Boundary value problems for higher order ordinary differential equations

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Abstract. Let $f : [a,b] \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a Carathéodory's function. Let $\{t_h\}$, with $t_h \in [a,b]$, and $\{x_h\}$ be two real sequences. In this paper, the family of boundary value problems

$$\begin{cases} x^{(k)} = f\left(t, x, x', \dots, x^{(n)}\right) \\ x^{(i)}(t_i) = x_i\,, \quad i = 0, 1, \dots, k-1 \end{cases} \qquad (k = n+1, n+2, n+3, \dots)$$

is considered. It is proved that these boundary value problems admit at least a solution for each $k \geq \nu$, where $\nu \geq n+1$ is a suitable integer. Some particular cases, obtained by specializing the sequence $\{t_h\}$, are pointed out. Similar results are also proved for the Picard problem.

Keywords: higher order ordinary differential equations, Nicoletti problem, Picard problem

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1. Introduction

In this paper we shall consider the ordinary differential equation

$$x^{(k)} = f\left(t, x, x', \dots, x^{(n)}\right)$$

where $k \geq n+1$. The function f: $[a,b] \times \mathbb{R}^{n+1} \to \mathbb{R}$ will be assumed to verify Carathéodory's type conditions. We shall be concerned with the existence of solutions of this differential equation satisfying the boundary values

$$x^{(i)}(t_i) = x_i\,, \quad t_i \in [a,b]\,,\ x_i \in \mathbb{R}\,, \quad i = 0, 1, \dots, k-1$$

or

$$x(t_i) = x_i\,, \quad a \leq t_0 < t_1 < \dots < t_{k-1} \leq b\,,\ x_i \in \mathbb{R}\,, \quad i = 0, 1, \dots, k-1\,.$$

The question of the existence of solutions of the previous boundary value problems was widely investigated (see, for instance, [2], [3], [8] and the references given therein). To the best of our knowledge, the most of the existence results assume growth conditions on the function $f(t, x, x', \dots, x^{(n)})$ with respect to the variables $x, x', \dots, x^{(n)}$ or require that $(b-a)$ is suitably small. When one attempts to establish the existence *in the large* of the solutions of boundary value problems, that is, existence without restricting the length of the smallest interval containing the points t_i, many difficulties arise.

In this paper no condition on $(b-a)$ neither further assumptions on f are imposed; existence theorems are established provided that the order k is greater than or equal to a suitable integer ν. Really, our assumptions also guarantee the existence of solutions for every $k \geq \nu$.

To give a more precise idea of these features, we present a result which follows immediately from Theorem 3 below, concerning the Cauchy problem.

Theorem A. *Let $\{x_h\}$ be a real bounded sequence and $\alpha = \sup_{h\geq 0} |x_h|$. Then, for every $\varrho > \alpha\, e^{b-a}$ there exists an integer $\nu \geq n+1$ such that, for every $k \geq \nu$, the problem*

$$\begin{cases} x^{(k)} = f\left(t, x, x', \dots, x^{(n)}\right) \\ x^{(i)}(a) = x_i\,, \quad i = 0, 1, \dots, k-1 \end{cases}$$

has at least a solution u_k with $u_k^{(k-1)}$ absolutely continuous in $[a,b]$ and $\|u_k\|_{C^n([a,b])} \leq \varrho$.

2. Notations and preliminary results

Let $[a,b]$ be a compact real interval and m a positive integer. In order to obtain our results, we introduce two classes of polynomials and describe some of their properties, which will be useful in the next section.

The Abel-Gontcharoff polynomials [10] constitute the first family.

Let $t_0, t_1, \dots, t_{m-1}$ m be arbitrary parameters belonging to $[a,b]$; we define, for every $t \in [a,b]$,

$$G_m(t; t_0, t_1, \dots, t_{m-1}) = \int_{t_0}^{t} ds_1 \int_{t_1}^{s_1} ds_2 \cdots \int_{t_{m-1}}^{s_{m-1}} ds_m\,, \tag{2.1}$$

and put $G_0(t) = 1$. Clearly, $G_m(t; t_0, t_1, \dots, t_{m-1})$ is an m-degree polynomial in t. It is immediate to verify that, if $h \leq m$,

$$D^h G_m(t; t_0, t_1, \dots, t_{m-1}) = G_{m-h}(t; t_h, t_{h+1}, \dots, t_{m-1})\,; \tag{2.2}$$

then $D^h G_m(t_h; t_0, t_1, \dots, t_{m-1}) = 0$ for $h = 0, 1, \dots, m-1$. By the definition (2.1), it is easy to prove that the following identities

$$G_m(t; t_0, t_1, \dots, t_{m-1}) = G_m(t+\tau; t_0+\tau, t_1+\tau, \dots, t_{m-1}+\tau)\,, \tag{2.3}$$

$$G_m(\lambda t; \lambda t_0, \lambda t_1, \dots, \lambda t_{m-1}) = \lambda^m G_m(t; t_0, t_1, \dots, t_{m-1}) \tag{2.4}$$

hold for all real numbers τ and λ. Useful bounds are given for these polynomials introducing the nonnegative functions $\gamma_0(t) = 1$ and, by recurrence,

$$\gamma_m(t; t_0, t_1, \dots, t_{m-1}) = \left| \int_{t_0}^{t} \gamma_{m-1}(s; t_1, t_2, \dots, t_{m-1})\, ds \right| .$$

Since,

$$|G_m(t; t_0, t_1, \dots, t_{m-1})| \le \gamma_m(t; t_0, t_1, \dots, t_{m-1}) ,$$

we are interested in finding upper bounds for the function γ_m. The following significative inequality was proved by Gontcharoff (see, for instance, [10, p. 38])

$$\gamma_m(t; t_0, t_1, \dots, t_{m-1}) \le \frac{1}{m!} \left(|t - t_0| + \sigma_{m-1}\right)^m , \tag{2.5}$$

where

$$\sigma_0 = 0, \quad \sigma_{m-1} = \sum_{h=0}^{m-2} |t_{h+1} - t_h| , \quad m \ge 2 .$$

Another inequality, which requires more details, will be recalled in the sequel.

The Lagrange polynomials constitute the other family. Assuming now that $a \le t_0 < t_1 < \cdots < t_{m-1} \le b$, one defines, for every $t \in [a, b]$ and $0 \le h \le m - 1$,

$$L_h(t; t_0, t_1, \dots, t_{m-1}) = \frac{(t - t_0)(t - t_1) \cdots (t - t_{h-1})(t - t_{h+1}) \cdots (t - t_{m-1})}{(t_h - t_0)(t_h - t_1) \cdots (t_h - t_{h-1})(t_h - t_{h+1}) \cdots (t_h - t_{m-1})} ,$$

that is, the $(m - 1)$-degree polynomial satisfying

$$L_h(t_j; t_0, t_1, \dots, t_{m-1}) = \begin{cases} 0 & \text{if } j \ne h \\ 1 & \text{if } j = h \end{cases} .$$

We denote by $C^{m-1}([a, b])$ the space of all real-valued functions defined on $[a, b]$ having continuous derivatives up to the order $m - 1$ on $[a, b]$; the norm in this space is given by

$$\|u\|_{C^{m-1}([a,b])} = \max_{0 \le i \le m-1} \left[\max_{t \in [a,b]} |u^{(i)}(t)| \right] , \quad u \in C^{m-1}([a, b]) .$$

The space of all real-valued functions $u \in C^{m-1}([a, b])$ such that $u^{(m-1)}$ is absolutely continuous in $[a, b]$, is denoted by $W^{m,1}([a, b])$.

Let $u \in W^{m,1}([a, b])$ and let $x_0, x_1, \dots, x_{m-1}$ be m given real numbers such that

$$u^{(i)}(t_i) = x_i , \quad i = 0, 1, \dots, m - 1 . \tag{2.6}$$

The function

$$\sigma(t) = u(t) - \sum_{h=0}^{m-1} x_h G_h(t; t_0, t_1, \dots, t_{h-1})$$

belongs to $W^{m,1}([a,b])$ and satisfies the conditions $\sigma^{(i)}(t_i) = 0$ $(i = 0, 1, \dots, m-1)$ and $\sigma^{(m)}(t) = u^{(m)}(t)$ almost everywhere in $[a,b]$. By the identities

$$\begin{aligned}
&\int_{t_0}^{t} ds_1 \int_{t_1}^{s_1} ds_2 \cdots \int_{t_{m-1}}^{s_{m-1}} \sigma^{(m)}(s_m)\, ds_m \\
&= \int_{t_0}^{t} ds_1 \int_{t_1}^{s_1} ds_2 \cdots \int_{t_{m-2}}^{s_{m-2}} \sigma^{(m-1)}(s_{m-1})\, ds_{m-1} \\
&= \int_{t_0}^{t} ds_1 \int_{t_1}^{s_1} ds_2 \cdots \int_{t_{m-3}}^{s_{m-3}} \sigma^{(m-2)}(s_{m-2})\, ds_{m-2} = \cdots = \sigma(t), \qquad t \in [a,b]\,,
\end{aligned}$$

we can write

$$\begin{aligned}
u(t) = &\sum_{h=0}^{m-1} x_h G_h(t; t_0, t_1, \dots, t_{h-1}) \\
&+ \int_{t_0}^{t} ds_1 \int_{t_1}^{s_1} ds_2 \cdots \int_{t_{m-1}}^{s_{m-1}} u^{(m)}(s)\, ds\,.
\end{aligned} \tag{2.7}$$

Using (2.2), it follows that, for $i < m$,

$$\begin{aligned}
u^{(i)}(t) = &\sum_{h=i}^{m-1} x_h G_{h-i}(t; t_i, t_{i+1}, \dots, t_{h-1}) \\
&+ \int_{t_i}^{t} ds_{i+1} \int_{t_{i+1}}^{s_{i+1}} ds_{i+2} \cdots \int_{t_{m-1}}^{s_{m-1}} u^{(m)}(s) ds_m\,.
\end{aligned}$$

This formula implies that the conditions (2.6) are included in (2.7).

In the particular case $t_0 = t_1 = \cdots = t_{m-1} = a$, the identity (2.7) becomes the Taylor's formula

$$u(t) = \sum_{h=0}^{m-1} \frac{(t-a)^h}{h!} x_h + \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} u^{(m)}(s)\, ds\,.$$

Another interpolation formula is given when $u \in W^{m,1}([a,b])$ and verifies the conditions

$$u(t_i) = x_i, \qquad i = 0, 1, \dots, m-1\,,$$

where $t_0 < t_1 < \cdots < t_{m-1}$ and x_i $(i = 0, 1, \ldots, m-1)$ are assigned real numbers. We have the identity

$$
\begin{aligned}
u(t) = & \sum_{h=0}^{m-1} \left[x_h - \frac{1}{(m-1)!} \int_a^{t_h} (t_h - s)^{m-1} u^{(m)}(s)\, ds \right] L_h(t; t_0, t_1, \ldots, t_{m-1}) \\
& + \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} u^{(m)}(s)\, ds\,, \quad t \in [a, b]\,.
\end{aligned}
\tag{2.8}
$$

3. Existence theorems

Let n be a nonnegative integer and f a real-valued function defined on $[a,b] \times \mathbb{R}^{n+1}$. On $\mathbb{R}^{n+1}$ we introduce the norm $\|z\| = \max_{0 \le i \le n} |z_i|$, where $z = (z_0, z_1, \ldots, z_n)$. We assume that f satisfies the Carathéodory's conditions

- (a_1) the function $t \to f(t, z)$ is measurable for every $z \in \mathbb{R}^{n+1}$;
- (a_2) the function $z \to f(t, z)$ is continuous for almost every $t \in [a, b]$;
- (a_3) the function $t \to \varphi(t, \varrho) = \sup_{\|z\| \le \varrho} |f(t, z)|$ is integrable on $[a, b]$ for every $\varrho > 0$.

We consider the ordinary differential equation

$$
x^{(k)} = f\left(t, x, x', \ldots, x^{(n)}\right)\,, \tag{3.1}
$$

where k is an integer greater than n. Since f satisfies the assumptions (a_1) and (a_2), it is reasonable to call a solution in $[a, b]$ of the equation (3.1) a function $u \in W^{k,1}([a,b])$ such that

$$
u^{(k)}(t) = f\left(t, u(t), u'(t), \ldots, u^{(n)}(t)\right)
$$

almost everywhere in $[a, b]$.

3.1 The Nicoletti problem.

The first result concerns the boundary value problem

$$
\begin{cases} x^{(k)} = f\left(t, x, x', \ldots, x^{(n)}\right) \\ x^{(i)}(t_i) = x_i\,, \quad i = 0, 1, \ldots, k-1 \end{cases} \tag{A_k}
$$

where $t_i \in [a, b]$ and $x_i \in \mathbb{R}$ $(i = 0, 1, \ldots, k-1)$.

Theorem 1. *Let f satisfy the assumptions* (a_1)–(a_3). *Let $\{t_h\}$ and $\{x_h\}$ be two real sequences, with $t_h \in [a, b]$ for every $h \ge 0$. If*

$$
\lim_{h \to \infty} \left[\max_{t \in [a,b]} \gamma_h(t; t_n, t_{n+1}, \ldots, t_{n+h-1}) \right] = 0 \tag{3.2}
$$

and, for every $k \geq n+1$,

$$\max_{t\in[a,b]}\left|\sum_{h=i}^{k-1} x_h G_{h-i}(t;t_i,t_{i+1},\dots,t_{h-1})\right| \leq M \ , \quad i=0,1,\dots,n \tag{3.3}$$

for some constant M, then, for every $\varrho > M$, there exists an integer $\nu \geq n+1$ such that the problem (A_k) *admits at least a solution $u_k \in W^{k,1}([a,b])$ for all $k \geq \nu$; moreover, $\|u_k\|_{C^n([a,b])} \leq \varrho$.*

Proof: Let $\varrho > M$. Owing to the assumptions, there exists an integer $\nu \geq n+1$ such that, for all $k \geq \nu$,

$$M + (b-a)^{n-i} \max_{t\in[a,b]} \gamma_{k-n-1}(t;t_n,t_{n+1},\dots,t_{k-2}) \int_a^b \varphi(s,\varrho)\, ds \leq \varrho\,, \qquad i=0,1,\dots,n\,. \tag{3.4}$$

Fix $k \geq \nu$. Taking into account (2.7), one can simply check that the problem (A_k) is equivalent to the problem

$$\begin{aligned} u(t) = {} & \sum_{h=0}^{k-1} x_h G_h(t;t_0,t_1,\dots,t_{h-1}) \\ & + \int_{t_0}^{t} ds_1 \int_{t_1}^{s_1} ds_2 \cdots \int_{t_{k-1}}^{s_{k-1}} f\left(s,u(s),u'(s),\dots,u^{(n)}(s)\right)\, ds\,, \\ & u \in C^n([a,b]). \end{aligned} \tag{3.5}$$

It is convenient to define

$$u_0(t) = \sum_{h=0}^{k-1} x_h G_h(t;t_0,t_1,\dots,t_{h-1})$$

and, for every $u \in C^n([a,b])$,

$$T(u)(t) = u_0(t) + \int_{t_0}^{t} ds_1 \int_{t_1}^{s_1} ds_2 \cdots \int_{t_{k-1}}^{s_{k-1}} f\left(s,u(s),u'(s),\dots,u^{(n)}(s)\right)\, ds\,, \qquad t\in[a,b]\,.$$

It is clear that $T(u) \in C^n([a,b])$ for every $u \in C^n([a,b])$. Since the problem (3.5) now can be written as $u = T(u)$, our aim is to find a fixed point of T in $C^n([a,b])$. Let us indicate with $\mathcal{B}_\varrho$ the set $\left\{u \in C^n([a,b]) \,:\, \|u\|_{C^n([a,b])} \leq \varrho\right\}$. We first show that $T\left(\mathcal{B}_\varrho\right) \subseteq \mathcal{B}_\varrho$. Indeed, if $u \in \mathcal{B}_\varrho$, $0 \le i \le n$ and $t \in [a,b]$, we have

$$\begin{aligned}
&\left|D^i T(u)(t)\right| \le \left|u_0^{(i)}(t)\right| \\
&+ \left|\int_{t_i}^{t} ds_{i+1} \left|\int_{t_{i+1}}^{s_{i+1}} ds_{i+2}\right| \cdots \left|\int_{t_{k-1}}^{s_{k-1}} f\left(s, u(s), u'(s), \dots, u^{(n)}(s)\right) ds\right| \cdots \right|\Bigg|\Bigg| \\
&\le M + \gamma_{k-i-1}(t; t_i, t_{i+1}, \dots, t_{k-2}) \int_a^b \varphi(s,\varrho)\, ds \\
&\le M + (b-a)^{n-i} \max_{t\in[a,b]} \gamma_{k-n-1}(t; t_n, t_{n+1}, \dots, t_{k-2}) \int_a^b \varphi(s,\varrho)\, ds \le \varrho\,.
\end{aligned}$$

Then, by (3.4),

$$\|T(u)\|_{C^n([a,b])} \le \varrho\,.$$

Next, if $u \in \mathcal{B}_\varrho$ then

$$\begin{aligned}
&\left|D^n\left[T(u)(t') - T(u)(t'')\right]\right| \\
&\le \left|u_0^{(n)}(t') - u_0^{(n)}(t'')\right| \\
&+ \left|\int_{t'}^{t''} ds_{n+1} \cdots \int_{t_{k-1}}^{s_{k-1}} f\left(s_k, u(s_k), u'(s_k), \dots, u^{(n)}(s_k)\right) ds_k\right| \\
&\le \left|u_0^{(n)}(t') - u_0^{(n)}(t'')\right| + \left|\int_{t'}^{t''} ds_{n+1}\right| \cdots \left|\int_{t_{k-1}}^{s_{k-1}} \varphi(s_k,\varrho)\, ds_k\right| \cdots \Big|\Big|
\end{aligned}$$

for every t', $t'' \in [a,b]$. Using the uniform continuity of the polynomial $u_0^{(n)}$ and the absolute continuity of the integral, we obtain that the set $\left\{D^n T(u) : u \in \mathcal{B}_\varrho\right\}$ is equicontinuous in $[a,b]$. By the Ascoli-Arzelà Theorem (see, for instance, [7, Theorem 54. IV]), it follows that the set $T(\mathcal{B}_\varrho)$ is relatively compact in $C^n([a,b])$.

Finally we prove that the operator T is continuous in $\mathcal{B}_\varrho$. Let $w \in \mathcal{B}_\varrho$ and $\{w_p\}$ be a sequence in $\mathcal{B}_\varrho$ such that

$$\lim_{p\to\infty} \left\|w_p - w\right\|_{C^n([a,b])} = 0\,.$$

Since

$$\left|f\left(s, w_p(s), w_p'(s), \dots, w_p^{(n)}(s)\right)\right| \le \varphi(s,\varrho)$$

for every $p \ge 0$ and $s \in [a,b]$, and

$$\lim_{p\to\infty} f\left(s, w_p(s), w_p'(s), \dots, w_p^{(n)}(s)\right) = f\left(s, w(s), w'(s), \dots, w^{(n)}(s)\right)$$

almost everywhere in $[a,b]$, by using the Lebesgue Dominated Convergence Theorem, we obtain

$$\text{(3.6)} \quad \lim_{p\to\infty}\int_a^b \left| f\left(s, w_p(s), w_p'(s), \dots, w_p^{(n)}(s)\right) - \right.$$
$$\left. - f\left(s, w(s), w'(s), \dots, w^{(n)}(s)\right)\right| ds = 0 .$$

Moreover, we have, for every $t \in [a,b]$, $0 \le i \le n$ and $p \ge 0$,

$$\begin{aligned}
&\left| D^i T(w_p)(t) - D^i T(w)(t) \right| \\
&= \left| \int_{t_i}^{t} ds_{i+1} \cdots \int_{t_{k-1}}^{s_{k-1}} f\left(s, w_p(s), w_p'(s), \dots, w_p^{(n)}(s)\right) ds \right. \\
&\left. - \int_{t_i}^{t} ds_{i+1} \cdots \int_{t_{k-1}}^{s_{k-1}} f\left(s, w(s), w'(s), \dots, w^{(n)}(s)\right) ds \right| \\
&\le \max_{t\in[a,b]} \gamma_{k-i-1}(t; t_i, t_{i+1}, \dots, t_{k-2}) \\
&\times \int_a^b \left| f\left(s, w_p(s), w_p'(s), \dots, w_p^{(n)}(s)\right) - f\left(s, w(s), w'(s), \dots, w^{(n)}(s)\right)\right| ds .
\end{aligned}$$

Hence, from (3.6), it follows

$$\lim_{p\to\infty} \left\| T(w_p) - T(w) \right\|_{C^n([a,b])} = 0 .$$

By the Schauder Fixed Point Theorem, there exists a function $u_k \in \mathcal{B}_\varrho$ such that $u_k = T(u_k)$. This completes the proof. $\square$

We think it useful to give some remarks to explain the assumptions of Theorem 1. In the trivial case $f(t,z) = 0$, the problem (A_k) has, for each fixed $k \ge n+1$, a unique solution u_k^0, given by

$$u_k^0(t) = \sum_{h=0}^{k-1} x_h G_h(t; t_0, t_1, \dots, t_{h-1}), \quad t \in [a,b] .$$

The family $\left\{u_k^0, k \ge n+1\right\}$ is equibounded in $C^n([a,b])$ if and only if (3.3) is satisfied. Therefore, this assumption is necessary in order to require that the problems (A_k) have an equibounded family of solutions.

In the sequel we shall show how, in some cases, the assumption (3.2) is verified for particular choices of the point t_i or if $b-a < \pi/2$. When $b-a > \pi/2$, we shall prove that (3.2) is a necessary condition for the equiboundedness of the solutions of (A_k) for arbitrary t_i.

3.2 Some particular cases.

A variety of special cases of the boundary value problem (A_k) is particularly interesting. We examine some of them, specializing the sequence $\{t_h\}$.

3.2.1. If the sequence $\{t_h\}$ is monotone, the inequality (2.5) implies

$$\gamma_h(t; t_n, t_{n+1}, \dots, t_{n+h-1}) \le \frac{2^h}{h!}(b-a)^h\,; \tag{3.7}$$

hence the assumption (3.2) is satisfied. Furthermore, for every $k \ge n+1$, we have

$$\begin{aligned}
&\max_{t\in[a,b]} \left|\sum_{h=i}^{k-1} x_h G_{h-i}(t; t_i, t_{i+1}, \dots, t_{h-1})\right| \\
&\le \sum_{j=0}^{k-i-1} |x_{i+j}| \max_{t\in[a,b]} \gamma_j(t; t_i, t_{i+1}, \dots, t_{i+j-1}) \\
&\le \sum_{j=0}^{k-i-1} |x_{i+j}| \frac{2^j}{j!}(b-a)^j\,, \quad i = 0, 1, \dots, n\,.
\end{aligned} \tag{3.8}$$

Therefore we can state the following

Theorem 2. *Let f satisfy the assumptions* (a_1)–(a_3) *and $\{x_h\}$ be a real sequence such that*

$$\sum_{h=0}^{\infty} |x_{h+i}| \frac{2^h}{h!}(b-a)^h \le M\,, \quad i = 0, 1, \dots, n$$

for some constant M. Then, for every $\varrho > M$, there exists an integer $\nu \ge n+1$ such that the problem (A_k)*, with $t_0, t_1, \dots, t_{k-1}$ monotone finite sequence in $[a,b]$, admits at least a solution $u_k \in W^{k,1}([a,b])$ for all $k \ge \nu$; moreover, $\|u_k\|_{C^n([a,b])} \le \varrho$.*

Proof: One can simply repeat the same arguments of Theorem 1, using now the explicit bounds given in (3.7) and (3.8). We point out only that the integer ν is furnished, for fixed $\varrho > M$, by the inequality

$$M + (b-a)^{k-i-1}\frac{2^{k-n-1}}{(k-n-1)!}\int_a^b \varphi(s,\varrho)\,ds \le \varrho\,, \quad i = 0, 1, \dots, n\,.$$

□

3.2.2. In the particular case $a = t_0 = t_1 = \dots = t_{k-1}$, the inequality (2.5) becomes

$$\gamma_h(t; t_n, t_{n+1}, \dots, t_{n+h-1}) \le \frac{(b-a)^h}{h!}$$

and, for $i = 0, 1, \dots, n$ and $k \ge n+1$, we obtain

$$\max_{t\in[a,b]} \left|\sum_{h=i}^{k-1} x_h G_{h-i}(t; t_i, t_{i+1}, \dots, t_{h-1})\right| \le \sum_{j=0}^{k-i-1} |x_{i+j}| \frac{(b-a)^j}{j!}\,.$$

So we have the following theorem, concerning the Cauchy problem.

Theorem 3. *Let f satisfy the assumptions* (a$_1$)–(a$_3$) *and* $\{x_h\}$ *be a real sequence such that*

$$\sum_{h=0}^{\infty} |x_{h+i}| \frac{(b-a)^h}{h!} \leq M\,, \quad i = 0, 1, \dots, n$$

for some constant M. Then, for every $\varrho > M$, there exists an integer $\nu \geq n+1$ such that the problem (A$_k$), *with $t_i = a$, admits at least a solution $u_k \in W^{k,1}([a,b])$ for all $k \geq \nu$; moreover, $\|u_k\|_{C^n([a,b])} \leq \varrho$.*

3.2.3. We now study the case when

$$|t_h - t_{h-1}| \leq \mu \leq \frac{1}{e} \quad \text{for every positive integer } h\,.$$

Using the inequality (2.5) and the Stirling's formula (see, for instance, [1, p. 257]), we get

$$\begin{aligned} \gamma_h(t; t_n, t_{n+1}, \dots, t_{n+h-1}) &\leq \frac{[b-a+\mu(h-1)]^h}{h!} \\ &\leq \frac{1}{\sqrt{2\pi h}} \left[e\frac{b-a}{h} + \mu e \frac{h-1}{h} \right]^h \leq \frac{1}{\sqrt{2\pi h}} e^{e(b-a)} \end{aligned}$$

and

$$\begin{aligned} &\max_{t\in[a,b]} \left| \sum_{h=i}^{k-1} x_h G_{h-i}(t; t_i, t_{i+1}, \dots, t_{h-1}) \right| \\ &\leq |x_i| + \sum_{j=1}^{k-i-1} |x_{i+j}| \frac{1}{\sqrt{2\pi j}} \left[e\frac{b-a}{j} + \mu e \right]^j . \end{aligned}$$

Hence we can formulate the following

Theorem 4. *Let f satisfy the assumptions* (a$_1$)–(a$_3$) *and* $\{x_h\}$ *be a real sequence such that*

$$|x_i| + \sum_{h=1}^{\infty} |x_{h+i}| \frac{1}{\sqrt{2\pi h}} \left[e\frac{(b-a)}{h} + \mu e \right]^h \leq M\,, \quad i = 0, 1, \dots, n\,,$$

where $\mu \leq \frac{1}{e}$ and M are constants. Then, for every $\varrho > M$, there exists an integer $\nu \geq n+1$ such that the problem (A$_k$), *with $|t_h - t_{h-1}| \leq \mu$, for $h = 1, 2, \dots, k-1$, admits at least a solution $u_k \in W^{k,1}([a,b])$ for all $k \geq \nu$; moreover, $\|u_k\|_{C^n([a,b])} \leq \varrho$.*

3.2.4. The last result is based on some inequalities obtained by Bernstein in [4], [5] and Schoenberg in [9]. Using the identities (2.3) and (2.4), we obtain

$$\begin{aligned} G_m(t;t_0,t_1,\dots,t_{m-1}) &= G_m(t-a;t_0-a,t_1-a,\dots,t_{m-1}-a) \\ &= (b-a)^m G_m\left(\frac{t-a}{b-a};\frac{t_0-a}{b-a},\frac{t_1-a}{b-a},\dots,\frac{t_{m-1}-a}{b-a}\right);\end{aligned}$$

hence, for arbitrary $t,t_0,t_1,\dots,t_{m-1}$ $(m\ge 1)$, it follows (see [9, inequality (12)])

$$\begin{aligned} \gamma_m(t;t_0,t_1,\dots,t_{m-1}) &\le |G_m(b;a,b,a,\dots)| = (b-a)^m\,|G_m(1;0,1,0,\dots)| \\ &= (b-a)^m\,|G_m(0;1,0,1,\dots)| \\ &= (b-a)^m \begin{cases} \frac{1}{m!}|E_m| & \text{if } m \text{ is even} \\ \frac{2^{m+1}}{(m+1)!}\left(2^{m+1}-1\right)|B_{m+1}| & \text{if } m \text{ is odd}\end{cases},\end{aligned}$$

where E_m and B_{m+1} are the Euler and Bernoulli numbers respectively. A more explicit bound is obtained taking into account the inequalities (see [1, p. 805])

$$2\left(\frac{2}{\pi}\right)^{m+1}\left(1-\frac{1}{3^{m+1}+1}\right) < \frac{|E_m|}{m!} < 2\left(\frac{2}{\pi}\right)^{m+1}, \qquad m \text{ even}$$

$$\frac{2(m+1)!}{(2\pi)^{m+1}} < |B_{m+1}| < \frac{2(m+1)!}{(2\pi)^{m+1}}\left(\frac{1}{1-2^{-m}}\right), \ m \text{ odd}.$$

Indeed, we have

$$\begin{aligned} &2\left(\frac{2}{\pi}\right)^{m+1}\left(1-\frac{1}{2^{m+1}}\right) \\ &< \frac{|G_m(b;a,b,\dots)|}{(b-a)^m} < 2\left(\frac{2}{\pi}\right)^{m+1}\left(1+\frac{1}{2^{m+1}-2}\right). \end{aligned} \tag{3.9}$$

Because

$$\max_{t\in[a,b]}\gamma_h(t;t_n,t_{n+1},\dots,t_{n+h-1}) < 3\left(\frac{2}{\pi}\right)^{h+1}(b-a)^h$$

for every sequence $\{t_h\}$ then, if $b-a<\pi/2$,

$$\lim_{h\to\infty}\left[\max_{t\in[a,b]}\gamma_h(t;t_n,t_{n+1},\dots,t_{n+h-1})\right] = 0\,,$$

that is the assumption (3.2) is always verified. Moreover,

$$\begin{aligned} &\max_{t\in[a,b]}\left|\sum_{h=i}^{k-1} x_h G_{h-i}(t;t_i,t_{i+1},\dots,t_{h-1})\right| \\ &\le 3\sum_{j=0}^{k-i-1}|x_{i+j}|\left(\frac{2}{\pi}\right)^{j+1}(b-a)^j\,, \quad i=0,1,\dots,n\,. \end{aligned}$$

Then, by means of the same arguments used in the previous proofs, one can verify the following

Theorem 5. *Assume that $b-a<\pi/2$. Let f satisfy the hypotheses* (a_1)–(a_3) *and $\{x_h\}$ be a real sequence such that*

$$\frac{6}{\pi}\sum_{h=0}^{\infty}|x_{h+i}|\left[\frac{2}{\pi}(b-a)\right]^h\le M\,,\quad i=0,1,\dots,n$$

for some constant M. Then, for every $\varrho>M$, there exists an integer $\nu\ge n+1$ such that the problem (A_k) *admits at least a solution $u_k\in W^{k,1}([a,b])$ for all $k\ge\nu$; moreover, $\|u_k\|_{C^n([a,b])}\le\varrho$.*

The constraint $b-a<\pi/2$ is not artificious, but it is related to the Abel-Gontcharoff interpolation polynomials. In fact, (2.7) can be regarded as an interpolation formula, where the term

$$R_m(t;t_0,t_1,\dots,t_{m-1})=\int_{t_0}^{t}ds_1\int_{t_1}^{s_1}ds_2\cdots\int_{t_{m-1}}^{s_{m-1}}u^{(m)}(s)\,ds$$

is the remainder. If $u^{(m)}(t)=1$ and $t_0=a$, $t_1=b$, $t_2=a,\dots$, then $R_m(t;t_0,t_1,\dots,t_{m-1})$ becomes $G_m(t;a,b,a,\dots)$. Hence, when $b-a>\pi/2$, taking into account (3.9), it is immediate to see that $\max_{t\in[a,b]}|R_m(t;a,b,a,\dots)|$ is great for large m.

The same constraint ($b-a<\pi/2$) appeared in a previous result (Theorem 1 of [6]) where a different technique, which avoids the use of the Abel-Gontcharoff polynomials, was applied.

Finally we shall show that, when $b-a>\pi/2$, the condition (3.2) is in general necessary in order to get the equiboundedness in $C^n([a,b])$ of the family $\{u_k\}$. To this aim, assume $b-a>\pi/2$ and consider the problem

$$(A_k^1)\qquad\begin{cases}x^{(k)}=1\\ x^{(i)}(t_i)=0\,,\quad i=0,1,\dots,k-1\end{cases}$$

where $t_i=a$ for i even and $t_i=b$ for i odd. Taking in mind the inequality (3.9), it is easy to check that (3.2) does not hold. The problem (A_k^1) has, for each $k\ge1$, a unique solution given by $G_k(t;a,b,a,\dots)$, $t\in[a,b]$. Again by (3.9), we obtain

$$\lim_{k\to\infty}\max_{t\in[a,b]}|G_k(t;a,b,a,\dots)|=\lim_{k\to\infty}2\left(\frac{2}{\pi}\right)^{k+1}(b-a)^k\,;$$

so that the solutions of (A_k^1) are not equibounded because $b-a>\pi/2$.

3.3 The Picard problem.

The other main result concerns the Picard problem

$$(B_k)\qquad\begin{cases}x^{(k)}=f\left(t,x,x',\dots,x^{(n)}\right)\\ x(t_i)=x_i\,,\quad i=0,1,\dots,k-1\end{cases}$$

where $a\le t_0<t_1\dots<t_{k-1}\le b$ and $x_i\in\mathbb{R}$ $(i=0,1,\dots,k-1)$.

Theorem 6. *Let f satisfy the assumptions* (a_1)–(a_3). *Let $\{t_h\}$ be an increasing sequence in $[a,b]$ and $\{x_h\}$ a real sequence such that for every $k \geq n+1$*

$$\max_{t\in[a,b]} \left| D^i \sum_{h=0}^{k-1} x_h L_h(t; t_0, t_1, \dots, t_{k-1}) \right| \leq M\,, \quad i = 0, 1, \dots, n \tag{3.10}$$

for some constant M. Then, for every $\varrho > M$, there exists an integer $\nu \geq n+1$ such that the problem (B_k) *admits at least a solution $v_k \in W^{k,1}([a,b])$ for all $k \geq \nu$; moreover, $\|v_k\|_{C^n([a,b])} \leq \varrho$.*

Proof: Let $\varrho > M$. Owing to the assumptions, there exists an integer $\nu \geq n+1$ such that, for every $k \geq \nu$,

$$M + \frac{(b-a)^{k-i-1}}{(k-i-1)!} \int_a^b \varphi(s, \varrho)\, ds \leq \varrho\,, \quad i = 0, 1, \dots, n\,.$$

Let us fix $k \geq \nu$. From the identity (2.8) it follows that (B_k) is equivalent to the problem

$$\begin{aligned} v(t) = \sum_{h=0}^{k-1} &\left[x_h - \frac{1}{(k-1)!} \int_a^{t_h} (t_h - s)^{k-1} f\left(s, v(s), v'(s), \dots, v^{(n)}(s)\right) ds \right] \\ &\times L_h(t; t_0, t_1, \dots, t_{k-1}) \\ &+ \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} f\left(s, v(s), v'(s), \dots, v^{(n)}(s)\right) ds\,, \\ &v \in C^n([a,b]). \end{aligned} \tag{3.11}$$

We look for a solution $v \in \mathcal{B}_\varrho$, where $\mathcal{B}_\varrho$ is the same set previously defined. For every $t \in [a,b]$ and every $v \in \mathcal{B}_\varrho$, we set

$$v_0(t) = \sum_{h=0}^{k-1} x_h L_h(t; t_0, t_1, \dots, t_{k-1})\,,$$

$$\begin{aligned} P(v)(t) = -\frac{1}{(k-1)!} &\sum_{h=0}^{k-1} L_h(t; t_0, t_1, \dots, t_{k-1}) \\ &\times \int_a^{t_h} (t_h - s)^{k-1} f\left(s, v(s), v'(s), \dots, v^{(n)}(s)\right) ds \end{aligned}$$

and

$$S(v)(t) = v_0(t) + P(v)(t) + \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} f\left(s, v(s), v'(s), \dots, v^{(n)}(s)\right) ds\,.$$

So the equation (3.11) becomes $v = S(v)$, $v \in \mathcal{B}_\varrho$. Since

$$P(v)(t) + \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} f\left(s, v(s), v'(s), \dots, v^{(n)}(s)\right) ds$$

vanishes when $t = t_i$ $(i = 0, 1, \ldots, k-1)$ and its k-order derivative is equal to $f\left(t, v(t), v'(t), \ldots, v^{(n)}(t)\right)$ for almost every $t \in [a, b]$, by applying a lemma of [11], we obtain

$$\begin{aligned}
&\left| D^i \left[P(v)(t) + \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} f\left(s, v(s), v'(s), \ldots, v^{(n)}(s)\right) ds \right] \right| \\
&\leq \frac{(b-a)^{k-i-1}}{(k-i-1)!} \int_a^b \left| f\left(s, v(s), v'(s), \ldots, v^{(n)}(s)\right) \right| ds \\
&\leq \frac{(b-a)^{k-i-1}}{(k-i-1)!} \int_a^b \varphi(s, \varrho)\, ds\,, \quad i = 0, 1, \ldots, n\,, \quad t \in [a, b]\,.
\end{aligned}$$

Therefore, by the assumption (3.10), we have

$$\left| D^i \left[S(v)(t) \right] \right| \leq M + \frac{(b-a)^{k-i-1}}{(k-i-1)!} \int_a^b \varphi(s, \varrho)\, ds \leq \varrho\,, \quad i = 0, 1, \ldots, n\,,\ t \in [a, b]\,,$$

that is $S(\mathcal{B}_\varrho) \subseteq \mathcal{B}_\varrho$. To prove that $S(\mathcal{B}_\varrho)$ is relatively compact in $C^n([a, b])$, it is sufficient to show that the set $\left\{ D^n S(v) : v \in \mathcal{B}_\varrho \right\}$ is equicontinuous. This is achieved immediately, noticing that $D^n \left[v_0(t) + P(v)(t) \right]$ is a polynomial with equibounded coefficients as $v \in \mathcal{B}_\varrho$, and using the inequality

$$\begin{aligned}
&\left| \int_a^{t'} (t'-s)^{k-n-1} f\left(s, v(s), v'(s), \ldots, v^{(n)}(s)\right) ds \right. \\
&\left. - \int_a^{t''} (t''-s)^{k-n-1} f\left(s, v(s), v'(s), \ldots, v^{(n)}(s)\right) ds \right| \\
&\leq (b-a)^{k-n-1} \left| \int_{t'}^{t''} \varphi(s, \varrho)\, ds \right| \\
&+ \int_a^b \left| (t'-s)^{k-n-1} - (t''-s)^{k-n-1} \right| \varphi(s, \varrho)\, ds\,, \quad t', t'' \in [a, b]\,.
\end{aligned}$$

The continuity in $\mathcal{B}_\varrho$ of the operator S can be established by means of the same arguments which we have used in the proof of Theorem 1. Owing to (3.11), the existence of a solution of the problem (B_k) is now obtained by applying the Schauder Fixed Point Theorem to the operator S. $\square$

The hypothesis (3.10) is satisfied in the significative case (see [2, Corollary 9.8]) $x_i = g(t_i)$, with $g \in C^\infty([a, b])$ and $\max_{t \in [a,b]} \left| g^{(j)}(t) \right| \leq \mu$ $(j = 0, 1, 2, \ldots)$ for some constant μ. In fact, if $k \geq n+1$, then the function

$$g(t) - \sum_{h=0}^{k-1} g(t_h) L_h(t; t_0, t_1, \ldots, t_{k-1})$$

vanishes in $t_0, t_1, \ldots, t_{k-1}$ and, by a lemma of [11], we have

$$\left|g^{(i)}(t) - D^i \sum_{h=0}^{k-1} g(t_h) L_h(t; t_0, t_1, \ldots, t_{k-1})\right| \\ \leq \frac{(b-a)^{k-i-1}}{(k-i-1)!} \int_a^b \left|g^{(k)}(t)\right| dt\,, \quad i = 0, 1, \ldots, n\,.$$

That is

$$\left|D^i \sum_{h=0}^{k-1} g(t_h) L_h(t; t_0, t_1, \ldots, t_{k-1})\right| \leq \mu + \frac{(b-a)^{k-i-1}}{(k-i-1)!} (b-a)\mu\,, \quad i = 0, 1, \ldots, n\,.$$

Therefore we can state the following

Theorem 7. *Let f satisfy the assumptions* (a_1)–(a_3) *and $g \in C^\infty([a,b])$ such that $\max_{t\in[a,b]} \left|g^{(j)}(t)\right| \leq \mu$ $(j = 0, 1, 2, \ldots)$ for some constant μ. Then, for every*

$$\varrho > \mu \left\{1 + \sup_{k \geq n+1} \left[\max_{0 \leq i \leq n} \frac{(b-a)^{k-i}}{(k-i-1)!}\right]\right\}\,,$$

there exists an integer $\nu \geq n+1$ such that the problem (B_k) *with arbitrary t_i and $x_i = g(t_i)$ $(i = 0, 1, \ldots, k-1)$ admits at least a solution $v_k \in W^{k,1}([a,b])$ for all $k \geq \nu$; moreover, $\|v_k\|_{C^n([a,b])} \leq \varrho$.*

4. Concluding remarks

In this paper we have pointed out that, for a given Carathéodory's real-valued function f defined in $[a,b] \times \mathbb{R}^{n+1}$, the existence of a solution for the boundary value problems (A_k) and (B_k) can be obtained provided that the order k of the differential equation (3.1) is appropriately large. To the best of our knowledge, the existence theorems for (A_k) and (B_k) previously established, deal with the case $k \geq n+1$ fixed and require that further conditions on the function f (for example, boundedness or growth conditions) or on the length of $[a,b]$ must be satisfied. We examine a different aspect of the problems (A_k) and (B_k). Nevertheless, the assumptions that are used to prove our results, can be rephrased as conditions on the function f or on the length of the segment $[a,b]$. In this way, one could obtain existence results, which are similar to the known ones.

Another remark concerns the possibility to extend our results to finite vector differential equations. This requires only small modifications in the proofs.

Finally it is possible to consider the case when f varies with k, provided that a condition as $t \to \sup_{k\geq n+1}[\sup_{\|z\|\leq\varrho} |f_k(t,z)|]$ *integrable* is assumed.

References

[1] Abramowitz M., Stegun I.A., *Handbook of Mathematical functions with Formulas, Graphs, and Mathematical Tables*, Dover Publ., New York, 1972.

[2] Agarwal R.P., *Boundary Value Problems for Higher Order Differential Equations*, World Sci. Publ., Singapore, 1986.

[3] Bernfeld S.R., Lakshmikantham V., *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.

[4] Bernstein S.N., *Sur les fonctions régulierèment monotones*, Atti Congresso Int. Mat. Bologna 1928, vol. 2 (1930), 267–275.

[5] ———, *On some properties of cyclically monotonic functions*, Izvestiya Akad. Nauk SSSR, Ser. Mat. **14** (1950), 381–404.

[6] Bonanno G., Marano S.A., *Higher order ordinary differential equations*, Differential Integral Equations **6** (1993), 1119–1123.

[7] Miranda C., *Istituzioni di Analisi Funzionale Lineare – I*, Unione Matematica Italiana, 1978.

[8] Piccinini L.C., Stampacchia G., Vidossich G., *Ordinary Differential Equations in* $\mathbb{R}^n$ (*Problems and Methods*), Springer-Verlag, New York, 1984.

[9] Schoenberg I.J., *On the zeros of successive derivatives of integral functions*, Trans. Amer. Math. Soc. **40** (1936), 12–23.

[10] Whittaker J.M., *Interpolatory Function Theory*, Stechert-Hafner Service Agency, New York, 1964.

[11] Zwirner G., *Su un problema di valori al contorno per equazioni differenziali ordinarie di ordine n*, Rend. Sem. Mat. Univ. Padova **12** (1941), 114–122.

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