A note on Boolean algebras

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Abstract. We show that splitting of elements of an independent family of infinite regular size will produce a full size independent set.

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Let us note that a family $\{b_\alpha : \alpha < \kappa\}$ of elements of a Boolean algebra is called independent if for any finite disjoint sets $I, J \subseteq K$ the meet

$$\bigwedge_{\alpha\in I} b_\alpha \;\wedge\; \bigwedge_{\beta\in J} (-b_\beta) \neq 0\,.$$

The following theorem gives a positive answer to a question raised by P. Koszmider.

Theorem. *Let κ be an infinite regular cardinal and suppose that in a Boolean algebra $\mathcal{A}$ there is an independent family $\{a_\alpha : \alpha < \kappa\}$ of size κ. Suppose also that we have $\{b_\alpha : \alpha < \kappa\}$, $\{c_\alpha : \alpha < \kappa\}$, subsets of $\mathcal{A}$, s.t. $\forall\, \alpha < \kappa\ b_\alpha \vee c_\alpha = a_\alpha, b_\alpha \wedge c_\alpha = 0$. Then there exist $I \in [\kappa]^\kappa$ and $\varphi : I \to B \cup C$ with $\varphi(\alpha) \in \{b_\alpha, c_\alpha\}$ such that $\{\varphi(\alpha) : \alpha \in I\}$ is independent in $\mathcal{A}$.*

Proof: We may assume that $\mathcal{A}$ is a field of sets, $\mathcal{A} \subset \mathcal{P}(X)$ for some set X. For every $x \in X$, define $f_x : \kappa \to 2$ by $f_x(\alpha) = 1 \Leftrightarrow x \in a_\alpha$. Let $F = \{f_x : x \in X\}$. Then F is a dense subspace of the Cantor Cube 2^κ.

Let $A_\alpha = \{f \in F : f(\alpha) = 1\} = \{f_x : x \in a_\alpha\}$, similarly $B_\alpha = \{f_x : x \in b_\alpha\}$, $C_\alpha = \{f_x : x \in c_\alpha\}$. Then $\{A_\alpha : \alpha < \kappa\}$ is an independent family of subsets of F (and of 2^κ) and $\forall\, \alpha\, A_\alpha = B_\alpha \dot\cup C_\alpha$.

We notice that it is sufficient to find an $I \in [\kappa]^\kappa$ and $\varphi \in \prod_{i\in I}\{B_i, C_i\}$ such that $\{\varphi(\alpha) : \alpha \in I\}$ is an independent family of subsets of F. These $I = \{i_\alpha : \alpha < \kappa\}$ and φ we will construct by induction on α so that if we stop at some stage $\alpha < \kappa$, we will have the required I and φ at once.

At a stage $\alpha < \kappa$ we have selected $I_\alpha = \{i_\beta : \beta < \alpha\}$ and $\varphi_\alpha \in \prod_{i\in I_\alpha}\{B_i, C_i\}$ so that, denoting by $\mathcal{K}_\alpha$ the set of all Boolean independence combinations from $\{\varphi(i) : i \in I_\alpha\}$, $\forall\, K \in \mathcal{K}_\alpha\ \bar{K} \supset$ some $U_K \leftarrow$ a clopen (basic) subset of 2^κ, and we fix the family $\mathcal{U}_\alpha = \{U_K : K \in \mathcal{K}_\alpha\}$. This is our induction hypothesis.

At the stage α we choose i_α and $\varphi(i_\alpha)$ to satisfy our inductive hypothesis on the larger sets

$$I_{\alpha+1}, \quad \mathcal{K}_{\alpha+1}, \quad \mathcal{U}_{\alpha+1}\,.$$

Suppose we cannot. Let $J \in [\kappa]^\kappa$ be disjoint from all indices of subbasic sets mentioned in the definitions of members of $\mathcal{U}_\alpha$. (So that e.g. $\forall U \in \mathcal{U}_\alpha\ U \restriction J = 2^J$). Then every i in J is a "bad" index, and for such i we must have

$$\exists\, K_1 \in \mathcal{K}_\alpha\ \exists K_2 \in \mathcal{K}_\alpha \text{ s.t.}$$

either $K_1 \cap B_i$ is nowhere dense in 2^κ or $K_2 \cap C_i$ is nowhere dense in 2^κ.

Then either

$$U_{K_1} \subset Int(\bar{K}_1) \subset \overline{C_i \cup (F \setminus A_i)}$$

or

$$U_{K_2} \subset Int(\bar{K}_2) \subset \overline{B_i \cup (F \setminus A_i)}\,,$$

and similarly for every i in J.

But since $|\mathcal{K}_\alpha| = |\,[\alpha]^{<\omega}\,| < \kappa$ and κ is regular, there is $I \in [J]^\kappa$, a fixed $K \in \mathcal{K}_\alpha$ and a function $\varphi \in \prod_{i\in I}\{B_i, C_i\}$ s.t. for every $i \in I$

$$U_K \subset Int(\bar{K}) \subset \overline{\varphi(i) \cup (F \setminus A_i)}\,.$$

Then $\{\varphi(i) : i \in I\}$ is an independent family of size κ.

Indeed, let L be a Boolean independence combination from this family, and let $\tilde{L}$ be the same combination with A_i's replacing $\varphi(i)$'s.

Then $\emptyset \neq \tilde{L} \cap U_K$ is an elementary basic open set in $F \subset 2^\kappa$ such that $(\tilde{L} \cap U_K) \setminus L$ is nowhere dense in F.

Hence $L \neq \emptyset$, as required. $\square$

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References

[NE] Negrepontis S., *Banach Spaces in Topology*, Handbook of Set-Theoretic Topology, North Holland, 1984, 1045–1142.

[Ko] Koppelberg S., *General Theory of Boolean Algebras*, in Handbook of Boolean Algebras vol. 1, J.D. Monk, R. Bonnet, eds., North Holland, 1989.

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