Bipolar barotropic nonnewtonian fluid

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Abstract. The paper describes the special situation of barotropic nonnewtonian fluid, where stress tensor can be written in the form of potentials which depend on e_{ij} and $(\frac{\partial e_{ij}}{\partial x_k})$. For this case, we prove the existence and uniqueness of weak solution.

Keywords: barotropic nonnewtonian fluid, bipolar fluid, existence, uniqueness, weak solution

Classification: 76N, 35Q

1. Introduction

In this paper we follow the ideas of the works of [2], [4], [8]. The main step is the study of nonnewtonian bipolar barotropic fluid. We investigate the properties of the momentum equations.

First we prove the existence of the weak solutions. We use the Galerkin method and the method of characteristics. We obtain apriori estimates. Now, similarly as in [5], we pass to the limit and we use a very useful Aubin lemma. Having obtained the existence of solution we prove its uniqueness.

2. Formulation of the problem

We consider the barotropic fluid, which means that the pressure $p \in C^1(0,+\infty)$ depends only on the density ϱ. The expression

$$P(\varrho) = \int_0^\varrho \frac{p'(\sigma)}{\sigma}\,\mathrm{d}\sigma \tag{2.1}$$

exists for every $\varrho > 0$.

Let $\Omega \subset \mathbb{R}^N$, $N = 2,3$, be a bounded domain with a smooth infinitely differentiable boundary and let $I = (0,T)$, $Q_T = I \times \Omega$ be the time-space cylinder. We assume that the body forces are given and

$$b \in L^\infty(Q_T). \tag{2.2}$$

Our aim is to find the velocity vector $v = (v_1,\dots,v_N) : Q_T \to \mathbb{R}^N$ and the density $\varrho : Q_T \to \mathbb{R}$ of compressible barotropic viscous fluid, the motion of which is governed by the continuity equation

$$\frac{\partial\varrho}{\partial t} + \frac{\partial(\varrho v_i)}{\partial x_i} = 0 \tag{2.3}$$

and the momentum equations

$$\frac{\partial(\varrho v_i)}{\partial t} + \frac{\partial}{\partial x_j}(\varrho v_i v_j + \delta_{ij} p - \tau^d_{ij}) = \varrho b_i, \qquad i = 1, 2, \dots, N. \tag{2.4}$$

Let us note that we always use the summation convention that one has to take the sum over an index occurring twice in some term. Now we will specify our bipolar nonnewtonian fluid.

A standard symmetric stress tensor τ_{ij} is considered such that

$$\tau_{ij} = -p\delta_{ij} + \tau^d_{ij}. \tag{2.5}$$

The deformation stress tensor is supposed to be expressed in the form of two potentials

$$\tau^d_{ij} = \frac{\partial V(e)}{\partial e_{ij}} - \frac{\partial}{\partial x_k}\Big(\frac{\partial W(De)}{\partial(\frac{\partial e_{ij}}{\partial x_k})}\Big), \tag{2.6}$$

where $e = \big(e_{ij}\big)^N_{i,j=1}$, $De = \Big(\dfrac{\partial e_{ij}}{\partial x_k}\Big)^N_{i,j,k=1}$, $e_{ij} = e_{ij}(v) = \frac{1}{2}\big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\big)$.
We also consider a third stress tensor τ^d_{ijk}, the form of which is the following

$$\tau^d_{ijk} = \frac{\partial W(De)}{\partial(\frac{\partial e_{ij}}{\partial x_k})}. \tag{2.7}$$

The Clausius-Duhem inequality implies (see [2])

$$\begin{aligned} &\tau^d_{ij}(v)e_{ij}(v) + \tau^d_{jj_1i}(v)\frac{\partial^2 v_j}{\partial x_{j_1}\partial x_i} + \\ &+ \frac{\partial}{\partial x_p}(\tau^d_{jip}(v))\frac{\partial v_j}{\partial x_i} \geq 0. \end{aligned} \tag{2.8}$$

The reader interested in the physical background is referred to [2]. We use (2.6), (2.7) and (2.8) and derive

$$\begin{aligned} &\Bigg(\frac{\partial V}{\partial e_{ij}}(e(v)) - \frac{\partial}{\partial x_k}\Bigg(\frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})}\big(De(v)\big)\Bigg)\Bigg)\frac{\partial v_j}{\partial x_i} + \\ &+ \frac{\partial}{\partial x_p}\Big(\frac{\partial W(De)}{\partial(\frac{\partial e_{ij}}{\partial x_p})}\Big)e_{ij}(v) + \frac{\partial W(De)}{\partial(\frac{\partial e_{jj_1}}{\partial x_i})}\frac{\partial^2 v_j}{\partial x_{j_1}\partial x_i} \geq 0. \end{aligned} \tag{2.8'}$$

We assume the following conditions:

$$\tag{2.9} c_1\Big(1+\Big|De\Big|\Big)^{q-2}|\xi|^2 \le \frac{\partial^2 W(De)}{\partial(\frac{\partial e_{ij}}{\partial x_k})\partial(\frac{\partial e_{i_1j_1}}{\partial x_{k_1}})}\xi^k_{ij}\xi^{k_1}_{i_1j_1} \le c_2\Big(1+\Big|De\Big|\Big)^{q-2}|\xi|^2, \qquad q>N;$$

$$\tag{2.9$'$} c_3(1+|e|)^{q-2}|\xi|^2 \le \frac{\partial^2 V(e)}{\partial e_{ij}\partial e_{i_1j_1}}\xi_{ij}\xi_{i_1j_1} \le c_4(1+|e|)^{q-2}|\xi|^2,$$

where c_1, c_2, c_3, c_4 are positive constants and $|\cdot|$ is a usual Euclidean norm of vector. Let us note that we will always suppose throughout the paper that $q>N$.

Assume

$$\tag{2.10} W(0)=0, \quad V(0)=0;$$

$$\tag{2.11} \frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})}(0)=0, \quad \frac{\partial V}{\partial e_{ij}}(0)=0.$$

The system (2.3), (2.4), is completed by the initial conditions

$$\tag{2.12} v(0)=v_0, \quad \varrho(0)=\varrho_0, \quad \varrho_0>0$$

and the boundary conditions

$$\tag{2.13} \tau^d_{ijk}\nu_j\nu_k = 0 \quad \text{on } (0,T)\times\partial\Omega,$$

$$\tag{2.14} v=0 \quad \text{on } (0,T)\times\partial\Omega.$$

Now, we are ready to give the weak formulation of (2.3), (2.4), (2.12)–(2.14).

Definition 2.15. A couple (ϱ, v) is said to be a weak solution to the problem (2.3), (2.4), (2.12)–(2.14), if the following conditions are fulfilled

(i) $$\varrho \in L^\infty\big(I; W^{1,q}(\Omega)\big),$$

(ii) $$\frac{\partial\varrho}{\partial t} \in L^\infty\big(I; L^q(\Omega)\big),$$

(iii) $$v \in L^\infty\big(I; W^{2,q}(\Omega)\cap W^{1,2}_0(\Omega)\big),$$

(iv) $$\frac{\partial v}{\partial t} \in L^2(Q_T),$$

(v) the continuity equation (2.3) is satisfied in the sense of distributions on Q_T,

(vi)
$$\int_\Omega \frac{\partial}{\partial t}(\varrho v_i)\varphi_i - \int_\Omega \varrho v_i v_j \frac{\partial \varphi_i}{\partial x_j} - \int_\Omega p \frac{\partial \varphi_i}{\partial x_i} + \\ + \int_\Omega \frac{\partial V(e(v))}{\partial e_{ij}} e_{ij}(\varphi) + \int_\Omega \frac{\partial W(De(v))}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)} \frac{\partial e_{ij}}{\partial x_k}(\varphi) = \\ = \int_\Omega \varrho b_i \varphi_i$$

holds for a.e. $t \in I$ and for every $\varphi \in W^{2,q}(\Omega;\mathbb{R}^N) \cap W_0^{1,2}(\Omega;\mathbb{R}^N)$,

(vii) the initial conditions (2.12), where $\varrho_0 \in C^1(\Omega)$ and $v_0 \in W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)$, are fulfilled.

3. A modified Galerkin method

First we construct a sequence of suitable approximations. Let us denote

$$V = W^{2,q}(\Omega;\mathbb{R}^N) \cap W_0^{1,2}(\Omega;\mathbb{R}^N), \\ W = W^{2,2}(\Omega;\mathbb{R}^N) \cap W_0^{1,2}(\Omega;\mathbb{R}^N). \tag{3.1}$$

It is easy to see that $V \subset W$, W is a Hilbert space. Let $<< . , . >>$ be a scalar product in W, $\{z^k\}_{k=1}^\infty$ be a complete orthogonal system of eigenfunctions in W which is given by the solution of the following eigenvalue problem

$$<< v, z^k >> = \lambda_k (v, z^k) \quad \forall v \in W,$$

where $(v, z^k) = \int_\Omega v z^k \, \mathrm{d}x$ and $0 < \lambda_1 \le \lambda_2 \le \dots$. From the regularity of elliptic equations (see [3]) we obtain $z^k \in C^\infty(\bar{\Omega}) \cap C_0(\Omega)$, $k = 1, 2, \dots$.

Let $L_m^2 = \operatorname{span}\{z^1, \dots, z^m\}$ in $L^2(\Omega;\mathbb{R}^N)$, $W_m = \operatorname{span}\{z^1, \dots, z^m\}$ in W and $V_m = \operatorname{span}\{z^1, \dots, z^m\}$ in V. Then we have an orthogonal projector P_m :

$$P_m v = \sum_{k=1}^m \lambda_k (v, z^k) z^k \qquad v \in L^2(\Omega) \tag{3.2}$$

from L^2 onto L_m^2 and also from W onto W_m. Moreover the continuity of P_m in V can be shown by Banach-Steinhaus theorem (see [7]), so

$$\|P_m v\|_{W^{2,q}(\Omega)} \le c \|v\|_{W^{2,q}(\Omega)} \qquad \forall v \in W^{2,q}(\Omega).$$

We put

$$v^m(t,x) = \sum_{k=1}^m c_k(t) z^k(x), \tag{3.3}$$

where $c = (c_1, \dots, c_m) \in C^1(\bar{I})$.

Let us look for $\varrho_m \in C^1(\bar{Q}_T)$ such that

$$\frac{\partial \varrho_m}{\partial t} + \frac{\partial}{\partial x_i}(\varrho_m v_i^m) = 0. \tag{3.4}$$

We shall suppose throughout the paper

$$\varrho_m(0,x) = \varrho_0(x) \in C^1(\bar{\Omega}); \quad \varrho_0(x) > 0 \text{ in } \bar{\Omega}.$$

The relation between Lagrangian and Euler coordinates leads us to the problem

$$\begin{aligned} \dot{x}^m(t) &= v^m\bigl(t, x^m(t)\bigr) \\ x^m(0) &= y, \quad y \in \bar{\Omega}. \end{aligned} \tag{3.5}$$

For every $t \in \bar{I}$, $y \mapsto x^m(t)$ is a diffeomorphism of $\bar{\Omega}$ onto $\bar{\Omega}$. For $\sigma_m = \ln \varrho_m$ we have

$$\frac{\partial \sigma_m}{\partial t} + \frac{\partial}{\partial x_i}(\sigma_m) v_i^m = \frac{\mathrm{d}}{\mathrm{d}t}\sigma_m\bigl(t, x^m(t)\bigr) = -\frac{\partial}{\partial x_i}\bigl(v_i^m(t, x^m)\bigr).$$

Hence

$$\varrho_m(t,x) = \varrho_0(y) \exp\biggl(- \int_0^t \frac{\partial v_i^m(\tau, x^m(\tau))}{\partial x_i} \,\mathrm{d}\tau \biggr), \tag{3.6}$$

where $y = x^m(0)$, $x = x^m(t)$.

Now, let us look for $\bar{v}^m$, $\bar{v}^m(t,x) = \sum_{k=1}^m \bar{c}_k(t) z^k(x)$ such that for every $t \in I$ we have:

$$\begin{aligned} &\int_\Omega \Bigl(\varrho_m \frac{\partial \bar{v}_i^m}{\partial t} + \varrho_m v_j^m \frac{\partial \bar{v}_i^m}{\partial x_j} + \frac{\partial p}{\partial x_i}\Bigr) z_i^\ell \,\mathrm{d}x = \\ &= -\int_\Omega \frac{\partial V}{\partial e_{ij}}(e(\bar{v}^m)) \cdot e_{ij}(z^\ell) - \int_\Omega \frac{\partial W\bigl(De(\bar{v}^m)\bigr)}{\partial\bigl(\frac{\partial e_{ij}}{\partial x_k}\bigr)} \cdot \frac{\partial e_{ij}}{\partial x_k}(z^\ell) + \int_\Omega \varrho_m b_i z_i^\ell, \\ &\quad \ell = 1, 2, \dots, m. \end{aligned} \tag{3.7}$$

The equations (3.7) will be completed by the initial conditions

$$\int_\Omega \sum_{k=1}^m \sum_{i=1}^N \bar{c}_k(0) z_i^k z_i^\ell \,\mathrm{d}x = \int_\Omega \sum_{i=1}^N \bar{v}_i(0,x) z_i^\ell(x) \,\mathrm{d}x, \tag{3.8}$$

where $\ell = 1, \dots, m$. Now, it is enough to assume that

$$v_0(x) = \bar{v}(0,x) \in L^2(\Omega; \mathbb{R}^N). \tag{3.9}$$

The main point is that for $t : 0 \le t \le T$

$$\det\Big(\int_\Omega \sum_{i=1}^N \varrho_m(x,t) z_i^\ell(x) z_i^k(x)\, \mathrm{d}x\Big)_{\ell,k=1}^m \ne 0.$$

So (3.7), (3.8), turn to a system of ordinary differential equations for $\bar{c}_i(t)$ that is uniquely solved.

Now, let us consider the mapping $c \to \bar{c}$ in a small enough interval $< 0, \alpha >$. If we start with $c_i(t)$ in the ball

$$\max_{\langle 0,\alpha\rangle} |c_i(t) - c_i(0)| \le 1; \quad i = 1, 2, \dots, m, \tag{3.10}$$

we get

$$\max_{\langle 0,\alpha\rangle} |\bar{c}_i(t) - c_i(0)| \le 1; \quad i = 1, 2, \dots, m,$$

so

$$\max_{\langle 0,\alpha\rangle} |\dot{\bar{c}}_i| \le K(\alpha),$$

$K(\alpha) > 0$ if α is small enough. The details of the proof can be found in [3]. Thus, applying the Schauder fixed point theorem, we find that $\bar{c}_i = c_i$ on $\langle 0, \alpha\rangle$, hence $\bar{v}^m = v^m$. For such a solution we get the following estimates:

$$\int_{\Omega_t} \varrho_m\, \mathrm{d}x = \int_{\Omega_0} \varrho_0\, \mathrm{d}x, \tag{3.11}$$

(3.12)
$$\frac{1}{2}\int_{\Omega_t} \varrho_m |v^m|^2 - \frac{1}{2}\int_{\Omega_0} \varrho_0 |v^m(0)|^2 + \int_{\Omega_t} P(\varrho_m) - \int_{\Omega_0} P(\varrho_0) +$$
$$+ \int_{Q_t} \frac{\partial V}{\partial e_{ij}} e_{ij}(v^m) + \frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})} \frac{\partial e_{ij}(v^m)}{\partial x_k} = \int_{Q_t} \varrho_m b_i v_i^m,$$

where $\Omega_t = \{(x,t); x \in \Omega\}$ for any $t \in I$, $\Omega_0 = \{(x,0); x \in \Omega\}$.

The equation (3.11) is obtained from (3.4) by integrating over Q_t, applying the Green theorem for the second term and using the boundary condition (2.14). The second estimate (3.12) can be derived from (3.7), where we put $v^m = \bar{v}^m$, multiply it by c^ℓ, sum for $\ell = 1, 2, \dots, m$ and integrate over Q_t, $t \in I$. Now let us multiply (3.7) ($v_m = \bar{v}_m$) by $\dot{c}_\ell$, summed for $\ell = 1, 2, \dots, m$ and integrated over Q_t, $t \in I$.

(3.13)
$$\int_{Q_t} \varrho_m \Big(\frac{\partial v_i^m}{\partial t}\Big)^2 + \int_{Q_t} \varrho_m \frac{\partial v_i^m}{\partial x_j} v_j^m \frac{\partial v_i^m}{\partial t} +$$
$$+ \int_{\Omega_t} V\big(e(v^m)\big) + \int_{\Omega_t} W\big(De(v^m)\big) + \int_{Q_t} \frac{\partial p}{\partial \varrho} \frac{\partial \varrho_m}{\partial x_i} \frac{\partial v_i^m}{\partial t} =$$
$$= \int_{\Omega_0} V\big(e(v^m(0))\big) + \int_{\Omega_0} W\big(De(v^m(0))\big) + \int_{Q_t} \varrho_m b_i \frac{\partial v_i^m}{\partial t}.$$

Remark. In the same way as in [6,10] from the properties of ordinary differential equations and from (3.12), it follows that $\alpha = T$.

Let us denote

$$((v,w)) = \int_\Omega \tau_{ij}^d(v) e_{ij}(w) \qquad v,w \in V. \tag{3.14}$$

The next proposition shows that the form $((.\,,.))$ is an elliptic one.

Proposition 3.15. *The Clasius-Duhem inequality implies*

(i) $$((v,v)) \geq 0 \quad \forall\, v \in V.$$

If moreover (2.9), (2.9′) *are fulfilled, then*

(ii) $$((v,v)) \geq c_1 \|v\|^q_{W^{2,q}(\Omega)} + c_2 \|v\|^2_{W^{2,2}(\Omega)}; \qquad \forall\, v \in V,\ c_1, c_2 > 0.$$

Proof: (i) Using (2.8) it is enough to show that

$$\int_\Omega \tau_{jj_1i}^d(v) \frac{\partial^2 v_j}{\partial x_{j_1} \partial x_i} + \frac{\partial}{\partial x_p}(\tau_{jip}^d(v)) \frac{\partial v_j}{\partial x_i} = 0.$$

But we have

$$\int_\Omega \frac{\partial}{\partial x_p}(\tau_{jip}^d(v)) \frac{\partial v_j}{\partial x_i} = -\int_\Omega \tau_{jip}^d(v) \frac{\partial^2 v_j}{\partial x_p \partial x_i} + \int_{\partial\Omega} \tau_{jip}^d(v) \frac{\partial v_j}{\partial x_i} \nu_p.$$

It can be proved (see [12]) that

$$\int_{\partial\Omega} \tau_{jip}^d(v) \frac{\partial v_j}{\partial x_i} \nu_p = \int_{\partial\Omega} \tau_{jip}^d(v) \frac{\partial v_j}{\partial \nu} \nu_i \nu_p = 0,$$

due to the boundary conditions (2.13), (2.14).

To prove (ii) we use L^p -version ($p \geq 1$) of the Korn inequality (see[13]) which has the following form

$$\int_\Omega |e(w)|^p \geq c \|w\|^p_{W^{1,p}(\Omega)} \qquad \forall w \in W_0^{1,p}(\Omega). \tag{3.16}$$

Moreover, a "version" of (3.16) for De can be proved, for more details see [13].

$$\int_\Omega |De(w)|^p \geq c \|w\|^p_{W^{2,p}(\Omega)} \qquad \forall w \in W^{2,p}(\Omega), \quad w = 0 \text{ on } \partial\Omega. \tag{3.16′}$$

Further, using the mean-value theorem, (2.9′), (2.11) we get

(3.17)
$$\frac{\partial V}{\partial e_{ij}}(e(v))\cdot e_{ij}(v)=\frac{\partial V}{\partial e_{ij}}(e(v))\cdot e_{ij}(v)-\frac{\partial V}{\partial e_{ij}}(0)\cdot e_{ij}(v)=$$
$$\int_0^1\frac{d}{d\tau}\frac{\partial V}{\partial e_{ij}}(\tau e(v))e_{ij}(v)\,\mathrm{d}\tau=\int_0^1\frac{\partial^2 V}{\partial e_{ij}\partial e_{i_1j_1}}(\tau e(v))e_{ij}(v)e_{i_1j_1}(v)\,\mathrm{d}\tau\geq$$
$$\int_0^1 c_3(1+\tau|e|)^{q-2}|e|^2 d\tau\geq c_1|e|^q+c_2|e|^2,\qquad c_1,c_2>0.$$

Similarly for $W(De)$ we obtain

$$\frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})}\big(De(v)\big)\cdot\frac{\partial e_{ij}}{\partial x_k}(v)\geq c_3|De|^q+c_4|De|^2, \tag{3.18}$$

where $c_3,c_4>0$.

$$((v,v))=\int_\Omega\frac{\partial V(e(v))}{\partial e_{ij}}e_{ij}(v)-\frac{\partial}{\partial x_k}\Big(\frac{\partial W(De(v))}{\partial(\frac{\partial e_{ij}}{\partial x_k})}\Big)e_{ij}(v)=$$
$$=(\text{ Green theorem and (2.13) })=\int_\Omega\frac{\partial V(e)}{\partial e_{ij}}e_{ij}+\int_\Omega\frac{\partial W(De)}{\partial(\frac{\partial e_{ij}}{\partial x_k})}\frac{\partial e_{ij}}{\partial x_k}$$
$$\geq\ (\text{ (3.17), (3.18) })\ \geq c_1\int_\Omega|e|^q+c_2\int_\Omega|e|^2+c_3\int_\Omega|De|^q+c_4\int_\Omega|De|^2\geq$$
$$\geq(\text{ (3.16), (3.16′) })\geq c_5\|v\|^q_{W^{1,q}(\Omega)}+c_6\|v\|^2_{W^{1,2}(\Omega)}+c_7\|v\|^q_{W^{2,q}(\Omega)}+$$
$$+c_8\|v\|^2_{W^{2,2}(\Omega)}\geq c_7\|v\|^q_{W^{2,q}(\Omega)}+c_8\|v\|^2_{W^{2,2}(\Omega)}.$$

□

The following proposition gives apriori estimates based on (3.11)–(3.13).

Proposition 3.19 (apriori estimates)**.** *Let $v_0\in V$. The solutions (ϱ_m,v^m) of (3.4), (3.7) ($v^m=\bar v^m$) satisfy the following estimates uniformly with respect to m.*

(i) $\|\varrho_m\|_{L^\infty\left(I;L^1(\Omega)\right)}\leq const.,$

(ii) $\|v^m\|_{L^q\left(I;W^{2,q}(\Omega)\right)}\leq const.,$

(iii) $\|v^m\|_{L^2\left(I;W^{2,2}(\Omega)\right)}\leq const.,$

(iv) $\|\varrho_m\|_{L^\infty(Q_T)}\leq const.,$

(v) $$\|\varrho_m\|_{L^\infty\left(I;W^{1,q}(\Omega)\right)} \leq const.,$$

(vi) $$\|\frac{\partial \varrho_m}{\partial t}\|_{L^\infty\left(I;L^q(\Omega)\right)} \leq const.,$$

(vii) $$\|\frac{\partial v^m}{\partial t}\|_{L^2(Q_T)} \leq const.,$$

(viii) $$\|v^m\|_{L^\infty\left(I;W^{2,q}(\Omega)\cap W_0^{1,2}(\Omega)\right)} \leq const..$$

PROOF:
(i): From (3.11) we have

$$\forall t \in I : \|\varrho_m\|_{L^1(\Omega_t)} \leq \|\varrho\|_{L^1(\Omega_0)},$$

that proves (i).

(ii) and (iii) follow from (3.12) and the ellipticity of the form $((.\,,.))$. More precisely,

(α) $$\int_{Q_t} \varrho_m b_i v_i^m \leq c \int_0^T \|\varrho_m\|_{L^1(\Omega)} \|b\|_{L^\infty(\Omega)} \|v^m\|_{W^{2,q}(\Omega)} \leq$$
$$\leq \text{(Young inequality)} \leq c_1(\varepsilon)\|\varrho_m\|^{q'}_{L^\infty(I;L^1(\Omega))}\|b\|^{q'}_{L^\infty(Q_T)}$$
$$+ \varepsilon\|v^m\|^q_{L^q(I;W^{2,q}(\Omega))},$$

where $q' : \frac{1}{q'} + \frac{1}{q} = 1$; $\varepsilon > 0$ arbitrary. From the properties of the function P, it follows (see [4,11])

(β) $$\int_{\Omega_t} P(\varrho_m) > 0.$$

Further,

(γ) $$\frac{1}{2}\int_{\Omega_0} \varrho_0 |v^m(0)|^2 \leq c\frac{1}{2}\int_{\Omega_0} \varrho_0|v_0|^2 \leq \text{const.} \quad \text{and} \quad \int_{\Omega_0} P(\varrho_0) \leq \text{const.}\ ,$$

due to $\varrho_0 \in C^1(\bar{\Omega})$, $v_0 \in V \hookrightarrow L^2(\Omega)$ and P_m is orthogonal on $L^2(\Omega)$, (2.1).

(δ) $$\int_0^t ((v^m, v^m)) \geq c_1\|v^m\|^q_{L^q((0,t);W^{2,q}(\Omega))} + c_2\|v^m\|^2_{L^2((0,t);W^{2,2}(\Omega))}.$$

Putting α, β, γ, δ, into (3.12) we find that

$$\|\varrho_m|v^m|^2\|_{L^\infty(I;L^1(\Omega))} \leq \text{ const.}$$

and

$$\|v^m\|_{L^2(I;W^{2,2}(\Omega))} \leq \text{ const.},$$
$$\|v^m\|_{L^q(I;W^{2,q}(\Omega))} \leq \text{ const..}$$

(iv): Since we can solve the continuity equation precisely, i.e.

$$\varrho_m(t,x) = \varrho\big(x(0)\big) \exp\Big(- \int_0^t \frac{\partial v_i^m}{\partial x_i}\big(\tau, x^m(\tau)\big)\, \mathrm{d}\tau \Big),$$

we get

$$\varrho_m(t,x) \geq c \exp\Big(-\gamma \int_0^T \|v^m\|_{W^{2,q}(\Omega)}\, \mathrm{d}\tau \Big) \geq c \exp(-\gamma_1 \cdot T^{1/q'}), \quad q' : \frac{1}{q} + \frac{1}{q'} = 1.$$

Similarly we obtain the estimate from above. These estimates imply (iv).
Analogously as in [5] we get from (3.6):

$$\text{(v)} \quad \varrho_m \in L^\infty\big(I; W^{1,q}(\Omega)\big) \quad \text{and} \quad \frac{\partial \varrho_m}{\partial t} \in L^q\big(I; L^q(\Omega)\big).$$

Now we will estimate terms in (3.13):

(α)
$$\int_{Q_t} \varrho_m \big(\frac{\partial v_i^m}{\partial t}\big)^2 \geq c_1 \Big\| \frac{\partial v^m}{\partial t} \Big\|^2_{L^2(Q_t)},$$
(β)
$$\int_{\Omega_t} V(e(v^m)) + W(De(v^m)) \geq c_2 \big(\|v^m(t)\|^q_{W^{2,q}(\Omega_t)} + \|v^m(t)\|^2_{W^{2,2}(\Omega_t)} \big),$$

here we used the same idea as in (3.17), L^p-version of Korn inequality and (2.9), (2.9′), (2.10), (2.11). In an analogous way, one can find

$$(\gamma) \qquad \int_\Omega V(e(v^m(0))) + W(De(v^m(0))) \leq c\big(\|v^m(0)\|^q_{W^{2,q}(\Omega)} +$$
$$+ \|v^m(0)\|^{q-1}_{W^{2,q-1}(\Omega)} + \cdots + \|v^m(0)\|^2_{W^{2,2}(\Omega)} \big) \leq$$
$$(\text{ using continuity of } P_m \text{ }) \leq c_3 \big(\|v_0\|^q_{W^{2,q}(\Omega)} +$$
$$+ \|v_0\|^{q-1}_{W^{2,q-1}(\Omega)} + \cdots + \|v_0\|^2_{W^{2,2}(\Omega)} \big).$$

Further,

$$(\delta)\qquad \int_{Q_t} \varrho_m b_i \frac{\partial v_i^m}{\partial t} \le c_4 \|\varrho_m\|_{L^\infty(Q_T)} \|b\|_{L^\infty(Q_T)} \|\frac{\partial v^m}{\partial t}\|_{L^2(Q_T)},$$

(ε)

$$-\int_{Q_t} \varrho_m \frac{\partial v_i^m}{\partial x_j} v_j^m \frac{\partial v_i^m}{\partial t} \le (\text{ Hölder inequality}) \le$$

$$\le \Big(\sum_i \int_{Q_t} \big(\varrho_m \frac{\partial v_i^m}{\partial x_j} v_j^m\big)^2 \Big)^{\frac{1}{2}} \cdot \Big\| \frac{\partial v^m}{\partial t} \Big\|_{L^2(Q_t)} \le$$

$$c_5 \|\varrho_m\|^{\frac{1}{2}}_{L^\infty(Q_t)} \|\varrho_m |v^m|^2\|^{\frac{1}{2}}_{L^\infty(I;L^1(\Omega))} \|v^m\|_{L^q((0,t);W^{2,q}(\Omega))} \Big\| \frac{\partial v^m}{\partial t} \Big\|_{L^2(Q_t)},$$

(ζ)

$$-\int_{Q_t} \frac{\partial p}{\partial \varrho} \frac{\partial \varrho_m}{\partial x_i} \frac{\partial v_i^m}{\partial t} \le c_6 \|\varrho_m\|_{L^\infty(I;W^{1,q}(\Omega))} \Big\| \frac{\partial v^m}{\partial t} \Big\|_{L^2(Q_t)}.$$

Putting all these results together into (3.13) we obtain

$$c_1 \Big\| \frac{\partial v^m}{\partial t} \Big\|^2_{L^2(Q_t)} + c_2 (\|v^m(t)\|^q_{W^{2,q}(\Omega_t)} + \|v^m(t)\|^2_{W^{2,2}(\Omega_t)}) \le$$

$$c_3 (\|v_0\|^q_{W^{2,q}(\Omega)} + \dots + \|v_0\|^2_{W^{2,2}(\Omega)}) + c_4 \|\varrho_m\|_{L^\infty(Q_t)} \|b\|_{L^\infty(Q_t)} \Big\| \frac{\partial v^m}{\partial t} \Big\|_{L^2(Q_t)} +$$

$$c_5 \|\varrho_m\|^{\frac{1}{2}}_{L^\infty(Q_t)} \|\varrho_m |v^m|^2\|^{\frac{1}{2}}_{L^\infty((0,t);L^1(\Omega))} \|v^m\|_{L^q((0,t);W^{2,q}(\Omega))} \Big\| \frac{\partial v^m}{\partial t} \Big\|_{L^2(Q_t)} +$$

$$c_6 \|\varrho_m\|_{L^\infty((0,t);W^{1,q}(\Omega))} \Big\| \frac{\partial v^m}{\partial t} \Big\|_{L^2(Q_t)}, \qquad t \in I.$$

Hence using (i)–(v) and the Young inequality we find that

$$\text{(vii)} \quad \Big\| \frac{\partial v^m}{\partial t} \Big\|_{L^2(Q_T)} \le c \qquad \text{and} \qquad \text{(viii)} \quad \|v^m\|_{L^\infty(I;W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega))} \le c.$$

To prove (vi) we show that $\frac{\partial}{\partial x_i}(\varrho_m v_i^m) \in L^\infty(I; L^q(\Omega))$ and the rest follows from the continuity equation. So

$$\Big(\int_{\Omega_t} \Big| \frac{\partial}{\partial x_i} (\varrho_m v_i^m) \Big|^q \,\mathrm{d}x \Big)^{1/q} \le \Big(\int_{\Omega_t} \Big| \frac{\partial \varrho_m}{\partial x_i} v_i^m \Big|^q \,\mathrm{d}x \Big)^{1/q} +$$

$$+ \Big(\int_{\Omega_t} |\varrho_m \frac{\partial v_i^m}{\partial x_i} \Big|^q \,\mathrm{d}x \Big)^{1/q} \le \|v^m\|_{C^1(\bar\Omega_t)} \|\nabla \varrho_m\|_{L^q(\Omega_t)} +$$

$$+ \|\nabla v^m\|_{C(\bar\Omega_t)} \|\varrho_m\|_{L^q(\Omega_t)} \le k \|v^m\|_{W^{2,q}(\Omega_t)} \|\varrho_m\|_{W^{1,q}(\Omega_t)} \le c.$$

□

4. Limit process

Our aim will be to prove the existence theorem. To this goal we will need the well-known Aubin lemma.

Lemma 4.1 (Aubin lemma). *Let B be a Banach space, B_i $(i = 0, 1)$ reflexive Banach spaces. Let $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$, $1 < p_i < \infty$. Let $W = \{v; v \in L^{p_0}(I, B_0); \frac{\partial v}{\partial t} \in L^{p_1}(I, B_1)\}$, then $W \hookrightarrow\hookrightarrow L^{p_0}(I, B)$. (Here, $\hookrightarrow\hookrightarrow$ denotes compact imbeddings.)*

Theorem 4.1. *Let $\varrho_0 \in C^1(\bar{\Omega})$, $\varrho_0 > 0$ in $\bar{\Omega}$, $v_0 \in V$. Let the assumptions* (2.8), (2.8′), (2.9)–(2.13) *be satisfied. Then we can choose a subsequence of solutions to* (3.4), (3.7) $\{\varrho^m, v^m\}_m$ *such that*

(i) $\varrho^m \to \varrho$ *strongly in* $L^2(Q_T)$;

(ii) $v^m \to v$ *strongly in* $L^2(I; W^{1,2}(\Omega))$;
$v^m \to v$ *weakly in* $L^q\big(I; W^{2,q}(\Omega)\big)$;

(iii) $\dfrac{\partial v^m}{\partial t} \to \dfrac{\partial v}{\partial t}$ *weakly in* $L^2(Q_T)$;

(iv) $\dfrac{\partial \varrho_m}{\partial t} \to \dfrac{\partial \varrho}{\partial t}$ *$*$-weakly in* $L^\infty\big(I; L^q(\Omega)\big)$;

(v) $\int_{Q_T} p(\varrho_m)\dfrac{\partial \phi_i}{\partial x_i} \to \int_{Q_T} p(\varrho)\dfrac{\partial \phi_i}{\partial x_i} \qquad \forall \phi \in L^q\big(I; W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)\big)$;

(vi) $\varrho_m v^m \to \varrho v$ *strongly in* $L^2(Q_T)$;

(vii) $\varrho_m v_i^m v_j^m \to \varrho v_i v_j$ *weakly in* $L^2(Q_T)$.

Proof: (i) follows from Lemma 4.1 with $B = B_1 = L^2(\Omega)$, $B_0 = W^{1,2}(\Omega)$, $p_1 = p_0 = 2$, and 3.19, the assertions (v), (vi).

(ii) can be proved analogously. In Lemma 4.1 we put $B_0 = W^{2,2}(\Omega)$, $B = W^{1,2}(\Omega)$, $B_1 = L^2(\Omega)$ and $p_0 = p_1 = 2$. The second assertion is obtained from (3.14)(ii).

(iii) and (iv) are the consequences of (3.14)(vii), (vi), respectively.

To prove (v) we will need the following lemma (see [6]).

Lemma 4.2. *Let G be a bounded domain in $\mathbb{R}^N \times \mathbb{R}$, q_m and q be functions from $L^{p'}(G)$, $1 < p' < \infty$ such that $\|q_m\|_{L^{p'}(G)} < c$ and $q_m \to q$ a.e. in G. Then $q_m \rightharpoonup q$ in $L^{p'}(G)$.*

Since $\varrho_m \to \varrho$ in $L^2(Q_T)$ we find a subsequence, still denoted by ϱ_m, such that $\varrho_m \to \varrho$ a.e. in Q_T. Because of (3.14)(iv) $\|p(\varrho_m)\|_{L^p(Q_T)} \le c$ for every $1 < p < \infty$. Moreover $p(\varrho_m) \to p(\varrho)$ a.e. in Q_T, $p \in C^1(0, \infty)$. Applying Lemma 4.2 we obtain $p(\varrho_m) \to p(\varrho)$ weakly in $L^p(Q_T)$ for every $1 < p < \infty$ that implies (v).

(vi) is a consequence of the following estimates:

$$\Big(\int_{Q_T} |\varrho_m v^m - \varrho v|^2\Big)^{1/2} \le \Big(\int_{Q_T} (\varrho_m(v^m - v))^2\Big)^{1/2} + \Big(\int_{Q_T} (v(\varrho_m - \varrho))^2\Big)^{1/2} \le$$
$$\le \|\varrho_m\|_{L^\infty(Q_T)}\|v^m - v\|_{L^2(Q_T)} + \|v\|_{L^\infty(I;C^1(\bar{\Omega}))}\|\varrho_m - \varrho\|_{L^2(Q_T)} \le$$
$$\le c\|v^m - v\|_{L^2(Q_T)} + \|v\|_{L^\infty(I;W^{2,q}(\Omega))}\|\varrho_m - \varrho\|_{L^2(Q_T)} \longrightarrow 0, \text{ as m } \to \infty,$$

due to (i) and (ii).

(vii) is shown in this way:

$$\begin{aligned}&\int_{Q_T}(\varrho_m v_i^m v_j^m-\varrho v_i v_j)\varphi\le\int_{Q_T}(\varrho_m v_i^m-\varrho v_i)v_j^m\varphi+\int_{Q_T}(\varrho v_i)(v_j^m-v_j)\varphi\le\\&\le\|\varrho_m v_i^m-\varrho v_i\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)}\|v_j^m\|_{L^\infty(I;V)}+\\&+\|v_j^m-v_j\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)}\|\varrho\|_{L^\infty(Q_T)}\cdot\|v_i\|_{L^\infty(I;V)},\end{aligned}$$

for every $\varphi\in L^2(Q_T)$. Both terms tend to zero due to (ii) and (vi). □

Now we are able to prove the existence of the weak solution to the problem (2.3), (2.4), (2.12)–(2.14), defined in (2.15).

Theorem 4.2. *Let the assumptions of Theorem* 4.1 *be satisfied. Then there exists at least one weak solution to the problem* (2.3), (2.4), (2.12)–(2.14), *s.t.*

$$\varrho\in L^\infty\big(I;W^{1,q}(\Omega)\big),\tag{4.3}$$

$$\frac{\partial\varrho}{\partial t}\in L^\infty\big(I;L^q(\Omega)\big),\tag{4.4}$$

$$v\in L^\infty\big(I;W^{2,q}(\Omega)\cap W_0^{1,2}(\Omega)\big),\tag{4.5}$$

$$\frac{\partial v}{\partial t}\in L^2(Q_T).\tag{4.6}$$

Proof: Let us multiply (3.4) by any $\varphi\in C_0^\infty(Q_T)$. Letting $m\to\infty$ and using (4.1)(iv); (vi) we easily verify that the continuity equation (2.2) holds in the sense of distributions and a.e. in Q_T.

Now we prove that (ϱ,v) satisfy (2.15)(vi) and also (2.4) in the sense of distributions on Q_T. In (3.7) let us put $v^m=\bar v^m$, instead of z_i^ℓ we multiply the equation by any $\varphi\in C^1(\bar I;V_m)$ s.t. $\varphi(T)=0$. By integrating over Q_T we obtain

$$\begin{aligned}&-\int_{Q_T}\varrho_m v_i^m\frac{\partial\varphi_i}{\partial t}-\int_{Q_T}\varrho_m v_i^m v_j^m\frac{\partial\varphi_i}{\partial x_j}-\int_{Q_T}p_m\frac{\partial\varphi_i}{\partial x_i}+\\&+\int_{Q_T}\frac{\partial V(e(v^m))}{\partial e_{ij}}e_{ij}(\varphi)+\int_{Q_T}\frac{\partial W}{\partial\big(\frac{\partial e_{ij}}{\partial x_k}\big)}\big(De(v^m)\big)\frac{\partial e_{ij}}{\partial x_k}(\varphi)\\&=\int_{Q_T}\varrho_m b_i\varphi_i+\int_\Omega\varrho_0 v_i^m(0)\varphi_i(0)\qquad\forall\varphi\in C^1(\bar I;V_m).\end{aligned}\tag{4.7}$$

Letting $m\to\infty$ in (4.7) we find that (2.15)(vi) holds a.e. in I and for any smooth test function from $\bigcup_{m=1}^\infty V_m$. The proof is based on the results from Theorem 4.1. Let us show more precisely the limit process in $((v^m,\varphi))$.

$$\begin{aligned}&\int_0^T\int_{\Omega_t}\Big(\frac{\partial V(e(v^m))}{\partial e_{ij}}-\frac{\partial V(e(v))}{\partial e_{ij}}\Big)e_{ij}(\varphi)=\\&=\int_0^T\int_{\Omega_t}\int_0^1\frac{\partial^2V}{\partial e_{ij}\partial e_{i_1j_1}}(e(v+\theta(v^m-v)))\,\mathrm{d}\theta\cdot e_{i_1j_1}(v^m-v)e_{ij}(\varphi).\end{aligned}$$

Put $w_\theta = v + \theta(v^m - v)$. Using (2.9′) we can estimate the second derivative of the potential V:

$$\int_0^T \int_{\Omega_t} \int_0^1 c(1 + |e(w_\theta)|)^{q-2} \, \mathrm{d}\theta |e(v^m - v)||e(\varphi)| \le$$

$$c\|\varphi\|_{C^1(\bar{I};V)} \|v^m - v\|_{L^2(I;W^{1,2}(\Omega))} \int_0^1 \Big(\int_{Q_T} (1 + |e(w_\theta)|)^{2(q-2)} \Big)^{1/2} \, \mathrm{d}\theta.$$

The last term is bounded by a constant, because

$$w_\theta \in L^\infty(I; W^{2,q}(\Omega)) \hookrightarrow L^\infty(I; C^1(\bar{\Omega})) \qquad \forall\, \theta \in \langle 0, 1\rangle\,.$$

Due to the strong convergence of v^m in $L^2(I; W^{1,2}(\Omega))$ the limit equals zero. Further,

$$\int_0^T \int_{\Omega_t} \left(\frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})} \Big(De(v^m) \Big) - \frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})} \Big(De(v) \Big) \right) \frac{\partial e_{ij}}{\partial x_k}(\varphi) =$$

$$= \int_0^T \int_{\Omega_t} \int_0^1 \frac{\partial^2 W}{\partial(\frac{\partial e_{ij}}{\partial x_k}) \partial(\frac{\partial e_{i_1 j_1}}{\partial x_{k_1}})} \Big(De(w_\theta) \Big) \, \mathrm{d}\theta \cdot \frac{\partial e_{i_1 j_1}}{\partial x_{k_1}}(v^m - v) \cdot \frac{\partial e_{ij}}{\partial x_k}(\varphi).$$

Now we use that $v^m \to v$ weakly in $L^q(I; W^{2,q}(\Omega))$ and show that for all $\theta \in \langle 0, 1\rangle$:

$$\int_0^T \int_{\Omega_t} |c(1 + |\frac{\partial e_{ij}}{\partial x_k}(w_\theta)|)^{q-2} \frac{\partial e_{ij}}{\partial x_k}(\varphi)|^p < c,$$

where $p : \frac{1}{p} + \frac{1}{q} = 1$. But this can be done by Hölder inequality and using the facts $w_\theta \in L^q(I; W^{2,q}(\Omega))$, $\varphi \in C^1(\bar{I}; \bigcup_m V_m) \subset C^1(\bar{I}; W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega))$. Notice that it is enough to take $\varphi \in C^1(\bar{I}; W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega))$.

Other terms in (4.7) are evaluated easily. We use (4.1)(vi), (vii), (v), (i), and the fact

$$\|v^m(0) - v_0\|_{L^2(\Omega)} \to 0 \qquad m \to \infty,$$

due to the definition of projector P_m. So, we can pass to the limit in (4.7) and obtain that (4.7) holds for (ϱ, v) and smooth test functions.

Now, it is just to realize that every term in (4.7) is a linear continuous functional on V. It means that we can complete the set of test functions: $\varphi \in C^1(\bar{I}; W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega))$. The necessary estimates in $((v, \varphi^n - \varphi))$ (where φ^n are smooth test functions and $\varphi^n \to \varphi$ -strongly in V) can be done in the same way as previously. We only change $v^m := v$, $v := 0$ and use (2.11). Other terms in (4.7) do not make problems. We finally obtain that (2.15)(vi) holds a.e. in I and for any test function from V. The momentum equation is satisfied also in the sense of distributions. The estimates (4.3)–(4.6) are fulfilled due to Proposition 3.19.

$\square$

5. Uniqueness

Theorem 5.1. *Let the assumptions of Theorem* 4.2 *be satisfied. Then there is the unique weak solution to the problem* (2.3), (2.4), (2.12)–(2.14).

Proof: Let (ϱ, v), $(\bar\varrho, \bar v)$ be two solutions. We denote $\xi = \varrho - \bar\varrho$ and $w = v - \bar v$. From the continuity equation we obtain

$$\frac{\partial \xi}{\partial t} = -\xi \frac{\partial v_j}{\partial x_j} - \bar\varrho \frac{\partial w_j}{\partial x_j} - \frac{\partial \xi}{\partial x_j} v_j - \frac{\partial \bar\varrho}{\partial x_j} w_j \quad \text{a.e. in } Q_T. \tag{5.1}$$

From the weak formulation of the momentum equation we get

(5.2)

$$\begin{aligned}
&\int_{\Omega_t} \bar\varrho \frac{\partial w_i}{\partial t}\phi_i + \int_{\Omega_t} \xi \frac{\partial v_i}{\partial t}\phi_i + \int_{\Omega_t} \xi v_j \frac{\partial v_i}{\partial x_j}\phi_i + \int_{\Omega_t} \bar\varrho w_j \frac{\partial v_i}{\partial x_j}\phi_i + \bar\varrho \bar v_j \frac{\partial w_i}{\partial x_j}\phi_i = \\
&\quad = \int_{\Omega_t} \int_0^1 \frac{\partial p}{\partial \varrho}(\bar\varrho + \vartheta\xi)\xi \, \mathrm{d}\theta \cdot \frac{\partial \phi_i}{\partial x_i} - \\
&\quad - \int_{\Omega_t} \int_0^1 \frac{\partial^2 V}{\partial e_{ij} \partial e_{k\ell}} \big(e(\bar v + \vartheta^1 w)\big) \, \mathrm{d}\vartheta^1 \cdot e_{k\ell}(w) e_{ij}(\phi) - \\
&\quad - \int_{\Omega_t} \int_0^1 \frac{\partial^2 W}{\partial(\frac{\partial e_{ij}}{\partial x_\ell})\partial(\frac{e_{i_1 j_1}}{\partial x_{\ell_1}})} \Big(De(\bar v + \vartheta^2 w)\Big) \, \mathrm{d}\vartheta^2 \cdot \frac{\partial e_{i_1 j_1}}{\partial x_{\ell_1}}(w) \frac{\partial e_{ij}}{\partial x_\ell}(\phi) + \int_{\Omega_t} \xi b_i \phi_i,
\end{aligned}$$

for every $\phi \in V$, $t \in I$. Now let us multiply (5.1) by ξ and integrate over Q_t, $t \in I$. We use Proposition 3.19, Young inequality and get the estimate

$$\frac{1}{2}\int_{\Omega_t} \xi^2 \, \mathrm{d}x \le c_1 K_1(\varepsilon) \int_{Q_T} \xi^2 \, \mathrm{d}x \, \mathrm{d}\tau + c_2 \varepsilon \int_0^t \|w\|^2_{W^{2,2}(\Omega)} \, \mathrm{d}\tau, \tag{5.3}$$

where

$$\begin{aligned}
&c_1 = c(\|v\|_{L^\infty(I;W^{2,q}(\Omega))} + \|\bar\varrho\|_{L^\infty(Q_T)} + \|\bar\varrho\|_{L^\infty(I;W^{1,q}(\Omega))});\\
&c_2 = c'(\|\bar\varrho\|_{L^\infty(Q_T)} + \|\bar\varrho\|_{L^\infty(I;W^{1,q}(\Omega))}),\\
&K_1(\varepsilon) > 0, \quad \varepsilon > 0 \text{ arbitrary .}
\end{aligned}$$

From (5.2) ($\phi = w$) using 3.19 and Young inequality

(5.4)

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_t} |w|^2 \, \mathrm{d}x + \int_{Q_t} \int_0^1 \frac{\partial^2 V}{\partial e_{ij} \partial e_{i_1 j_1}} (e(\bar v + \vartheta_1 w)) \, \mathrm{d}\vartheta_1 \cdot e_{i_1 j_1}(w) e_{ij}(w) \, \mathrm{d}x \, \mathrm{d}\tau + \\
&+ \int_{Q_t} \int_0^1 \frac{\partial^2 W}{\partial(\frac{\partial}{\partial x_k} e_{ij})\partial(\frac{\partial}{\partial x_{k_1}} e_{i_1 j_1})} \Big(De(\bar v + \vartheta_2 w)\Big) \, \mathrm{d}\vartheta_2 \cdot \\
&\cdot \frac{\partial}{\partial x_k} e_{ij}(w) \frac{\partial}{\partial x_{k_1}} e_{i_1 j_1}(w) \, \mathrm{d}x \, \mathrm{d}\tau \le \\
&\le c_3 K_3(\varepsilon) \int_{Q_T} \xi^2 \, \mathrm{d}x \, \mathrm{d}\tau + c_4 K_4(\varepsilon) \int_{Q_T} |w|^2 \, \mathrm{d}x \, \mathrm{d}\tau + c_5 \varepsilon \int_0^t \|w\|^2_{W^{2,2}(\Omega)} \, \mathrm{d}\tau,
\end{aligned}$$

where

$$c_3 = c\Big(\Big\|\frac{\partial v}{\partial t}\Big\|^2_{L^2(Q_T)} + \|v\|^2_{L^\infty(I;W^{2,q}(\Omega))} + \max_J\Big|\frac{\mathrm{d}p}{\mathrm{d}\varrho}\Big| + \|b\|^2_{L^\infty(Q_T)}\Big);$$

$$J = \Big\langle 0, \|\bar\varrho\|_{L^\infty(Q_T)} + \|\varrho\|_{L^\infty(Q_T)}\Big\rangle;$$

$$c_4 = c\Big(\|\bar\varrho\|_{L^\infty(Q_T)}\|v\|_{L^\infty(I;W^{2,q}(\Omega))} + \|\bar\varrho\|_{L^\infty(Q_T)}\|\bar v\|_{L^\infty(I;W^{2,q}(\Omega))} + \|v\|^2_{L^\infty(I;W^{2,q}(\Omega))}\Big);$$

$$c_5 = c\Big(\|\bar\varrho\|_{L^\infty(Q_T)}\|\bar v\|_{L^\infty(I;W^{2,q}(\Omega))} + \max_J\Big|\frac{\mathrm{d}p}{\mathrm{d}\varrho}\Big| + \Big\|\frac{\partial\bar\varrho}{\partial t}\Big\|_{L^\infty(I;L^q(\Omega))}\Big),$$

$K_2(\varepsilon),\ K_3(\varepsilon) > 0,\quad \varepsilon > 0$ arbitrary.

Similarly as in 3.15 we can show that

$$\int_{Q_t}\int_0^1 \frac{\partial^2 V}{\partial e_{ij}\partial e_{i_1j_1}}(e(\bar v + \vartheta_1 w))\ \mathrm{d}\vartheta_1 \cdot e_{i_1j_1}(w)e_{ij}(w)\ \mathrm{d}x\ \mathrm{d}\tau + \int_{Q_t}\int_0^1 \frac{\partial^2 W}{\partial(\frac{\partial}{\partial x_k}e_{ij})\partial(\frac{\partial}{\partial x_{k_1}}e_{i_1j_1})}\Big(De(\bar v + \vartheta_2 w)\Big)\ \mathrm{d}\vartheta_2 \cdot \frac{\partial}{\partial x_k}e_{ij}(w)\frac{\partial}{\partial x_{k_1}}e_{i_1j_1}(w)\ \mathrm{d}x\ \mathrm{d}\tau \geq c\int_0^t \|w\|^2_{W^{2,2}(\Omega)}.$$

Now we add (5.3), (5.4), and find

$$\frac{1}{2}\int_{\Omega_t}|w|^2 + |\xi|^2\ \mathrm{d}x \leq c\int_0^T\int_\Omega |w|^2 + |\xi|^2\ \mathrm{d}x\ \mathrm{d}t. \tag{5.5}$$

Hence Gronwall inequality concludes that

$$\xi = 0, \quad w = 0 \qquad \text{a.e.} \in Q_T.$$

This finishes the proof. □

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