Topological spaces with a selected subset-cardinal invariants and inequalities

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Abstract. Cardinal functions for topological spaces in which a subset is selected in a certain way are defined and studied. Most of the main cardinal inequalities are generalized for such spaces.

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All spaces are assumed to be Hausdorff and standard notations following [5] and [7] are used.

From now on let X be a topological space and X_0 be a selected subset of X.

The subspace X_0 is said to be compact (or Lindelöf) in X (see [3]), if from each open cover γ of X a finite (countable) $\gamma' \subseteq \gamma$ could be chosen that covers X_0.

From the other side in [9] Sun Shu-Hao introduced the invariant $kL(X) = \omega \cdot \min\{\tau :$ there is an $A \subseteq X$, $|A| \leq 2^\tau$ such that
$(*)$: for each open cover γ of X there are $\gamma' \in [\gamma]^{\leq\tau}$ and $B \in [A]^{\leq\tau}$ such that $X = \bigcup\gamma' \cup \overline{B}\}$
and observed that $kL(X) \leq \min\{d(X), L(X), s(X)\}$. He also proved that for a Hausdorff space X we have $|X| \leq \exp kL(X) \cdot \psi_C(X) \cdot t(X)$, where $\psi_C(X) = \omega \cdot \min\{\tau :$ for each $X \in X$ there is a family of open neighborhoods $\{U_\alpha(x) : \alpha \in \tau\}$ of x such that $\{x\} = \bigcap\{\overline{U_\alpha(x)} : \alpha \in \tau\}\}$. To do this he used the following:

Lemma 1. *If X is a Hausdorff topological space, $L \in [X]^{\leq \exp\tau}$ and $\psi_C(X) \cdot t(X) \leq \tau$, then $|\overline{L}| \leq 2^\tau$.*

Figuratively speaking the above two notions show that we could have some “bad” part of a certain space, but if the cardinality of this part is not “too big”, we still could get some results about the cardinality of the main space.

In this way we come to the following concept: we say that $L(X, X_0) = \omega \cdot \min\{\tau :$ for each open cover γ of X there is $\gamma' \in [\gamma]^{\leq\tau}$ such that $\bigcup\gamma' \supseteq X \setminus X_0\}$.

We have that $L(X, \emptyset) = L(X)$, $L(X, X_0) \leq L(X)$, $L(X, X_0) \leq L(X \setminus X_0) \leq hL(X)$, $L(X) \leq L(X_0) \cdot L(X, X_0)$ and if $X \setminus X_0$ is Lindelöf in X then $L(X, X_0) \leq \omega$. In the conditions of Lemma 1 there is $X_0 \in [X]^{\leq \exp\tau}$ such that $L(X, X_0) \leq \tau$. In

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that sense we could look at $L(X, X_0)$ as a generalization of the notions mentioned in the beginning.

We shall prove the following:

Theorem 1. *If X is a Hausdorff topological space then:*

$$|X \setminus X_0| \leq \exp L(X, X_0) \cdot \psi_C(X) \cdot t(X).$$

Proof: Let $L(X, X_0) \cdot \psi_C(X) \cdot t(X) \leq \tau$ and for every $x \in X$ let us fix a family of neighborhoods of the point $x - \mathcal{W}(x)$ with $|\mathcal{W}(x)| \leq \tau$ such that $\{x\} = \bigcap\{\overline{U} : U \in \mathcal{W}(x)\}$. By transfinite induction we shall define two families – $\{H_\alpha : \alpha \in \tau^+\} \subseteq \exp(X \setminus X_0)$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ such that:

(1) $H_\alpha = \overline{H_\alpha}^{X \setminus X_0}$.
(2) $|H_\alpha| \leq 2^\tau$ for every $\alpha \in \tau^+$.
(3) $H_\alpha \subseteq H_{\alpha'}$ if $\alpha \leq \alpha' \in \tau^+$.
(4) If $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ are already defined then $\mathcal{B}_\alpha = \bigcup\{\mathcal{W}(x) : x \in \bigcup\{\overline{H_\beta} : \beta \in \alpha\}\}$.
(5) If $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leq\tau}$ and $X \setminus (\cup\mathcal{W} \cup X_0) \neq \emptyset$ then $H_\alpha \setminus (\cup\mathcal{W} \cup X_0) \neq \emptyset$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ and $\{\mathcal{B}_\beta : \beta \in \alpha\}$ be already defined with properties (1)–(5).

Let $E_\alpha = \{\mathcal{W} : \mathcal{W} \in [\mathcal{B}_\alpha]^{\leq\tau}$ and $X \setminus (\cup\mathcal{W} \cup X_0) \neq \emptyset\}$. For every $\mathcal{W} \in E_\alpha$ we choose a point $\phi(\mathcal{W}) \in X \setminus (\cup\mathcal{W} \cup X_0) \neq \emptyset$ and let $C_\alpha = \{\phi(\mathcal{W}) : \mathcal{W} \in E_\alpha\}$. Since $|E_\alpha| \leq 2^\tau$ we have that $|C_\alpha| \leq 2^\tau$. Finally we put $H_\alpha = \overline{C_\alpha \cup \cup\{H_\beta : \beta \in \alpha\}}^{X \setminus X_0}$. Since $|C_\alpha \cup \cup\{H_\beta : \beta \in \alpha\}| \leq 2^\tau$ and $\psi_c(X) \cdot t(X) \leq \tau$, using Lemma 1 we obtain that $|H_\alpha| \leq 2^\tau$. It can be easily seen that the conditions (1)–(5) are satisfied.

Let $H = \bigcup\{H_\alpha : \alpha \in \tau^+\}$. H is closed in $X \setminus X_0$ and $\overline{H} = \bigcup\{\overline{H_\alpha} : \alpha \in \tau^+\}$.

Let us show that $X \setminus X_0 = H$. Suppose there is a $q \in X \setminus H \setminus X_0$. Then $q \notin \overline{H}$, hence for every $p \in \overline{H}$, we can choose $V_p \in \mathcal{W}(p)$ such that $q \notin \overline{V}_p$. Let $\mu = \{V_p : p \in \overline{H}\} \cup \{X \setminus \overline{H}\}$. We have that $\bigcup \mu \supseteq X$ and from $L(X, X_0) \leq \tau$ we can choose $\mu_0 \in [\mu]^{\leq\tau}$ such that $H \subseteq X \setminus X_0 \subseteq \bigcup \mu_0$. We have that $\mu_0 = \{V_p : p \in \overline{H} \in [\overline{H}]^{\leq\tau}\} \cup \{X \setminus \overline{H}\}$. Hence $H \subseteq \bigcup\{V_p : p \in H' \in [\overline{H}]^{\leq\tau}\}$. Let $\mu' = \{V_p : p \in H' \in [\overline{H}]^{\leq\tau}\}$. From the regularity of τ^+ and the fact that $|\mu'| \leq \tau$ there is an $\alpha_0 \in \tau^+$ such that $\mu' \subseteq B_{\alpha_0}$. Then we have already chosen a point $\phi(\mu') \in (X \setminus (\cup\mu' \cup X_0)) \cap H_\alpha \subseteq H$ and at the same time $\bigcup \mu' \supseteq H$ – a contradiction. □

In fact we have proved:

Theorem 1*. *If $\overline{X \setminus X_0}$ is Hausdorff then:*

$$|\overline{X \setminus X_0}| \leq \exp L(X, X_0) \cdot \psi_C(\overline{X \setminus X_0}) \cdot t(\overline{X \setminus X_0}).$$

Theorem 1.** *Let X be a Hausdorff topological space, $X_0 \subseteq X$, $L(X, X_0) \cdot t(X) \leq \tau$ and if $H \in [X \setminus X_0]^{\leq \exp \tau}$ then $|\overline{H}^{X \setminus X_0}| \leq 2^\tau$. Then $|\overline{X \setminus X_0}| \leq 2^\tau$.*

Corollary 1.1 ([1]). *For every Hausdorff topological space X we have that $|X| \leq \exp L(X) \cdot \psi(X) \cdot t(X)$.*

Corollary 1.2 ([8]). *For every regular topological space X we have that $|X| \leq \exp kL(X) \cdot \chi(X)$.*

Corollary 1.3 ([9]). *For every Hausdorff topological space X we have that $|X| \leq \exp kL(X) \cdot \psi_C(X) \cdot t(X)$.*

Now let us define: $wL(X, X_0) = \omega \cdot \min\{\tau :$ for each open cover γ of X there is $\gamma' \in [\gamma]^{\leq \tau}$ such that $\overline{\bigcup \gamma'} \supseteq X \setminus X_0\}$ and $qL(X, X_0) = \omega \cdot \min\{\tau :$ for each $F = \overline{F}^{X \setminus X_0} \subseteq X \setminus X_0$ and for each open cover γ of $\overline{F}$ there is $\gamma' \in [\gamma]^{\leq \tau}$ such that $\overline{\bigcup \gamma'} \supseteq F\}$. We have that $wL(X, X_0) \leq qL(X, X_0) \leq L(X, X_0)$, $wl(X, X_0) \leq wL(X \setminus X_0)$, $wL(X, \emptyset) = wL(X)$, $wL(X, X_0) \leq wL(X)$, $qL(X, X_0) \leq qL(X)$, $qL(X, X_0) \leq qL(X \setminus X_0)$ and $qL(X, \emptyset) = qL(X)$. We also have the following lemma:

Lemma 2. *If X is normal then $wL(X, X_0) = qL(X, X_0)$.*

Proof: Let $wL(X, X_0) \leq \tau$, let $F = \overline{F}^{X \setminus X_0} \subset X \setminus X_0$ and let γ be an open in X cover of $\overline{F}$. From the normality of X we have that there is an open U such that $\overline{F} \subset U \subset \overline{U} \subset \bigcup \gamma$. Let $\gamma_1 = \gamma \cup \{X \setminus \overline{U}\}$. Then $\bigcup \gamma_1 = X$ and therefore there is $\gamma_1' \in [\gamma_1]^{\leq \tau}$ such that $\overline{\bigcup \gamma_1'} \supseteq X \setminus X_0$. We have that $\gamma_1' = \gamma' \cup \{X \setminus \overline{U}\}$, where $\gamma' \in [\gamma]^{\leq \tau}$. Therefore $\overline{\bigcup \gamma_1'} = \overline{\bigcup \gamma'} \cup \overline{X \setminus \overline{U}}$. Since $\overline{X \setminus \overline{U}} \cap F = \emptyset$ we have that $F \subseteq \overline{\bigcup \gamma'}$ i.e. $qL(X, X_0) \leq \tau$. □

Theorem 2. *If X is a regular topological space then:*

$$|X \setminus X_0| \leq \exp qL(X, X_0) \cdot \chi(X).$$

Proof: Let $qL(X, X_0) \cdot \chi(X) \leq \tau$ and for every $x \in X$, let us fix a local base at the point $x - \mathcal{W}(x)$ with $|\mathcal{W}(x)| \leq \tau$. By transfinite induction we shall define two families – $\{H_\alpha : \alpha \in \tau^+\} \subseteq \exp(X \setminus X_0)$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ such that:

(1) $H_\alpha = \overline{H_\alpha}^{X \setminus X_0}$.
(2) $|H_\alpha| \leq 2^\tau$ for every $\alpha \in \tau^+$.
(3) $H_\alpha \subseteq H_{\alpha'}$ if $\alpha \leq \alpha' \in \tau^+$.
(4) If $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ are already defined then $\mathcal{B}_\alpha = \bigcup\{\mathcal{W}(x) : x \in \bigcup\{\overline{H_\beta} : \beta \in \alpha\}\}$.
(5) If $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leq \tau}$ and $X \setminus (\overline{\cup \mathcal{W}} \cup X_0) \neq \emptyset$ then $H_\alpha \setminus (\overline{\cup \mathcal{W}} \cup X_0) \neq \emptyset$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ and $\{\mathcal{B}_\beta : \beta \in \alpha\}$ be already defined with properties (1)–(5).

Let $E_\alpha = \{\mathcal{W} : \mathcal{W} \in [\mathcal{B}_\alpha]^{\leq\tau}$ and $X \setminus (\overline{\cup\mathcal{W}} \cup X_0) \neq \emptyset\}$. For every $\mathcal{W} \in E_\alpha$ we choose a point $\phi(\mathcal{W}) \in X \setminus (\overline{\cup\mathcal{W}} \cup X_0) \neq \emptyset$ and let $C_\alpha = \{\phi(\mathcal{W}) : \mathcal{W} \in E_\alpha\}$. Since $|E_\alpha| \leq 2^\tau$ we have that $|C_\alpha| \leq 2^\tau$. Finally we put $H_\alpha = \overline{C_\alpha \cup \cup\{H_\beta : \beta \in \alpha\}}^{X\setminus X_0}$. Since $|C_\alpha \cup \cup\{H_\beta : \beta \in \alpha\}| \leq 2^\tau$ and $\chi(X) \leq \tau$ then $|H_\alpha| \leq 2^\tau$. It can be easily seen that the conditions (1)–(5) are satisfied.

Let $H = \bigcup\{H_\alpha : \alpha \in \tau^+\}$. H is closed in $X \setminus X_0$ and $\overline{H} = \bigcup\{\overline{H_\alpha} : \alpha \in \tau^+\}$.

Let us show that $X \setminus X_0 = H$. Suppose there is a $q \in X \setminus H \setminus X_0$. Then $q \notin \overline{H}$ and from the regularity of X there is an open V such that $q \in V \subseteq \overline{V} \subseteq X \setminus \overline{H} \subseteq X \setminus H$ i.e. $\overline{V} \cap \overline{H} = \emptyset$. For every $p \in \overline{H}$ we can choose $V_p \in \mathcal{W}(p)$ such that $V_p \cap \overline{V} = \emptyset$. Let $\mu = \{V_p : p \in \overline{H}\}$. We have that $\bigcup\mu \supseteq \overline{H}$ and from $qL(X, X_0) \leq \tau$ we can choose $\mu_0 \in [\mu]^{\leq\tau}$ such that $H \subseteq \overline{\bigcup\mu_0}$. We have that $\mu_0 = \{V_p : p \in H' \in [\overline{H}]^{\leq\tau}\}$. From the regularity of τ^+ there is an $\alpha_0 \in \tau^+$ such that $\mu' \subseteq B_{\alpha_0}$ and $q \in X \setminus (X_0 \overline{\cup\mu_0})$. Then we have already chosen a point $\phi(\mu_0) \in (X \setminus (\overline{\cup\mu_0} \cup X_0)) \cap H_\alpha \subseteq H$ and at the same time $\overline{\bigcup\mu_0} \supseteq H$ – a contradiction. ☐

In fact we have proved:

Theorem 2*. *If $\overline{X \setminus X_0}$ is regular then:*

$$|\overline{X \setminus X_0}| \leq \exp qL(X, X_0) \cdot \chi(\overline{X \setminus X_0}).$$

Corollary 2.1. *For every normal topological space X we have that $|X \setminus X_0| \leq \exp wL(X \setminus X_0) \cdot \chi(X)$.*

Corollary 2.2 ([2]). *If X is a regular topological space then $|X| \leq \exp qL(X) \cdot \chi(X)$.*

Corollary 2.3 ([4]). *If X is a normal topological space then $|X| \leq \exp wL(X) \cdot \chi(X)$.*

Corollary 2.4 ([10]). *For every regular topological space X we have that $|X| \leq \exp qkL(X) \cdot \chi(X)$.*

Corollary 2.5 ([10]). *For every normal topological space X we have that $|X| \leq \exp wkL(X) \cdot \chi(X)$.*

Now let us define: $c(X, X_0) \leq \tau$ iff for every $F \subseteq X \setminus X_0$ and every canonically open set $V \supseteq F$ and every open cover γ of the set $F \cup (\overline{F} \cap X_0 \cap V)$ there is $\gamma' \in [\gamma]^{\leq\tau}$ such that $\overline{\bigcup\gamma'} \supseteq F$. We have that $qL(X, X_0) \leq c(X, X_0)$, $c(X, X_0) \leq c(X \setminus X_0)$, $c(X, \emptyset) = c(X)$ and $c(X, X_0) \leq c(X)$.

Theorem 3. *If X is a Hausdorff topological space then:*

$$|X \setminus X_0| \leq c(X, X_0) \cdot \chi(X).$$

Proof: Let $c(X,X_0)\cdot\chi(X)\le\tau$ and for every $x\in X$, let us fix a local base in the point $x - \mathcal{W}(x)$ with $|\mathcal{W}(x)|\le\tau$. By transfinite induction we shall define two families – $\{H_\alpha:\alpha\in\tau^+\}\subseteq\exp(X\setminus X_0)$ and $\{\mathcal{B}_\alpha:\alpha\in\tau^+\}$ such that:

(1) $|H_\alpha|\le 2^\tau$ for every $\alpha\in\tau^+$.
(2) $H_\alpha\subseteq H_{\alpha'}$ if $\alpha\le\alpha'\in\tau^+$.
(3) If $\alpha\in\tau^+$ and $\{H_\beta:\beta\in\alpha\}$ are already defined then $B_\alpha=\bigcup\{\mathcal{W}(x):x\in\overline{\bigcup\{H_\beta:\beta\in\alpha\}}\}$.
(4) If $\mathcal{W}_\delta\in[\mathcal{B}_\alpha]^{\le\tau}$ for every $\delta\in\tau$, $W=\bigcup\{\overline{\bigcup\{U:U\in\mathcal{W}_\delta\}}:\delta\in\tau\}$ and $X\setminus(W\cup X_0)\ne\emptyset$ then $H_\alpha\setminus(W\cup X_0)\ne\emptyset$.

Let $\alpha\in\tau^+$ and $\{H_\beta:\beta\in\alpha\}$ and $\{\mathcal{B}_\beta:\beta\in\alpha\}$ be already defined with properties (1)–(4).

Let $E_\alpha=\{W:W=\bigcup\{\overline{\bigcup\{U:U\in E_\delta\}}:\delta\in\tau,\ \mathcal{W}_\delta\in[\mathcal{B}_\alpha]^{\le\tau}$ for every $\delta\in\tau$ and $X\setminus(W\cup X_0)\ne\emptyset\}$. For every $W\in E_\alpha$ we choose a point $\phi(W)\in X\setminus(W\cup X_0)\ne\emptyset$ and let $C_\alpha=\{\phi(W):W\in E_\alpha\}$. Since $|E_\alpha|\le 2^\tau$ we have that $|C_\alpha|\le 2^\tau$. Finally we put $H_\alpha=C_\alpha\cup\bigcup\{H_\beta:\beta\in\alpha\}$. It can be easily seen that the conditions (1)–(4) are satisfied.

Let $H=\bigcup\{H_\alpha:\alpha\in\tau^+\}$. Then $\overline{H}=\bigcup\{\overline{H}_\alpha:\alpha\in\tau^+\}$ and $|H|\le 2^\tau$. Therefore $|\overline{H}|\le 2^\tau$.

Let us show that $X\setminus X_0=H$. Suppose there is a $q\in X\setminus H\setminus X_0$. We have that $\{q\}=\bigcap\{\overline{V}:V\in\mathcal{W}(q)\}$ and let $H(V,q)=H\setminus\overline{V}$, for every $V\in\mathcal{W}(q)$. Let us note that $H=\bigcup\{H(V,q):V\in\mathcal{W}(q)\}$. For every $x\in H(V,q)$ there is $U(x)\in\mathcal{W}(x)$ such that $U(x)\subseteq X\setminus\overline{V}$. Let $\mu'(V)=\{U(x):x\in H(V,q)\}$. We have that $X\setminus\overline{V}$ is canonically open and contains $H(V,q)\subseteq H\subseteq X\setminus X_0$. We consider $W(V,q)=\overline{H(V,q)}\cap X_0\cap(X\setminus\overline{V})\subseteq\overline{H}$. Hence $|W(V,q)|\le 2^\tau$. For every $z\in W(V,q)$ we choose a $W(z)\in\mathcal{W}(z)$ such that $W(z)\subseteq X\setminus\overline{V}$. Let $\mu''(V)=\{W(z):z\in W(V,q)\}$ and let $\mu'''(V)=\mu'(V)\cup\mu''(V)$. Then $\mu'''(V)$ covers $H(V,q)\cup W(V,q)$ and from $c(X,X_0)\le\tau$ we can choose $\mu_0(V)\in[\mu'''(V)]^{\le\tau}$ such that $H(V,q)\subseteq\overline{\bigcup\mu_0(V)}$. But $\bigcup\mu_0(V)\subseteq X\setminus\overline{V}\subseteq X\setminus V$; then $\overline{\bigcup\mu_0(V)}\subseteq X\setminus\overline{V}$. Then we can choose an $\alpha_0\in\tau^+$ such that $\mu_0(V)\subseteq\mathcal{B}_{\alpha_0}$ for every $V\in\mathcal{W}(q)$. We have that $W=\bigcup\{\overline{\bigcup\mu_0(V)}:V\in\mathcal{W}(q)\}\supseteq H$ and $q\notin W$. So we have already chosen a point $\phi(W)\in(X\setminus(W\cup X_0))\cap H_\alpha\subseteq H$ and at the same time $W\supseteq H$ – a contradiction. □

In fact we have proved:

Theorem 3*. *If $\overline{X\setminus X_0}$ is a Hausdorff topological space then:*
$$|\overline{X\setminus X_0}|\le\exp c(X,X_0)\cdot\chi(\overline{X\setminus X_0}).$$

Corollary 3.1 ([6]). *If X is a Hausdorff topological space then:*
$$|X|\le\exp c(X)\cdot\chi(X).$$

We could also consider another generalization of $c(X)$ i.e. $c_1(X,X_0)\le\tau$ iff for every open in X family γ there is a $\gamma'\in[\gamma]^{\le\tau}$ such that $\overline{\bigcup\gamma'}\supseteq(\bigcup\gamma)\cap(X\setminus X_0)$. In the same way as in Theorem 3 we obtain:

Theorem 4. *If X is a Hausdorff topological space then:*

$$|X \setminus X_0| \leq \exp c_1(X, X_0) \cdot \chi(X).$$

Theorem 4*. *If $\overline{X \setminus X_0}$ is a Hausdorff topological space then:*

$$|\overline{X \setminus X_0}| \leq \exp c_1(X, X_0) \cdot \chi(\overline{X \setminus X_0}).$$

Examples

1. If X is a Lindelöf not hereditarily Lindelöf space and $X \setminus X_0$ is a non-Lindelöf subspace of X then $L(X, X_0) < L(X \setminus X_0)$.

2. If X is the one-point compactification of a discrete space $D(\tau)$ where $\tau > \aleph_0$ and $X \setminus X_0 = D(\tau)$, then $c(X, X_0) < c(X \setminus X_0)$ and $L(X, X_0) < L(X \setminus X_0)$. Also if X is a space with $c(X) \leq \aleph_0$ and $s(X) > \aleph_0$ and if $X \setminus X_0$ is an uncountable discrete subspace of X then $c(X, X_0) < c(X \setminus X_0)$.

3. Let $X = R_I \cup (\bigcup\{R_Q^\alpha : \alpha \in \tau\})$ for any $\tau > \aleph_0$, where R_I is the set of irrationals in R and $R_Q^\alpha = R_Q$ for every $\alpha \in \tau$. Let the points of R_Q^α have their usual neighborhoods and if $x \in R_I$ then let sets of the form $U(x, \varepsilon) = (R_I \cap (x - \varepsilon, x + \varepsilon)) \cup (\bigcup\{R_Q^\alpha \cap (x - \varepsilon, x + \varepsilon) : \alpha \in \tau\})$ be the local base in x. Then X is regular, $\chi(X) \leq \aleph_0$, $wL(X, X_0) \leq \aleph_0$ and $wL(X \setminus X_0) = \tau$, where $X_0 = R_I$.

References

[1] Archangel'skii A.V., *On the cardinality of bicompacta satisfying the first axiom of countability*, Soviet Math. Dokl. **10** (1969), 951–955.

[2] ———, *A theorem about cardinality*, Uspehi Matem. Nauk **34** (1979), 177–178.

[3] Archangel'skii A.V., M.M. Genedi Hamdi, *The position of subspaces in a topological space: relative compactness, Lindelöfness and axioms of separation*, Vestnik Moskovskogo Universiteta, ser I., vol. 6, 1989, 67–69.

[4] Bell M., Ginsburgh J., Woods G., *Cardinal inequalities for topological spaces involving the weak Lindelöf number*, Pacific J. Math. **79** (1978), 37–45.

[5] Engelking R., *General Topology*, Warszawa, 1976.

[6] Hajnal A., Juhasz I., *Discrete Subspaces of Topological Spaces I & II*, Pc. Koninkl. Nederl. Akad. Wet., Ser. A, 70 (1967), 343–356; 72 (1969), 18–30.

[7] Juhasz I., *Cardinal Functions in Topology – Ten Years Later*, Math. Centre Tracts 123, Amsterdam, 1980.

[8] Liu Xiao Shi, *Two cardinal functions of topological spaces and improvements of some famous cardinal inequalities*, Acta Math. Sinica **29** (1986), 494–497.

[9] Sun Shu-Hao, *A note on Archangel'skii's inequality*, J. Math. Soc. Japan **39** (1987), 363–365.

[10] Stavrova D.N., *A New Inequality for the Cardinality of topological Spaces*, Proc. of the XX-th Conf. of the UBM, 1991.

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