ and the generalized derivatives $u_{x^st^r}$ $(0 \leq s+4r \leq 2k,\ r = 0,1,2,\dots)$.*

Proof: From Lemmas 3 and 4 we know that

$$\begin{aligned} &u_{Nx^s} \in L_\infty([0,T] \times \Omega), \qquad 0 \leq s \leq 2k-1, \\ &u_{Nx^st} \in L_\infty([0,T] \times \Omega), \qquad 0 \leq s \leq 2k-5. \end{aligned}$$

If $k \geq 3$, then we can select a subsequence still denoted by $\{u_N(x,t)\}$ from $\{u_N(x,t)\}$ such that there exists a function $u(x,t)$ and when $N \to \infty$ the subsequence $\{u_N(x,t)\}$ uniformly converges to the limiting function $u(x,t)$ in Q_T. The corresponding subsequence of the derivatives $\{u_{Nx}(x,t)\}$ also uniformly converges to $u_x(x,t)$. The subsequences $\{u_{Nx^s}(x,t)\}$ $(0 \leq s \leq 2k)$ and $\{u_{Nx^st}(x,t)\}$ $(0 \leq s \leq 2(k-2))$ weakly converge to the generalized derivatives u_{x^s} $(0 \leq s \leq 2k)$ and u_{x^st} $(0 \leq s \leq 2(k-2))$ in $L_2(Q_T)$ respectively. Therefore when $k \geq 3$ there exists a generalized global solution $u(x,t)$ of the initial boundary value problems (1), (3). If $k \geq 5$, then from Lemma 5 it follows that $u_{x^st^r} \in L_2(Q_T)$ $(0 \leq s+4r \leq 2k)$ and $u_{x^st^r} \in L_\infty(Q_T)$ $(0 \leq s \leq 2(k-2r)-1)$, $r = 2,3,\dots$. Hence there exists a classical global solution $u(x,t)$ of the initial boundary value problems (1), (3), and this solution has the regularities as those stated in Theorem 1. It is easy to prove the uniqueness of solutions for the problems (1), (3). This completes the proof of the theorem. □

Theorem 2. *Suppose that the following conditions are satisfied:*

(1) *There exist constants* $a_0 > 0$, $b_0 > 0$ *such that* $A(t) \geq a_0 > 0$, $B(t) \geq b_0 > 0$ *in* $[0,\infty)$*;*
(2) $g \in C^1$ *and* $g'(s) \geq 0$, $\forall s \in \mathbb{R}$*;* $f \in C$, $F(u) = \int_0^u f(\xi)\,d\xi$*;* $h \in C^1$, $h(0) = 0$ *and* $h'(\xi) \geq 0$, $\forall \xi \in \mathbb{R}$*;*
(3) $G \in C^1$, $G(0) = 0$ *and there exists a constant* $\gamma_0 > 0$ *such that* $G'(\xi) \leq -\gamma_0$, $\forall \xi \in \mathbb{R}$.

Then the generalized or classical solution $u(x,t)$ *of the initial boundary value problems* (1), (3), *has the asymptotic behavior*

$$\lim_{t\to\infty} |u(\cdot,t)|_{L_2(\Omega)} = 0.$$

Proof: Multiplying (1) by u and integrating in Ω, integrating by parts and by the argument proved in Lemma 1, we can obtain

$$\frac{d}{dt}|u(\cdot,t)|^2_{L_2(\Omega)} + 2a_0|u_{x^2}(\cdot,t)|^2_{L_2(\Omega)} + 2b_0|u_x(\cdot,t)|^2_{L_2(\Omega)} \leq -2\gamma_0|u(\cdot,t)|^2_{L_2(\Omega)}\,. \tag{61}$$

By separation of variables from (61) we deduce

$$|u(\cdot,t)|^2_{L_2(\Omega)} \leq |\varphi|^2_{L_2(\Omega)}e^{-2\gamma_0 t}\,. \tag{62}$$

Theorem 2 is proved. □

3. Initial boundary value problems (1), (4)

In this section we again consider the initial boundary value problems (1), (4) by the Galerkin method. Let $\{y_n(x)\}$ be the orthonormal complete system composed of the eigenfunctions of the following boundary problem of the ordinary differential equation [2]

$$\begin{cases} y^{(4)} = \lambda y, \\ y'(-\ell) = y'(\ell) = 0, \quad y'''(-\ell) = y'''(\ell) = 0 \end{cases}$$

corresponding to eigenvalues λ_n $(n = 1, 2, \dots)$. Observe that the orthonormal complete system $\{y_n(x)\}$ on the boundary points of Ω has the properties

$$y_s^{(L)}(-\ell) = y_s^{(L)}(\ell) = 0, \quad L = 2\nu + 1, \ \nu = 0, 1, \dots \ . \tag{63}$$

By the method in Section 2 we can obtain the following theorems.

Theorem 3. *Suppose that the following conditions are satisfied:*

(1) *There exist constants $a_0 > 0$, $b > 0$, such that $A(t) \geq a_0 > 0$, $-b \leq B(t) \leq b$ on $[0, T]$ and let $k = 2r + p_{r-1}$, $p_{r-1} \geq 0$ $(r = 1, 2, \dots)$ and $A(t)$, $B(t)$ are continuously differentiable for r-times in $[0, T]$;*
(2) *$g \in C^{2k}$ $(k \geq 1)$; $g'(s) \geq 0$, $\forall\, s \in \mathbb{R}$ and $|g'(s)| \leq K_1|s|^{\xi+1}$, $|g''(s)| \leq K_1|s|^{\xi}$, where $0 < \xi < 3$ and $K_1 > 0$ is a constant;*
(3) *$h \in C^{2k-1}$, $h'(s) \geq 0$, $\forall\, s \in \mathbb{R}$ and $|h(s)| \leq K_2|s|^{\mu+1}$, $|h'(s)| \leq K_2|s|^{\mu}$, where $0 < \mu < \frac{4}{3}$ and K_2 is a constant;*
(4) *$G \in C^{2k-1}$, $G'(s) \leq \gamma$, $\forall\, s \in \mathbb{R}$, where γ is a constant;*
(5)

$$\frac{\partial^\beta}{\partial x^\beta}[g(u)_{x^2}]\,|_{\,x=-\ell} = \frac{\partial^\beta}{\partial x^\beta}[g(u)_{x^2}]\,|_{\,x=\ell} = 0, \ \beta = 1, 3, \dots, 2k-1, \tag{64}$$

$$\frac{\partial^\beta}{\partial x^\beta}[h(u_x)_x]\,|_{\,x=-\ell} = \frac{\partial^\beta}{\partial x^\beta}[h(u_x)_x]\,|_{\,x=\ell} = 0, \ \beta = 1, 3, \dots, 2k-1, \tag{65}$$

$$\frac{\partial^\beta}{\partial x^\beta}G(u)\,|_{\,x=-\ell} = \frac{\partial^\beta}{\partial x^\beta}G(u)\,|_{\,x=\ell} = 0, \ \beta = 1, 3, \dots, 2k-1; \tag{66}$$

(6) *$\varphi \in V_{2k}$, and φ satisfies the boundary conditions.*

If $k \geq 3$, then there exists a unique generalized global solution $u(x, t)$ of the initial boundary value problems (1), (4), and the solution has continuous derivatives u_{x^s} $(0 \leq s \leq 2k-5)$ and the generalized derivatives $u_{x^s t^r}$ $(0 \leq s + 4r \leq 2k$, $r = 0, 1)$. If $k \geq 5$, then there exists a unique classical global solution $u(x, t)$ of the initial boundary value problems (1), (4), and the solution $u(x, t)$ has continuous derivatives $u_{x^s t^r}$ $(0 \leq s + 4r \leq 2k - 5$, $r = 0, 1, \dots)$ and the generalized derivatives $u_{x^s t^r}$ $(0 \leq s + 4r \leq 2k$, $r = 0, 1, \dots)$.

Theorem 4. *Suppose that the following conditions are satisfied:*

(1) *There exist constants $a_0 > 0$, $b_0 > 0$, such that $A(t) \geq a_0 > 0$, $B(t) \geq b_0 > 0$ in $[0, \infty)$;*
(2) *$g \in C^1$ and $g'(s) \geq 0$, $\forall\, s \in \mathbb{R}$; $h \in C^1$, $h'(\xi) \geq 0$, $\forall\, \xi \in \mathbb{R}$ and $h(0) = 0$;*
(3) *$G \in C^1$, $G(0) = 0$ and there exists a constant $\gamma_0 > 0$ such that $G'(\xi) \leq -\gamma_0$, $\forall\, \xi \in \mathbb{R}$.*

Then the generalized or classical solution $u(x,t)$ *of the initial boundary value problems* (1), (4), *has the asymptotic behavior*

$$\lim_{t\to\infty} |u(\cdot,t)|_{L_2(\Omega)} = 0.$$

Remark. For example, let $G(u) = cu^7$ in the equation (2), where $c < 0$ is a constant, then $g(u)$ $(= au^3)$ and $G(u)$ satisfy all conditions of Theorem 1–4. If $a_2 > 0$ and $\varphi \in V_{2k}$ $(k \geq 1)$, then the initial boundary value problems (2), (3) or (2), (4), have the conclusions of Theorems 1–4.

References

[1] Cohen D.S., Murray J.D., *A generalized diffusion model for growth and dispersal in population*, J. Math. Biol. **12** (1981), 237–249.
[2] Naimark M.A., *Linear Differential Operators*, Moscow, 1954.
[3] Maz'ja V.G., *Sobolev Spaces*, Springer-Verlag, 1985.
[4] Zhou Yulin, Fu Hongyuan, *The nonlinear hyperbolic systems of higher order of generalized Sine-Gordon type* (in Chinese), Acta Math. Sinica **26** (1983), 234–249.
[5] Chen Guowang, *First boundary problems for nonlinear parabolic and hyperbolic coupled systems of higher order*, Chinese Journal of Contemporary Mathematics **9** (1988), 98–116.
[6] Liu Baoping, Pao C.V., *Integral representation of generalized diffusion model in population problems*, Journal of Integral Equations **6** (1984), 175–185.
[7] Chen Guowang, *Initial value problem for a class of nonlinear parabolic system of fourth-order*, Acta Mathematica Scientia **11** (1991), 393–400.

Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China

(*Received* December 21, 1993)